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Introduction to Mathematics

Taibah University, Preparatory Year Program

MATH 101

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Contents

1 Review of Basic Concepts

Positive and negative numbers, used to represent gains and losses on a board such as this one, are examples of *real numbers* encountered in applications of mathematics.

Chapter objectives

- Define the concept of sets
- Distinguish between different type of numbers
- Identify different factoring methods for polynomials
- Name the basic rational and radical expressions

1.1 Sets

- Basic Definitions
- Operations on Sets

Basic Definitions

A **set** is a collection of objects. The objects that belong to a set are called the **elements,** or **members,** of the set. In algebra, the elements of a set are usually numbers. Sets are commonly written using **set braces,** $\{\ \}$. For example, the set containing the elements 1, 2, 3, and 4 is written as follows.

$$\{1, 2, 3, 4\}$$

Since the order in which the elements are listed is not important, this same set can also be written as $\{4, 3, 2, 1\}$ or with any other arrangement of the four numbers.

To show that 4 is an element of the set $\{1, 2, 3, 4\}$, we use the symbol $\in$.

$$4 \in \{1, 2, 3, 4\}$$

Since 5 is *not* an element of this set, we place a slash through the symbol $\in$.

$$5 \notin \{1, 2, 3, 4\}$$

It is customary to name sets with capital letters. If S is used to name the se above, then we write it as follows.

$$S = \{1, 2, 3, 4\}$$

Set S was written by listing its elements. Set S might also be described as

"the set containing the first four counting numbers."

In this example, the notation $\{1, 2, 3, 4\}$, with the elements listed between s braces, is briefer than the verbal description.

The set F, consisting of all fractions between 0 and 1, is an example of a **infinite set,** one that has an unending list of distinct elements. A **finite set** one that has a limited number of elements. The process of counting its elemen comes to an end. Some infinite sets can be described by listing. For example, th set of numbers N used for counting, called the **natural numbers,** or the **coun ing numbers,** can be written as follows.

$$\mathbf{N = \{1, 2, 3, 4, \ldots\}} \quad \text{Natural (counting) numbers}$$

The three dots (*ellipsis points*) show that the list of elements of the set continu according to the established pattern.

Sets are often written using a variable to represent an arbitrary element the set. For example,

$$\{x \mid x \text{ is a natural number between 2 and 7}\} \quad \text{Set-builder notation}$$

(which is read "the set of all elements x such that x is a natural number betwee 2 and 7") uses **set-builder notation** to represent the set $\{3, 4, 5, 6\}$. The number 2 and 7 are *not* between 2 and 7.

EXAMPLE 1 Using Set Notation and Terminology

Identify each set as *finite* or *infinite*. Then determine whether 10 is an elen of the set.

(a) $\{7, 8, 9, \ldots, 14\}$

(b) $\{1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots\}$

(c) $\{x \mid x \text{ is a fraction between 1 and 2}\}$

(d) $\{x \mid x \text{ is a natural number between 9 and 11}\}$

SOLUTION

(a) The set is finite, because the process of counting its elements 7, 8, 9, 10, 11, 12, 13, and 14 comes to an end. The number 10 does belong to the set, and this is written as follows.

$$10 \in \{7, 8, 9, \ldots, 14\}$$

(b) The set is infinite, because the ellipsis points indicate that the pattern continues forever. In this case,

$$10 \notin \{1, \tfrac{1}{4}, \tfrac{1}{16}, \tfrac{1}{64}, \ldots\}.$$

(c) Between any two distinct natural numbers there are infinitely many fractions, so this set is infinite. The number 10 is not an element.

(d) There is only one natural number between 9 and 11, namely 10. So the set is finite, and 10 is an element.

HOMEWORK 1 **Listing the Elements of a Set**

Use set notation, and write the elements belonging to each set.

(a) $\{x \mid x$ is a natural number less than $5\}$

(b) $\{x \mid x$ is a natural number greater than 7 and less than $14\}$

When we are discussing a particular situation or problem, the **universal set** (whether expressed or implied) contains all the elements included in the discussion. The letter U is used to represent the universal set. The **null set, or empty set,** is the set containing no elements. We write the null set by either using the special symbol $\emptyset$, or else writing set braces enclosing no elements, $\{\ \}$.

CAUTION Do not combine these symbols. ***$\{\emptyset\}$ is not the null set.***

Every element of the set $S = \{1, 2, 3, 4\}$ is a natural number. S is an example of a *subset* of the set N of natural numbers, and this is written

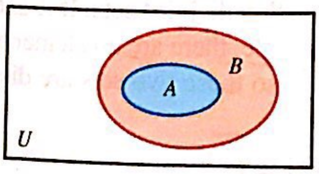
$$S \subseteq N.$$

By definition, set A is a **subset** of set B if every element of set A is also an element of set B. For example, if $A = \{2, 5, 9\}$ and $B = \{2, 3, 5, 6, 9, 10\}$, then $A \subseteq B$. However, there are some elements of B that are not in A, so B is not a subset of A, which is written

$$B \not\subseteq A.$$

By the definition, every set is a subset of itself. Also, by definition, $\emptyset$ is a subset of every set.

If A is any set, then $\emptyset \subseteq A$.



$A \subseteq B$

Figure 1

Figure 1 shows a set A that is a subset of set B. The rectangle in the drawing represents the universal set U. Such diagrams are called **Venn diagrams.**

Two sets A and B are equal whenever $A \subseteq B$ and $B \subseteq A$. Equivalently, $A = B$ if the two sets contain exactly the same elements. For example,

$$\{1, 2, 3\} = \{3, 1, 2\}$$

is true, since both sets contain exactly the same elements. However,

$$\{1, 2, 3\} \neq \{0, 1, 2, 3\}$$

since the set $\{0, 1, 2, 3\}$ contains the element 0, which is not an element of $\{1, 2, 3\}$.

EXAMPLE 2 Examining Subset Relationships

Let $U = \{1,3,5,7,9,11,13\}$, $A = \{1,3,5,7,9,11\}$, $B = \{1,3,7,9\}$, $C = \{3,9,11\}$, and $D = \{1,9\}$. Determine whether each statement is *true* or *false*.

(a) $D \subseteq B$ **(b)** $B \subseteq D$ **(c)** $C \nsubseteq A$ **(d)** $U = A$

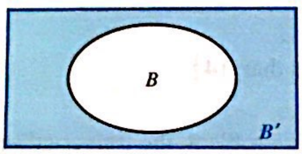
SOLUTION

(a) All elements of D, namely 1 and 9, are also elements of B, so D is a subset of B, and $D \subseteq B$ is true.

(b) There is at least one element of B (for example, 3) that is not an element of D, so B is *not* a subset of D. Thus, $B \subseteq D$ is false.

(c) C is a subset of A, because every element of C is also an element of A. Thus, $C \subseteq A$ is true, and as a result, $C \nsubseteq A$ is false.

(d) U contains the element 13, but A does not. Therefore, $U = A$ is false.



B'

Figure 2

Operations on Sets Given a set A and a universal set U, the set of all elements of U that do not belong to set A is called the **complement** of set A. For example, if set A is the set of all students in your class 30 years old or older, and set U is the set of all students in the class, then the complement of A would be the set of all the students in the class younger than age 30. The complement of set A is written A' (read **"A-prime"**). The Venn diagram in **Figure 2** shows a set B. Its complement, B', is in color.

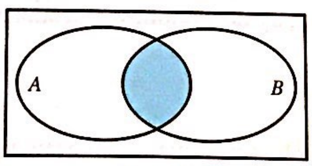
HOMEWORK 2 Finding the Complement of a Set

Let $U = \{1,2,3,4,5,6,7\}$, $A = \{1,3,5,7\}$, and $B = \{3,4,6\}$. Find each set.

(a) A' **(b)** B' **(c)** $\emptyset'$ **(d)** U'

Given two sets A and B, the ~~set~~ of all elements belonging both to set A *and* to set B is called the **intersection** of the two sets, written $A \cap B$. For example, if $A = \{1,2,4,5,7\}$ and $B = \{2,4,5,7,9,11\}$, then we have the following.

$$A \cap B = \{1,2,4,5,7\} \cap \{2,4,5,7,9,11\} = \{2,4,5,7\}$$



$A \cap B$

Figure 3

The Venn diagram in **Figure 3** shows two sets A and B. Their intersection, $A \cap B$, is in color.

Two sets that have no elements in common are called **disjoint sets.** If A and B are any two disjoint sets, then $A \cap B = \emptyset$. For example, there are no elements common to both $\{50,51,54\}$ and $\{52,53,55,56\}$, so these two sets are disjoint.

$$\{50,51,54\} \cap \{52,53,55,56\} = \emptyset$$

EXAMPLE 3 Finding the Intersection of Two Sets

Find each of the following.

(a) $\{9, 15, 25, 36\} \cap \{15, 20, 25, 30, 35\}$ **(b)** $\{2, 3, 4, 5, 6\} \cap \{1, 2, 3, 4\}$

(c) $\{1, 3, 5\} \cap \{2, 4, 6\}$

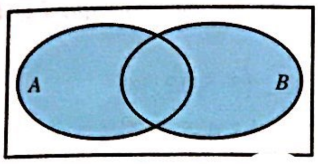
SOLUTION

(a) $\{9, 15, 25, 36\} \cap \{15, 20, 25, 30, 35\} = \{15, 25\}$

The elements 15 and 25 are the only ones belonging to both sets.

(b) $\{2, 3, 4, 5, 6\} \cap \{1, 2, 3, 4\} = \{2, 3, 4\}$

(c) $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$ Disjoint sets



$A \cup B$

Figure 4

The set of all elements belonging to set A ***or*** to set B (or to both) is called the **union** of the two sets, written $A \cup B$. For example, if $A = \{1, 3, 5\}$ and $B = \{3, 5, 7, 9\}$ then we have the following.

$$A \cup B = \{1, 3, 5\} \cup \{3, 5, 7, 9\} = \{1, 3, 5, 7, 9\}$$

The Venn diagram in **Figure 4** shows two sets A and B. Their union, $A \cup B$, is in color.

HOMEWORK 3 Finding the Union of Two Sets

Find each of the following.

(a) $\{1, 2, 5, 9, 14\} \cup \{1, 3, 4, 8\}$ **(b)** $\{1, 3, 5, 7\} \cup \{2, 4, 6\}$

(c) $\{1, 3, 5, 7, \ldots\} \cup \{2. 4. 6, \ldots\}$

The **set operations** are summarized below.

Set Operations

Let A and B be sets, with universal set U.

The **complement** of set A is the set A' of all elements in the universal set that do *not* belong to set A.

$$A' = \{x \mid x \in U, \quad x \notin A\}$$

The **intersection** of sets A and B, written $A \cap B$, is made up of all the elements belonging to both set A *and* set B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The **union** of sets A and B, written $A \cup B$, is made up of all the elements belonging to set A *or* to set B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

1.1 Exercises

Identify each set as finite *or* infinite. *Then determine whether* 10 *is an element of the set.* **See Example 1.**

1. $\{1, 2, 3, 4, 5, \ldots, 75\}$
2. $\{4, 5, 6, \ldots\}$
3. $\{x \mid x$ is a natural number greater than or equal to $10\}$
4. $\{x \mid x$ is an even natural number$\}$

Use set notation, and list all the elements of each set. **See Example 2.**

5. $\{8, 9, 10, \ldots, 17\}$
6. $\{3, 9, 27, \ldots, 729\}$
7. $\{74, 68, 62, \ldots, 38\}$
8. $\{x \mid x$ is a natural number not greater than $4\}$

Insert $\in$ *or* $\notin$ *in each blank to make the resulting statement true.* **See Example 1 and Homework 1.**

9. 9 ____ $\{3, 2, 5, 9, 8\}$
10. -12 ____ $\{3, 5, 12, 14\}$
11. 0 ____ $\{0, 5, 6, 7, 8, 10\}$
12. $\{5\}$ ____ $\{3, 4, 5, 6, 7\}$
13. $\{2\}$ ____ $\{2, 4, 6, 8\}$
14. $\emptyset$ ____ $\emptyset$

Determine whether each statement is true *or* false. **See Examples 1–2 and Homework 1.**

15. $6 \in \{-2, 5, 8, 9\}$
16. $12 \in \{18, 17, 15, 13, 12\}$
17. $3 \notin \{7, 6, 5, 4\}$
18. $\{3, 0, 9, 6, 2\} = \{2, 9, 0, 3, 6\}$
19. $\{3, 7, 12, 14\} = \{3, 7, 12, 14, 0\}$
20. $\{x \mid x$ is a natural number greater than $10\} = \{11, 12, 13, \ldots\}$

Let $A = \{2, 4, 6, 8, 10, 12\}$, $B = \{2, 4, 8, 10\}$, $C = \{4, 10, 12\}$, $D = \{2, 10\}$, *and* $U = \{2, 4, 6, 8, 10, 12, 14\}$.

Determine whether each statement is true *or* false. **See Example 2.**

21. $C \subseteq U$
22. $D \subseteq A$
23. $B \subseteq C$
24. $\emptyset \subseteq \emptyset$
25. $\{0, 2\} \subseteq D$
26. $A \subseteq C$

Insert $\subseteq$ *or* $\not\subseteq$ *in each blank to make the resulting statement true.* **See Example 2.**

27. $\{1, 5\}$ ____ $\{0, -1, 2, 3, 1, 5\}$
28. $\{5, 6, 7, 8\}$ ____ $\{1, 2, 3, 4, 5, 6, 7\}$
29. $\emptyset$ ____ $\emptyset$

Determine whether each statement is true *or* false. **See Homework 2–3 and Example 3.**

30. $\{8, 11, 15\} \cap \{8, 11, 19, 20\} = \{8, 11\}$
31. $\{6, 12, 14, 16\} \cup \{6, 14, 19\} = \{6, 14\}$
32. $\{8, 9, 6\} \cup \{9, 8, 6\} = \{8, 9\}$
33. $\{3, 5, 9, 10\} \cup \emptyset = \{3, 5, 9, 10\}$
34. $\{1, 2, 4\} \cap \{1, 2, 4\} = \emptyset$
35. $\emptyset \cap \emptyset = \emptyset$

Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, $M = \{0, 2, 4, 6, 8\}$, $N = \{1, 3, 5, 7, 9, 11, 13\}$, $Q = \{0, 2, 4, 6, 8, 10, 12\}$, *and* $R = \{0, 1, 2, 3, 4\}$.

Use these sets to find each of the following. Identify any disjoint sets. **See Homework 2–3 and Example 3.**

36. $M \cap U$
37. $M \cup R$
38. $U \cap N$
39. $M \cup Q$
40. Q'
41. $Q \cap R'$
42. $\emptyset \cap Q$
43. $R \cup \emptyset$
44. $(N \cup R) \cap M$
45. $(R \cup N) \cap M$
46. $Q \cap (M \cup N)$
47. $(U \cap \emptyset') \cup R$

1.2 Real Numbers and Their Properties

- Sets of Numbers and the Number Line
- Exponents
- Order of Operations
- Properties of Real Numbers
- Order on the Number Line
- Absolute Value

Sets of Numbers and the Number Line

As mentioned in the previous section, the set of **natural numbers** is written in set notation as follows.

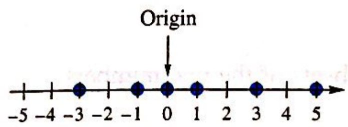
$$\{1, 2, 3, 4, \ldots\} \quad \text{Natural numbers (Section 1.1)}$$

Including 0 with the set of natural numbers gives the set of **whole numbers.**

$$\{0, 1, 2, 3, 4, \ldots\} \quad \text{Whole numbers}$$

Including the negatives of the natural numbers with the set of whole numbers gives the set of **integers.**

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \quad \text{Integers}$$



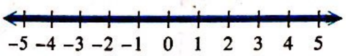
Graph of the Set $\{-3, -1, 0, 1, 3, 5\}$

Figure 5

Integers can be **graphed** on a **number line.** See **Figure 5.** Every number corresponds to one and only one point on the number line, and each point corresponds to one and only one number. The number associated with a given point is called the **coordinate** of the point. This correspondence forms a **coordinate system.**

The result of dividing two integers (with a nonzero divisor) is called a *rational number,* or *fraction.* A **rational number** is an element of the set defined as follows.

$$\left\{\frac{p}{q} \,\middle|\, p \text{ and } q \text{ are integers and } q \neq 0\right\} \quad \text{Rational numbers}$$

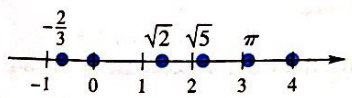


Graph of the Set of Real Numbers

Figure 6

The set of rational numbers includes the natural numbers, the whole numbers, and the integers. For example, the integer -3 is a rational number because it can be written as $\frac{-3}{1}$. Numbers that can be written as repeating or terminating decimals are also rational numbers. For example, $0.\overline{6} = 0.66666\ldots$ represents a rational number that can be expressed as the fraction $\frac{2}{3}$.

The set of all numbers that correspond to points on a number line is the **real numbers,** shown in **Figure 6.** Real numbers can be represented by decimals. Since every fraction has a decimal form—for example, $\frac{1}{4} = 0.25$—real numbers include rational numbers.



$\sqrt{2}$, $\sqrt{5}$, and π are irrational. Since $\sqrt{2}$ is approximately equal to 1.41, it is located between 1 and 2, slightly closer to 1.

Figure 7

Some real numbers cannot be represented by quotients of integers. These numbers are **irrational numbers.** The set of irrational numbers includes $\sqrt{3}$ and $\sqrt{5}$. Another irrational number is π, which is *approximately* equal to 3.14159. The numbers in the set $\left\{-\frac{2}{3}, 0, \sqrt{2}, \sqrt{5}, \pi, 4\right\}$ can be located on a number line, as shown in **Figure 7.**

The sets of numbers discussed so far are summarized as follows.

Sets of Numbers [redacted]

Set	Description
Natural numbers	$\{1, 2, 3, 4, \ldots\}$
Whole numbers	$\{0, 1, 2, 3, 4, \ldots\}$
Integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers	$\left\{\frac{p}{q} \,\middle|\, p \text{ and } q \text{ are integers and } q \neq 0\right\}$
Irrational numbers	$\{x \mid x \text{ is real but not rational}\}$
Real numbers	$\{x \mid x \text{ corresponds to a point on a number line}\}$

EXAMPLE 1 Identifying Sets of Numbers

Let $A = \left\{-8, -6, -\frac{12}{4}, -\frac{3}{4}, 0, \frac{3}{8}, \frac{1}{2}, 1, \sqrt{2}, \sqrt{5}, 6\right\}$. List the elements from A that belong to each set.

(a) Natural numbers **(b)** Whole numbers **(c)** Integers

(d) Rational numbers **(e)** Irrational numbers **(f)** Real numbers

SOLUTION

(a) Natural numbers: 1 and 6

(b) Whole numbers: 0, 1, and 6

(c) Integers: -8, -6, $-\frac{12}{4}$ (or -3), 0, 1, and 6

(d) Rational numbers: -8, -6, $-\frac{12}{4}$ (or -3), $-\frac{3}{4}$, 0, $\frac{3}{8}$, $\frac{1}{2}$, 1, and 6

(e) Irrational numbers: $\sqrt{2}$ and $\sqrt{5}$

(f) All elements of A are real numbers.

Figure 8 shows the relationships among the subsets of the real numbers.

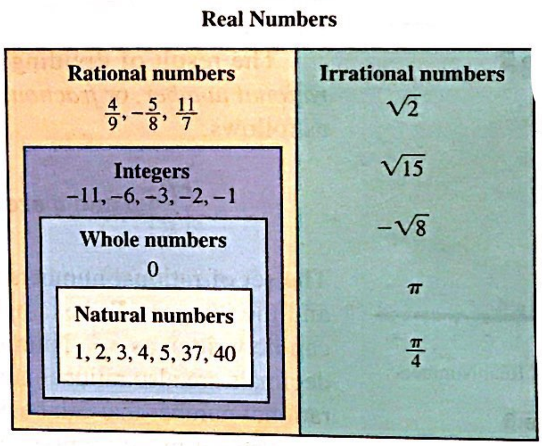


Figure 8

Exponents The product $2 \cdot 2 \cdot 2$ can be written as 2^3, where the 3 shows that three factors of 2 appear in the product.

Exponential Notation

If n is any positive integer and a is any real number, then the nth power of a is written using exponential notation as follows.

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}$$

That is, a^n means the product of n factors of a. The integer n is the **exponent**, a is the **base**, and a^n is a **power** or an **exponential expression** (or simply an **exponential**). Read a^n as **"a to the nth power,"** or just **"a to the nth."**

HOMEWORK 1 Evaluating Exponential Expressions

Evaluate each exponential expression and identify the base and the exponent.

(a) 4^3 **(b)** $(-6)^2$ **(c)** -6^2 **(d)** $4 \cdot 3^2$ **(e)** $(4 \cdot 3)^2$

Order of Operations When a problem involves more than one operation symbol, we use the following order of operations.

Order of Operations

If grouping symbols such as parentheses, square brackets, absolute value bars, or fraction bars are present, begin as follows.

Step 1 Work separately above and below each **fraction bar.**

Step 2 Use the rules below within each set of **parentheses** or **square brackets.** Start with the innermost set and work outward.

If no grouping symbols are present, follow these steps.

Step 1 Simplify all **powers** and **roots.** ***Work from left to right.***

Step 2 Do any **multiplications** or **divisions** in order. ***Work from left to right.***

Step 3 Do any **negations, additions,** or **subtractions** in order. ***Work from left to right.***

EXAMPLE 2 **Using Order of Operations**

Evaluate each expression.

(a) $6 \div 3 + 2^3 \cdot 5$ **(b)** $(8 + 6) \div 7 \cdot 3 - 6$

(c) $\dfrac{4 + 3^2}{6 - 5 \cdot 3}$ **(d)** $\dfrac{-(-3)^3 + (-5)}{2(-8) - 5(3)}$

SOLUTION

(a)
$$\begin{aligned} 6 \div 3 + 2^3 \cdot 5 &= 6 \div 3 + 8 \cdot 5 && \text{Evaluate the exponential.}\\ &= 2 + 8 \cdot 5 && \text{Divide.}\\ &= 2 + 40 && \text{Multiply.}\\ &= 42 && \text{Add.} \end{aligned}$$

Multiply or divide *in order from left to right.*

(b)
$$\begin{aligned} (8 + 6) \div 7 \cdot 3 - 6 &= 14 \div 7 \cdot 3 - 6 && \text{Work inside parentheses.}\\ &= 2 \cdot 3 - 6 && \text{Divide.}\\ &= 6 - 6 && \text{Multiply.}\\ &= 0 && \text{Subtract.} \end{aligned}$$

Be careful to divide *before* multiplying here.

(c)
$$\begin{aligned} \frac{4 + 3^2}{6 - 5 \cdot 3} &= \frac{4 + 9}{6 - 15} && \text{Evaluate the exponential and multiply.}\\ &= \frac{13}{-9}, \text{ or } -\frac{13}{9} && \text{Add and subtract; } \tfrac{a}{-b} = -\tfrac{a}{b}. \end{aligned}$$

(d)
$$\begin{aligned} \frac{-(-3)^3 + (-5)}{2(-8) - 5(3)} &= \frac{-(-27) + (-5)}{2(-8) - 5(3)} && \text{Evaluate the exponential.}\\ &= \frac{27 + (-5)}{-16 - 15} && \text{Multiply.}\\ &= \frac{22}{-31}, \text{ or } -\frac{22}{31} && \text{Add and subtract: } \tfrac{a}{-b} = -\tfrac{a}{b}. \end{aligned}$$

HOMEWORK 2 **Using Order of Operations**

Evaluate each expression for $x = -2$, $y = 5$, and $z = -3$.

(a) $-4x^2 - 7y + 4z$ **(b)** $\dfrac{2(x-5)^2 + 4y}{z+4}$ **(c)** $\dfrac{\frac{x}{2} - \frac{y}{5}}{\frac{3z}{9} + \frac{8y}{5}}$

Properties of Real Numbers The following basic properties can be generalized to apply to expressions with variables.

Properties of Real Numbers

Let a, b, and c represent real numbers.

Property	Description
Closure Properties $a + b$ is a real number. ab is a real number.	The sum or product of two real numbers is a real number.
Commutative Properties $a + b = b + a$ $ab = ba$	The sum or product of two real numbers is the same regardless of their order.
Associative Properties $(a + b) + c = a + (b + c)$ $(ab)c = a(bc)$	The sum or product of three real numbers is the same no matter which two are added or multiplied first.
Identity Properties There exists a unique real number 0 such that $a + 0 = a$ and $0 + a = a$. There exists a unique real number 1 such that $a \cdot 1 = a$ and $1 \cdot a = a$.	The sum of a real number and 0 is that real number, and the product of a real number and 1 is that real number.
Inverse Properties There exists a unique real number $-a$ such that $a + (-a) = 0$ and $-a + a = 0$. If $a \neq 0$, there exists a unique real number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1$.	The sum of any real number and its negative is 0, and the product of any nonzero real number and its reciprocal is 1.
Distributive Properties $a(b + c) = ab + ac$ $a(b - c) = ab - ac$	The product of a real number and the sum (or difference) of two real numbers equals the sum (or difference) of the products of the first number and each of the other numbers.

The multiplication property of zero says that $0 \cdot a = a \cdot 0 = 0$ for all real numbers a.

CAUTION With the commutative properties, the ***order*** changes, but with the associative properties, the ***grouping*** changes.

Commutative Properties	Associative Properties
$(x + 4) + 9 = (4 + x) + 9$	$(x + 4) + 9 = x + (4 + 9)$
$7 \cdot (5 \cdot 2) = (5 \cdot 2) \cdot 7$	$7 \cdot (5 \cdot 2) = (7 \cdot 5) \cdot 2$

EXAMPLE 3 Simplifying Expressions

Use the commutative and associative properties to simplify each expression.

(a) $6 + (9 + x)$ **(b)** $\frac{5}{8}(16y)$ **(c)** $-10p\left(\frac{6}{5}\right)$

SOLUTION

(a) $6 + (9 + x) = (6 + 9) + x$ Associative property

$= 15 + x$ Add.

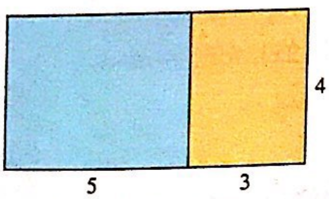
(b) $\frac{5}{8}(16y) = \left(\frac{5}{8} \cdot 16\right)y$ Associative property

$= 10y$ Multiply.

(c) $-10p\left(\frac{6}{5}\right) = \frac{6}{5}(-10p)$ Commutative property

$= \left[\frac{6}{5}(-10)\right]p$ Associative property

$= -12p$ Multiply.



Geometric Model of the Distributive Property

Figure 9

Figure 9 helps to explain the distributive property. The area of the entire region shown can be found in two ways, as follows.

$$4(5 + 3) = 4(8) = 32$$

or

$$4(5) + 4(3) = 20 + 12 = 32$$

The result is the same. This means that

$$4(5 + 3) = 4(5) + 4(3).$$

HOMEWORK 3 Using the Distributive Property

Rewrite each expression using the distributive property and simplify, if possible.

(a) $3(x + y)$ **(b)** $-(m - 4n)$ **(c)** $\frac{1}{3}\left(\frac{4}{5}m - \frac{3}{2}n - 27\right)$ **(d)** $7p + 21$

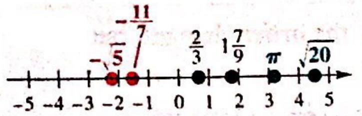
Order on the Number Line If the real number a is to the left of the real number b on a number line, then

***a* is less than *b*,** written $a < b$.

If a is to the right of b, then

***a* is greater than *b*,** written $a > b$.

The inequality symbol must point toward the lesser number.



$-\sqrt{5}$ is to the left of $-\frac{11}{7}$ on the number line, so $-\sqrt{5} < -\frac{11}{7}$, and $\sqrt{20}$ is to the right of π, indicating that $\sqrt{20} > \pi$.

Figure 10

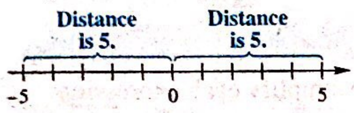


Figure 11

Figure 10 illustrates this with several pairs of numbers. Statements involving these symbols, as well as the symbols less than or equal to, $\leq$, and greater than or equal to, $\geq$, are called **inequalities.** The inequality $a < b < c$ says that b is ***between*** a and c since $a < b$ and $b < c$.

Absolute Value The distance on the number line from a number to 0 is called the **absolute value** of that number. The absolute value of the number a is written $|a|$. For example, the distance on the number line from 5 to 0 is 5, as is the distance from -5 to 0. See **Figure 11.** Therefore, both of the following are true.

$$|5| = 5 \quad \text{and} \quad |-5| = 5$$

NOTE *Since distance cannot be negative, the absolute value of a number is always positive or* **0.**

The algebraic definition of absolute value follows.

Absolute Value

Let a represent a real number.

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

That is, the absolute value of a positive number or **0** *equals that number, while the absolute value of a negative number equals its negative (or opposite).*

EXAMPLE 4 Evaluating Absolute Values

Evaluate each expression.

(a) $\left|-\frac{5}{8}\right|$ **(b)** $-|8|$ **(c)** $-|-2|$ **(d)** $|2x|$, for $x = \pi$

SOLUTION

(a) $\left|-\frac{5}{8}\right| = \frac{5}{8}$

(b) $-|8| = -(8) = -8$

(c) $-|-2| = -(2) = -2$

(d) $|2\pi| = 2\pi$

Absolute value is useful in applications where only the *size* (or magnitude), not the *sign,* of the difference between two numbers *is* important.

NOTE As seen in **Homework 4(b),** absolute value bars can also act as symbols of inclusion. Remember this when applying the rules for order of operations.

HOMEWORK 4 Evaluating Absolute Value Expressions

Let $x = -6$ and $y = 10$. Evaluate each expression.

(a) $|2x - 3y|$ **(b)** $\dfrac{2|x| - |3y|}{|xy|}$

Distance between Points on a Number Line

If P and Q are points on a number line with coordinates a and b, respectively, then the distance $d(P, Q)$ between them is given by the following.

$$d(P,Q) = |b - a| \quad \text{or} \quad d(P,Q) = |a - b|$$

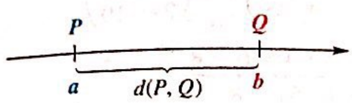


Figure 12

That is, the distance between two points on a number line is the absolute value of the difference between their coordinates in either order. See **Figure 12.**

EXAMPLE 5 **Finding the Distance between Two Points**

Find the distance between -5 and 8.

SOLUTION Use the first formula above, with $a = -5$ and $b = 8$.

$$|b - a| = |8 - (-5)| = |8 + 5| = |13| = 13$$

Alternatively, for $a = 8$ and $b = -5$, we obtain the same result.

$$|b - a| = |(-5) - 8| = |-13| = 13$$

1.2 Exercises

1. *Concept Check* Match each number from Column I with the letter or letters of the sets of numbers from Column II to which the number belongs. There may be more than one choice, so give all choices.

I		II	
(a) 0	**(b)** 34	**A.** Natural numbers	**B.** Whole numbers
(c) $-\frac{9}{4}$	**(d)** $\sqrt{36}$	**C.** Integers	**D.** Rational numbers
(e) $\sqrt{13}$	**(f)** 2.16	**E.** Irrational numbers	**F.** Real numbers

2. Explain why no answer in **Exercise 1** can contain both D and E as choices.

Concept Check Decide whether each statement is true *or* false. *If it is false, tell why.*

3. Every integer is a whole number.
4. Every natural number is an integer.
5. Every irrational number is an integer.
6. Every integer is a rational number.
7. Every natural number is a whole number.
8. Some rational numbers are irrational.
9. Some rational numbers are whole numbers.
10. Some real numbers are integers.

Let set $A = \left\{-6, -\frac{12}{4}, -\frac{5}{8}, -\sqrt{3}, 0, \frac{1}{4}, 1, 2\pi, 3, \sqrt{12}\right\}$. *List all the elements of A that belong to each set. See* ***Example 1.***

11. Natural numbers
12. Whole numbers
13. Integers
14. Rational numbers
15. Irrational numbers
16. Real numbers

Evaluate each expression. See ***Homework 1.***

17. -2^4
18. -3^5
19. $(-2)^4$
20. $(-2)^6$
21. $(-3)^5$
22. $(-2)^5$
23. $-2 \cdot 3^4$
24. $-4 \cdot 5^3$

Evaluate each expression. **See Example 2.**

25. $-2 \cdot 5 + 12 \div 3$

26. $9 \cdot 3 - 16 \div 4$

27. $-4(9-8) + (-7)(2)^3$

28. $6(-5) - (-3)(2)^4$

29. $(4 - 2^3)(-2 + \sqrt{25})$

30. $(5 - 3^2)(\sqrt{16} - 2^3)$

31. $\left(-\frac{2}{9} - \frac{1}{4}\right) - \left[-\frac{5}{18} - \left(-\frac{1}{2}\right)\right]$

32. $\left[-\frac{5}{8} - \left(-\frac{2}{5}\right)\right] - \left(\frac{3}{2} - \frac{11}{10}\right)$

33. $\dfrac{-8 + (-4)(-6) \div 12}{4 - (-3)}$

34. $\dfrac{15 \div 5 \cdot 4 \div 6 - 8}{-6 - (-5) - 8 \div 2}$

Evaluate each expression for $p = -4$, $q = 8$, *and* $r = -10$. **See Homework 2.**

35. $-p^2 - 7q + r^2$

36. $-p^2 - 2q + r$

37. $\dfrac{q + r}{q + p}$

38. $\dfrac{p + r}{p + q}$

39. $\dfrac{3q}{r} - \dfrac{5}{p}$

40. $\dfrac{3r}{q} - \dfrac{2}{r}$

41. $\dfrac{5r}{2p - 3r}$

42. $\dfrac{3q}{3p - 2r}$

43. $\dfrac{\frac{q}{2} - \frac{r}{3}}{\frac{3p}{4} + \frac{q}{8}}$

44. $\dfrac{\frac{q}{4} - \frac{r}{5}}{\frac{p}{2} + \frac{q}{2}}$

45. $\dfrac{-(p + 2)^2 - 3r}{2 - q}$

46. $\dfrac{-(q - 6)^2 - 2p}{4 - p}$

47. $\dfrac{3p + 3(4 + p)^3}{r + 8}$

48. $\dfrac{5q + 2(1 + p)^3}{r + 3}$

Identify the property illustrated in each statement. Assume all variables represent real numbers. **See Example 3 and Homework 3.**

49. $6 \cdot 12 + 6 \cdot 15 = 6(12 + 15)$

50. $8(m + 4) = 8m + 32$

51. $(t - 6) \cdot \left(\frac{1}{t - 6}\right) = 1$, if $t - 6 \neq 0$

52. $\dfrac{2 + m}{2 - m} \cdot \dfrac{2 - m}{2 + m} = 1$, if $m \neq 2$ or -2

53. $(7.5 - y) + 0 = 7.5 - y$

54. $1 \cdot (3x - 7) = 3x - 7$

55. $5(t + 3) = (t + 3) \cdot 5$

56. $-7 + (x + 3) = (x + 3) + (-7)$

57. $(5x)\left(\frac{1}{x}\right) = 5\left(x \cdot \frac{1}{x}\right)$

58. $(38 + 99) + 1 = 38 + (99 + 1)$

59. $5 + \sqrt{3}$ is a real number.

60. 5π is a real number.

61. Is there a commutative property for subtraction? That is, in general, is $a - b$ equal to $b - a$? Support your answer with examples.

62. Is there an associative property for subtraction? That is, does $(a - b) - c$ equal $a - (b - c)$ in general? Support your answer with examples.

Simplify each expression. **See Example 3 and Homework 3.**

63. $\frac{10}{11}(22z)$

64. $\left(\frac{3}{4}r\right)(-12)$

65. $(m + 5) + 6$

66. $8 + (a + 7)$

67. $\frac{3}{8}\left(\frac{16}{9}y + \frac{32}{27}z - \frac{40}{9}\right)$

68. $-\frac{1}{4}(20m + 8y - 32z)$

Use the distributive property to rewrite sums as products and products as sums. **See Homework 3.**

69. $8p - 14p$

70. $15x - 10x$

71. $-4(z - y)$

72. $-3(m + n)$

Concept Check *Use the distributive property to calculate each value mentally.*

73. $72 \cdot 17 + 28 \cdot 17$

74. $32 \cdot 80 + 32 \cdot 20$

75. $123\frac{5}{8} \cdot 1\frac{1}{2} - 23\frac{5}{8} \cdot 1\frac{1}{2}$

76. $17\frac{2}{5} \cdot 14\frac{3}{4} - 17\frac{2}{5} \cdot 4\frac{3}{4}$

Concept Check *Decide whether each statement is* true *or* false. *If false, correct the statement so it is true.*

77. $|6 - 8| = |6| - |8|$

78. $|(-3)^3| = -|3^3|$

79. $|-5| \cdot |6| = |-5 \cdot 6|$

80. $\frac{|-14|}{|2|} = \left|\frac{-14}{2}\right|$

81. $|a - b| = |a| - |b|$, if $b > a > 0$

82. If a is negative, then $|a| = -a$.

Evaluate each expression. **See Example 4.**

83. $|-10|$

84. $|-15|$

85. $-\left|\frac{4}{7}\right|$

86. $-\left|\frac{7}{2}\right|$

87. $-|-8|$

88. $-|-12|$

1.3 Polynomials [redacted]

- Rules for Exponents
- Polynomials
- Addition and Subtraction
- Multiplication
- Division

Rules for Exponents From **Section 1.2,** the notation a^m (where m is a positive integer and a is a real number) means that a appears as a factor m times. In the same way, a^n (where n is a positive integer) means that a appears as a factor n times. In the product $a^m \cdot a^n$, the base a would appear $m + n$ times, so the **product rule** states the following.

$$a^m \cdot a^n = a^{m+n} \quad \text{Product rule}$$

Also consider the expression $(2^5)^3$, which can be written as follows

$$(2^5)^3 = 2^5 \cdot 2^5 \cdot 2^5 \quad \text{Definition of exponent}$$

$$= 2^{5+5+5}, \text{ or } 2^{15} \quad \text{Generalization of the product rule}$$

The exponent 15 could have been obtained by multiplying 5 and 3. This example suggests the first of the **power rules** below. The others are found in a similar way.

1. $(a^m)^n = a^{mn}$ **2.** $(ab)^m = a^m b^m$ **3.** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ $(b \neq 0)$

Power rules for positive integers m and n and real numbers a and b

Rules for Exponents

For all positive integers m and n and all real numbers a and b, the following rules hold.

Rule	Description
Product Rule $a^m \cdot a^n = a^{m+n}$	When multiplying powers of like bases, keep the base and add the exponents.
Power Rule 1 $(a^m)^n = a^{mn}$	To raise a power to a power, multiply the exponents.

Power Rule 2
$(ab)^m = a^m b^m$

To raise a product to a power, raise each factor to that power.

Power Rule 3
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$

To raise a quotient to a power, raise the numerator and the denominator to that power.

EXAMPLE 1 Using the Product Rule

Find each product.

(a) $y^4 \cdot y^7$ **(b)** $(6z^5)(9z^3)(2z^2)$

SOLUTION

(a) $y^4 \cdot y^7 = y^{4+7} = y^{11}$ Product rule: Keep the base and add the exponents.

(b) $(6z^5)(9z^3)(2z^2) = (6 \cdot 9 \cdot 2) \cdot (z^5 z^3 z^2)$ Commutative and associative properties (Section 1.2)

$= 108z^{5+3+2}$ Multiply. Apply the product rule.

$= 108z^{10}$ Add.

HOMEWORK 1 Using the Power Rules

Simplify. Assume all variables represent nonzero real numbers.

(a) $(5^3)^2$ **(b)** $(3^4x^2)^3$ **(c)** $\left(\frac{2^5}{b^4}\right)^3$ **(d)** $\left(\frac{-2m^6}{t^2z}\right)^5$

CAUTION The expressions mn^2 and $(mn)^2$ are ***not*** equivalent. The second power rule can be used only with the second expression:

$$(mn)^2 = m^2n^2.$$

A zero exponent is defined as follows.

Zero Exponent

For any nonzero real number a, $\mathbf{a^0 = 1.}$

***That is, any nonzero number with a zero exponent equals* 1.**

To illustrate why a^0 is defined to equal 1, consider the product

$$a^n \cdot a^0, \quad \text{for} \quad a \neq 0.$$

We want the definition of a^0 to be consistent so that the product rule applies. Now apply this rule.

$$a^n \cdot a^0 = a^{n+0} = a^n$$

The product of a^n and a^0 must be a^n, and thus a^0 is acting like the identity element 1. So for consistency, we *define* a^0 to equal 1. **(0^0 is undefined.)**

EXAMPLE 2 **Using the Definition of a^0**

Evaluate each power.

(a) 4^0 **(b)** $(-4)^0$ **(c)** -4^0 **(d)** $-(-4)^0$ **(e)** $(7r)^0$

SOLUTION

(a) $4^0 = 1$ Base is 4.

(b) $(-4)^0 = 1$ Base is -4.

(c) $-4^0 = -(4^0) = -1$ Base is 4.

(d) $-(-4)^0 = -(1) = -1$ Base is -4.

(e) $(7r)^0 = 1,\ r \neq 0$ Base is $7r$.

Polynomials Any collection of numbers or variables joined by the basic operations of addition, subtraction, multiplication, or division (except by 0), or the operations of raising to powers or taking roots, formed according to the rules of algebra, is an **algebraic expression.**

$$-2x^2 + 3x, \quad \frac{15y}{2y-3}, \quad \sqrt{m^3 - 64}, \quad (3a+b)^4 \qquad \text{Algebraic expressions}$$

The product of a real number and one or more variables raised to powers is a **term.** The real number is the **numerical coefficient,** or just the **coefficient,** of the variables. The coefficient of the variable in $-3m^4$ is -3, while the coefficient in $-p^2$ is -1. **Like terms** are terms with the same variables each raised to the same powers.

$$-13x^3, \quad 4x^3, \quad -x^3 \quad \text{Like terms} \qquad 6y, \quad 6y^2, \quad 4y^3 \quad \text{Unlike terms}$$

A **polynomial** is defined as a term or a finite sum of terms, with only positive or zero integer exponents permitted on the variables. If the terms of a polynomial contain only the variable x, then the polynomial is a **polynomial in x.**

$$5x^3 - 8x^2 + 7x - 4, \quad 9p^5 - 3, \quad 8r^2, \quad 6 \qquad \text{Polynomials}$$

The terms of a polynomial cannot have variables in a denominator.

$$9x^2 - 4x + \frac{6}{x} \qquad \text{Not a polynomial}$$

The **degree of a term** with one variable is the exponent on the variable. For example, the degree of $2x^3$ is 3, and the degree of $17x$ (that is, $17x^1$) is 1. The greatest degree of any term in a polynomial is the **degree of the polynomial.** For example,

$$4x^3 - 2x^2 - 3x + 7$$

has degree 3, because the greatest degree of any term is 3. A nonzero constant such as -6, equivalent to $-6x^0$, has degree 0. (The polynomial 0 has no degree.)

A polynomial can have more than one variable. A term containing more than one variable has degree equal to the sum of all the exponents appearing on the variables in the term. For example, $-3x^4y^3z^5$ has degree $4 + 3 + 5 = 12$. The degree of a polynomial in more than one variable is equal to the greatest degree of any term appearing in the polynomial. By this definition, the polynomial

$$2x^4y^3 - 3x^5y + x^6y^2$$

has degree 8, because the x^6y^2 term has the greatest degree, 8.

A polynomial containing exactly three terms is a **trinomial.** A two-term polynomial is a **binomial.** A single-term polynomial is a **monomial.**

HOMEWORK 2 **Classifying Polynomials**

The table classifies several polynomials.

Polynomial	Degree	Type
$9p^7 - 4p^3 + 8p^2$	7	Trinomial
$29x^{11} + 8x^{15}$	15	Binomial
$-10r^6s^8$	14	Monomial
$5a^3b^7 - 3a^5b^5 + 4a^2b^9 - a^{10}$	11	None of these

Addition and Subtraction Since the variables used in polynomials represent real numbers, a polynomial represents a real number. This means that all the properties of the real numbers mentioned in **Section 1.2** hold for polynomials. In particular, the distributive property holds.

$$3m^5 - 7m^5 = (3 - 7)m^5 = -4m^5 \quad \text{Distributive property}$$

Thus, polynomials are added by adding coefficients of like terms, and they are subtracted bv subtracting coefficients of like terms.

EXAMPLE 3 **Adding and Subtracting Polynomials**

Add or subtract, as indicated.

(a) $(2y^4 - 3y^2 + y) + (4y^4 + 7y^2 + 6y)$

(b) $(-3m^3 - 8m^2 + 4) - (m^3 + 7m^2 - 3)$

(c) $(8m^4p^5 - 9m^3p^5) + (11m^4p^5 + 15m^3p^5)$

(d) $4(x^2 - 3x + 7) - 5(2x^2 - 8x - 4)$

SOLUTION

(a) $(2y^4 - 3y^2 + y) + (4y^4 + 7y^2 + 6y)$ $\quad y = 1y$

$= (2 + 4)y^4 + (-3 + 7)y^2 + (1 + 6)y$ Add coefficients of like terms.

$= 6y^4 + 4y^2 + 7y$ Work inside parentheses.

(b) $(-3m^3 - 8m^2 + 4) - (m^3 + 7m^2 - 3)$

$= (-3 - 1)m^3 + (-8 - 7)m^2 + [4 - (-3)]$ Subtract coefficients of like terms.

$= -4m^3 - 15m^2 + 7$ Simplify.

(c) $(8m^4p^5 - 9m^3p^5) + (11m^4p^5 + 15m^3p^5) = 19m^4p^5 + 6m^3p^5$

(d) $4(x^2 - 3x + 7) - 5(2x^2 - 8x - 4)$

$= 4x^2 - 4(3x) + 4(7) - 5(2x^2) - 5(-8x) - 5(-4)$ Distributive property (Section 1.2)

$= 4x^2 - 12x + 28 - 10x^2 + 40x + 20$ Multiply.

$= -6x^2 + 28x + 48$ Add like terms.

As shown in **Examples 3(a), (b), and (d),** polynomials in one variable are often written with their terms in **descending order** (or descending degree). Thus, the term of greatest degree is first, the one with the next greatest degree is next, and so on.

Multiplication One way to find the product of two polynomials, such as $3x - 4$ and $2x^2 - 3x + 5$, is to treat $3x - 4$ as a single expression and use the distributive property.

$$\begin{aligned}
&(3x-4)(2x^2-3x+5)\\
&\quad = (3x-4)(2x^2) - (3x-4)(3x) + (3x-4)(5)\\
&\quad = 3x(2x^2) - 4(2x^2) - 3x(3x) - (-4)(3x) + 3x(5) - 4(5)\\
&\quad = 6x^3 - 8x^2 - 9x^2 + 12x + 15x - 20\\
&\quad = 6x^3 - 17x^2 + 27x - 20
\end{aligned}$$

Another method is to write such a product vertically, similar to the method used in arithmetic for multiplying whole numbers.

Place like terms in the same column.

$$\begin{array}{rrrrl}
 & 2x^2 & -\ 3x & +\ 5 & \\
 & & 3x & -\ 4 & \\ \hline
 & -8x^2 & +\ 12x & -\ 20 & \leftarrow -4(2x^2-3x+5)\\
6x^3 & -\ 9x^2 & +\ 15x & & \leftarrow 3x(2x^2-3x+5)\\ \hline
6x^3 & -\ 17x^2 & +\ 27x & -\ 20 & \text{Add in columns.}
\end{array}$$

HOMEWORK 3 Multiplying Polynomials

Multiply $(3p^2 - 4p + 1)(p^3 + 2p - 8)$.

The **FOIL method** is a convenient way to find the product of two binomials. The memory aid **FOIL** (for **F**irst, **O**utside, **I**nside, **L**ast) gives the pairs of terms to be multiplied to find the product, as shown in the next example.

EXAMPLE 4 Using the FOIL Method to Multiply Two Binomials

Find each product.

(a) $(6m + 1)(4m - 3)$ **(b)** $(2x + 7)(2x - 7)$ **(c)** $r^2(3r + 2)(3r - 2)$

SOLUTION

(a) $(6m+1)(4m-3) = \overset{\text{F}}{6m(4m)} + \overset{\text{O}}{6m(-3)} + \overset{\text{I}}{1(4m)} + \overset{\text{L}}{1(-3)}$

$= 24m^2 - 14m - 3$ $\quad -18m + 4m = -14m$

(b) $(2x+7)(2x-7) = 4x^2 - 14x + 14x - 49$ FOIL

$= 4x^2 - 49$ Combine like terms.

(c) $r^2(3r+2)(3r-2) = r^2(9r^2 - 6r + 6r - 4)$ FOIL

$= r^2(9r^2 - 4)$ Combine like terms.

$= 9r^4 - 4r^2$ Distributive property

In **Example 4(a),** the product of two binomials is a trinomial, while in **Examples 4(b) and (c),** the product of two binomials is a binomial. ***The product of two binomials of the forms $x + y$ and $x - y$ is always a binomial.*** The squares of binomials, $(x + y)^2$ and $(x - y)^2$, are also special products.

Special Products

Product of the Sum and Difference of Two Terms	$(x+y)(x-y) = x^2 - y^2$
Square of a Binomial	$(x+y)^2 = x^2 + 2xy + y^2$
	$(x-y)^2 = x^2 - 2xy + y^2$

HOMEWORK 4 **Using the Special Products**

Find each product.

(a) $(3p + 11)(3p - 11)$ **(b)** $(5m^3 - 3)(5m^3 + 3)$

(c) $(9k - 11r^3)(9k + 11r^3)$ **(d)** $(2m + 5)^2$

(e) $(3x - 7y^4)^2$

CAUTION See **Homework 4(d) and (e).** ***The square of a binomial has three terms.*** Do **not** give $x^2 + y^2$ as the result of expanding $(x + y)^2$, or $x^2 - y^2$ as the result of expanding $(x - y)^2$.

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$

Remember to include the middle term.

EXAMPLE 5 **Multiplying More Complicated Binomials**

Find each product.

(a) $[(3p - 2) + 5q][(3p - 2) - 5q]$ **(b)** $(x + y)^3$ **(c)** $(2a + b)^4$

SOLUTION

(a) $[(3p - 2) + 5q][(3p - 2) - 5q]$

$= (3p - 2)^2 - (5q)^2$ Product of the sum and difference of two terms

$= 9p^2 - 12p + 4 - 25q^2$ Square both quantities.

(b) $(x + y)^3 = (x + y)^2(x + y)$

This does *not* equal $x^3 + y^3$.

$= (x^2 + 2xy + y^2)(x + y)$ Square $x + y$.

$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$ Multiply.

$= x^3 + 3x^2y + 3xy^2 + y^3$ Combine like terms.

(c) $(2a + b)^4 = (2a + b)^2(2a + b)^2$

$= (4a^2 + 4ab + b^2)(4a^2 + 4ab + b^2)$ Square each $2a + b$.

$= 16a^4 + 16a^3b + 4a^2b^2 + 16a^3b + 16a^2b^2$ Distributive property
$\quad + 4ab^3 + 4a^2b^2 + 4ab^3 + b^4$

$= 16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$ Combine like terms.

Division The quotient of two polynomials can be found with an algorithm (that is, a step-by-step procedure) for long division similar to that used for dividing whole numbers. ***Both polynomials must be written in descending order to use this algorithm.***

HOMEWORK 5 **Dividing Polynomials**

Divide $4m^3 - 8m^2 + 5m + 6$ by $2m - 1$.

When a polynomial has a missing term, we allow for that term by inserting a term with a 0 coefficient for it.

EXAMPLE 6 **Dividing Polynomials with Missing Terms**

Divide $3x^3 - 2x^2 - 150$ by $x^2 - 4$.

SOLUTION Both polynomials have missing first-degree terms. Insert each missing term with a 0 coefficient.

$$\begin{array}{r} 3x - 2 \\ x^2 + 0x - 4\overline{)3x^3 - 2x^2 + 0x - 150} \\ \underline{3x^3 + 0x^2 - 12x} \\ -2x^2 + 12x - 150 \\ \underline{-2x^2 + 0x + 8} \\ 12x - 158 \end{array}$$

Missing term

Missing term

Insert placeholders for missing terms.

Remainder

The division process ends when the remainder is 0 or the degree of the remainder is less than that of the divisor. Since $12x - 158$ has lesser degree than the divisor $x^2 - 4$, it is the remainder. Thus, the entire quotient is written as follows.

$$\frac{3x^3 - 2x^2 - 150}{x^2 - 4} = 3x - 2 + \frac{12x - 158}{x^2 - 4}$$

1.3 Exercises

Simplify each expression. See Example 1.

1. $(-4x^5)(4x^2)$
2. $(3y^4)(-6y^3)$
3. $n^6 \cdot n^4 \cdot n$
4. $a^8 \cdot a^5 \cdot a$
5. $9^3 \cdot 9^5$
6. $4^2 \cdot 4^8$
7. $(-3m^4)(6m^2)(-4m^5)$
8. $(-8t^3)(2t^6)(-5t^4)$
9. $(5x^2y)(-3x^3y^4)$

10. ***Concept Check*** Decide whether each expression has been simplified correctly. If not, correct it. Assume all variables represent nonzero real numbers.

(a) $(mn)^2 = mn^2$ **(b)** $y^2 \cdot y^5 = y^7$ **(c)** $\left(\frac{k}{5}\right)^3 = \frac{k^3}{5}$ **(d)** $3^0y = 0$

(e) $4^5 \cdot 4^2 = 16^7$ **(f)** $(a^2)^3 = a^5$ **(g)** $cd^0 = 1$ **(h)** $(2b)^4 = 8b^4$

Simplify each expression. Assume variables represent nonzero real numbers. See Examples 1–2 and Homework 1.

11. $(2^2)^5$
12. $(6^4)^3$
13. $(-6x^2)^3$
14. $(-2x^5)^5$
15. $-(4m^3n^0)^2$
16. $-(2x^0y^4)^3$
17. $\left(\frac{r^8}{s^2}\right)^3$
18. $\left(\frac{p^4}{q}\right)^2$
19. $\left(\frac{-4m^2}{tp^2}\right)^4$
20. $\left(\frac{-5n^4}{r^2}\right)^3$
21. $-\left(\frac{x^3y^5}{z}\right)^0$
22. $-\left(\frac{p^2q^3}{r^3}\right)^0$

Match each expression in Column I with its equivalent in Column II. See Example 2.

	I	II
23.	(a) 6^0	A. 0
	(b) -6^0	B. 1
	(c) $(-6)^0$	C. -1
	(d) $-(-6)^0$	D. 6
		E. -6

	I	II
24.	(a) $3p^0$	A. 0
	(b) $-3p^0$	B. 1
	(c) $(3p)^0$	C. -1
	(d) $(-3p)^0$	D. 3
		E. -3

25. Explain why $x^2 + x^2$ is not equivalent to x^4.

26. Explain why $(x + y)^2$ is not equivalent to $x^2 + y^2$.

Identify each expression as a polynomial *or* not a polynomial. *For each polynomial give the degree and identify it as a* monomial, binomial, trinomial, *or* none of these. ***See Homework 2.***

27. $-5x^{11}$

28. $-4y^5$

29. $6x + 3x^4$

30. $-9y + 5y^3$

31. $-7z^5 - 2z^3 + 1$

32. $-9t^4 + 8t^3 - 7$

33. $15a^2b^3 + 12a^3b^8 - 13b^5 + 12b^6$

34. $-16x^5y^7 + 12x^3y^8 - 4xy^9 + 18x^{10}$

35. $\frac{3}{8}x^5 - \frac{1}{x^2} + 9$

36. $\frac{2}{3}t^6 + \frac{3}{t^5} + 1$

37. 5

38. 9

Find each sum or difference. ***See Example 3.***

39. $(5x^2 - 4x + 7) + (-4x^2 + 3x - 5)$

40. $(3m^3 - 3m^2 + 4) + (-2m^3 - m^2 + 6)$

41. $2(12y^2 - 8y + 6) - 4(3y^2 - 4y + 2)$

42. $3(8p^2 - 5p) - 5(3p^2 - 2p + 4)$

43. $(6m^4 - 3m^2 + m) - (2m^3 + 5m^2 + 4m) + (m^2 - m)$

44. $-(8x^3 + x - 3) + (2x^3 + x^2) - (4x^2 + 3x - 1)$

Find each product. ***See Homework 3–4 and Example 4.***

45. $(4r - 1)(7r + 2)$

46. $(5m - 6)(3m + 4)$

47. $x^2\left(3x - \frac{2}{3}\right)\left(5x + \frac{1}{3}\right)$

48. $m^3\left(2m - \frac{1}{4}\right)\left(3m + \frac{1}{2}\right)$

49. $4x^2(3x^3 + 2x^2 - 5x + 1)$

50. $2b^3(b^2 - 4b + 3)$

51. $(2z - 1)(-z^2 + 3z - 4)$

52. $(3w + 2)(-w^2 + 4w - 3)$

53. $(m - n + k)(m + 2n - 3k)$

54. $(r - 3s + t)(2r - s + t)$

55. $(2x + 3)(2x - 3)(4x^2 - 9)$

56. $(3y - 5)(3y + 5)(9y^2 - 25)$

57. $(x + 1)(x + 1)(x - 1)(x - 1)$

58. $(t + 4)(t + 4)(t - 4)(t - 4)$

Find each product. ***See Homework 4 and Example 5.***

59. $(2m + 3)(2m - 3)$

60. $(8s - 3t)(8s + 3t)$

61. $(4x^2 - 5y)(4x^2 + 5y)$

62. $(2m^3 + n)(2m^3 - n)$

63. $(4m + 2n)^2$

64. $(a - 6b)^2$

65. $(5r - 3t^2)^2$

66. $(2z^4 - 3y)^2$

67. $[(2p - 3) + q]^2$

68. $[(4y - 1) + z]^2$

69. $[(3q+5)-p][(3q+5)+p]$

70. $[(9r-s)+2][(9r-s)-2]$

71. $[(3a+b)-1]^2$

72. $[(2m+7)-n]^2$

73. $(y+2)^3$

74. $(z-3)^3$

75. $(q-2)^4$

76. $(r+3)^4$

Perform the indicated operations. **See Examples 3–5 and Homework 3–4.**

77. $(p^3-4p^2+p)-(3p^2+2p+7)$

78. $(x^4-3x^2+2)-(-2x^4+x^2-3)$

79. $(7m+2n)(7m-2n)$

80. $(3p+5)^2$

81. $-3(4q^2-3q+2)+2(-q^2+q-4)$

82. $2(3r^2+4r+2)-3(-r^2+4r-5)$

83. $p(4p-6)+2(3p-8)$

84. $m(5m-2)+9(5-m)$

85. $-y(y^2-4)+6y^2(2y-3)$

86. $-z^3(9-z)+4z(2+3z)$

Perform each division. **See Homework 5 and Example 6.**

87. $\dfrac{-4x^7-14x^6+10x^4-14x^2}{-2x^2}$

88. $\dfrac{-8r^3s-12r^2s^2+20rs^3}{-4rs}$

89. $\dfrac{4x^3-3x^2+1}{x-2}$

90. $\dfrac{3x^3-2x+5}{x-3}$

91. $\dfrac{6m^3+7m^2-4m+2}{3m+2}$

92. $\dfrac{10x^3+11x^2-2x+3}{5x+3}$

93. $\dfrac{x^4+5x^2+5x+27}{x^2+3}$

94. $\dfrac{k^4-4k^2+2k+5}{k^2+1}$

Relating Concepts

For individual or collaborative investigation *(Exercises 95–98)*

The special products can be used to perform selected multiplications. On the left, we use $(x+y)(x-y)=x^2-y^2$. *On the right,* $(x-y)^2=x^2-2xy+y^2$.

$$\begin{aligned} 51\times 49 &= (50+1)(50-1) \\ &= 50^2-1^2 \\ &= 2500-1 \\ &= 2499 \end{aligned}$$

$$\begin{aligned} 47^2 &= (50-3)^2 \\ &= 50^2-2(50)(3)+3^2 \\ &= 2500-300+9 \\ &= 2209 \end{aligned}$$

Use special products to evaluate each expression.

95. 99×101

96. 63×57

97. 102^2

98. 71^2

1.4 Factoring Polynomials

- Factoring Out the Greatest Common Factor
- Factoring by Grouping
- Factoring Trinomials
- Factoring Binomials
- Factoring by Substitution

The process of finding polynomials whose product equals a given polynomial is called **factoring.** Unless otherwise specified, we consider only integer coefficients when factoring polynomials. For example, since

$$4x + 12 = 4(x + 3)$$

both 4 and $x + 3$ are **factors** of $4x + 12$ and $4(x + 3)$ is a **factored form** of $4x + 12$.

A polynomial with variable terms that cannot be written as a product of two polynomials of lower degree is a **prime polynomial.** A polynomial is **factored completely** when it is written as a product of prime polynomials.

Factoring Out the Greatest Common Factor To factor $6x^2y^3 + 9xy^4 + 18y^5$, we look for a monomial that is the greatest common factor (GCF) of the three terms.

$$6x^2y^3 + 9xy^4 + 18y^5 = 3y^3(2x^2) + 3y^3(3xy) + 3y^3(6y^2) \quad \text{GCF} = 3y^3$$

$$= 3y^3(2x^2 + 3xy + 6y^2) \quad \text{Distributive property (Section 1.2)}$$

EXAMPLE 1 Factoring Out the Greatest Common Factor

Factor out the greatest common factor from each polynomial.

(a) $9y^5 + y^2$ **(b)** $6x^2t + 8xt + 12t$

(c) $14(m + 1)^3 - 28(m + 1)^2 - 7(m + 1)$

SOLUTION

(a) $9y^5 + y^2 = y^2(9y^3) + y^2(1)$ GCF $= y^2$

$= y^2(9y^3 + 1)$ Distributive property

Remember to include the 1.

CHECK Multiply out the factored form: $y^2(9y^3 + 1) = \overbrace{9y^5 + y^2}^{\text{Original polynomial}}$. ✓

(b) $6x^2t + 8xt + 12t = 2t(3x^2 + 4x + 6)$ GCF $= 2t$

CHECK $2t(3x^2 + 4x + 6) = 6x^2t + 8xt + 12t$ ✓

(c) $14(m + 1)^3 - 28(m + 1)^2 - 7(m + 1)$

$= 7(m + 1)[2(m + 1)^2 - 4(m + 1) - 1]$ GCF $= 7(m + 1)$

$= 7(m + 1)[2(m^2 + 2m + 1) - 4m - 4 - 1]$ Square $m + 1$ (Section 1.3); distributive property

Remember the middle term.

$= 7(m + 1)(2m^2 + 4m + 2 - 4m - 4 - 1)$ Distributive property

$= 7(m + 1)(2m^2 - 3)$ Combine like terms.

CAUTION In **Example 1(a),** the 1 is essential in the answer, since

$$y^2(9y^3) \neq 9y^5 + y^2.$$

Factoring can always be checked by multiplying.

Factoring by Grouping When a polynomial has more than three terms, it can sometimes be factored using **factoring by grouping.** Consider this example.

$$ax + ay + 6x + 6y = \overbrace{(ax + ay)}^{\text{Terms with common factor } a} + \overbrace{(6x + 6y)}^{\text{Terms with common factor } 6} \qquad \text{Group the terms so that each group has a common factor.}$$

$$= a(x + y) + 6(x + y) \qquad \text{Factor each group.}$$

$$= (x + y)(a + 6) \qquad \text{Factor out } x + y.$$

It is not always obvious which terms should be grouped. In cases like the one above, group in pairs. Experience and repeated trials are the most reliable tools.

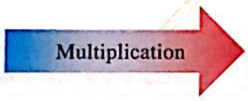
HOMEWORK 1 Factoring by Grouping

Factor each polynomial by grouping.

(a) $mp^2 + 7m + 3p^2 + 21$ **(b)** $2y^2 + az - 2z - ay^2$

(c) $4x^3 + 2x^2 - 2x - 1$

Factoring Trinomials As shown here, factoring is the opposite of multiplication.



$$(2x + 1)(3x - 4) = 6x^2 - 5x - 4$$

One strategy in factoring trinomials requires using the FOIL method in reverse.

EXAMPLE 2 Factoring Trinomials

Factor each trinomial, if possible.

(a) $4y^2 - 11y + 6$ **(b)** $6p^2 - 7p - 5$

(c) $2x^2 + 13x - 18$ **(d)** $16y^3 + 24y^2 - 16y$

SOLUTION

(a) To factor this polynomial, we must find integers a, b, c, and d such that

$$4y^2 - 11y + 6 = (ay + b)(cy + d). \qquad \text{FOIL}$$

Using FOIL, we see that $ac = 4$ and $bd = 6$. The positive factors of 4 are 4 and 1 or 2 and 2. Since the middle term has a negative coefficient, we consider only negative factors of 6. The possibilities are -2 and -3 or -1 and -6.

Now we try various arrangements of these factors until we find one that gives the correct coefficient of y.

$$(2y - 1)(2y - 6) = 4y^2 - 14y + 6 \qquad \text{Incorrect}$$

$$(2y - 2)(2y - 3) = 4y^2 - 10y + 6 \qquad \text{Incorrect}$$

$$(y - 2)(4y - 3) = 4y^2 - 11y + 6 \qquad \text{Correct}$$

Therefore, $4y^2 - 11y + 6$ factors as $(y - 2)(4y - 3)$.

CHECK $(y-2)(4y-3) = 4y^2 - 3y - 8y + 6$ FOIL

$= 4y^2 - 11y + 6$ ✓ Original polynomial

(b) Again, we try various possibilities to factor $6p^2 - 7p - 5$. The positive factors of 6 could be 2 and 3 or 1 and 6. As factors of -5 we have only -1 and 5 or -5 and 1.

$(2p-5)(3p+1) = 6p^2 - 13p - 5$ Incorrect

$(3p-5)(2p+1) = 6p^2 - 7p - 5$ Correct

Thus, $6p^2 - 7p - 5$ factors as $(3p-5)(2p+1)$.

(c) If we try to factor $2x^2 + 13x - 18$ as above, we find that none of the pairs of factors gives the correct coefficient of x.

$(2x+9)(x-2) = 2x^2 + 5x - 18$ Incorrect

$(2x-3)(x+6) = 2x^2 + 9x - 18$ Incorrect

$(2x-1)(x+18) = 2x^2 + 35x - 18$ Incorrect

Additional trials are also unsuccessful. Thus, this trinomial cannot be factored with integer coefficients and is prime.

(d) $16y^3 + 24y^2 - 16y = 8y(2y^2 + 3y - 2)$ Factor out the GCF, $8y$.

$= 8y(2y-1)(y+2)$ Factor the trinomial.

Remember to include the common factor in the final form.

NOTE In **Example 2**, we chose positive factors of the positive first term. We could have used two negative factors, but the work is easier if positive factors are used.

Each of the special patterns for multiplication given in **Section 1.3** can be used in reverse to get a pattern for factoring. Perfect square trinomials can be factored as follows.

Factoring Perfect Square Trinomials

$$x^2 + 2xy + y^2 = (x+y)^2$$
$$x^2 - 2xy + y^2 = (x-y)^2$$

HOMEWORK 2 **Factoring Perfect Square Trinomials**

Factor each trinomial.

(a) $16p^2 - 40pq + 25q^2$

(b) $36x^2y^2 + 84xy + 49$

Factoring Binomials Check first to see whether the terms of a binomial have a common factor. If so, factor it out. The binomial may also fit one of the following patterns.

Factoring Binomials

Difference of Squares	$x^2 - y^2 = (x+y)(x-y)$
Difference of Cubes	$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
Sum of Cubes	$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

CAUTION ***There is no factoring pattern for a sum of squares in the real number system.*** In particular, $x^2 + y^2$ **does not factor as** $(x + y)^2$, for real numbers x and y.

EXAMPLE 3 **Factoring Differences of Squares**

Factor each polynomial.

(a) $4m^2 - 9$ **(b)** $256k^4 - 625m^4$ **(c)** $(a + 2b)^2 - 4c^2$

(d) $x^2 - 6x + 9 - y^4$ **(e)** $y^2 - x^2 + 6x - 9$

SOLUTION

(a) $4m^2 - 9 = (2m)^2 - 3^2$ Write as a difference of squares.

$= (2m + 3)(2m - 3)$ Factor.

Check by multiplying.

(b) $256k^4 - 625m^4 = (16k^2)^2 - (25m^2)^2$ Write as a difference of squares.

Don't stop here. $= (16k^2 + 25m^2)(16k^2 - 25m^2)$ Factor.

$= (16k^2 + 25m^2)(4k + 5m)(4k - 5m)$ Factor $16k^2 - 25m^2$.

CHECK $(16k^2 + 25m^2)(4k + 5m)(4k - 5m)$

$= (16k^2 + 25m^2)(16k^2 - 25m^2)$ Multiply the last two factors.

$= 256k^4 - 625m^4$ ✓ Original polynomial

(c) $(a + 2b)^2 - 4c^2 = (a + 2b)^2 - (2c)^2$ Write as a difference of squares.

$= [(a + 2b) + 2c][(a + 2b) - 2c]$ Factor.

$= (a + 2b + 2c)(a + 2b - 2c)$

Check by multiplying.

(d) $x^2 - 6x + 9 - y^4 = (x^2 - 6x + 9) - y^4$ Group terms.

$= (x - 3)^2 - y^4$ Factor the trinomial.

$= (x - 3)^2 - (y^2)^2$ Write as a difference of squares.

$= [(x - 3) + y^2][(x - 3) - y^2]$ Factor.

$= (x - 3 + y^2)(x - 3 - y^2)$

Check by multiplying.

(e) $y^2 - x^2 + 6x - 9 = y^2 - (x^2 - 6x + 9)$ Factor out the negative sign and group the last three terms.

Be careful with signs. This is a perfect square trinomial.

$= y^2 - (x - 3)^2$ Write as a difference of squares.

$= [y - (x - 3)][y + (x - 3)]$ Factor.

$= (y - x + 3)(y + x - 3)$ Distributive property

Check by multiplying.

CAUTION When factoring as in **Example 3(e),** be careful with signs. Inserting an open parenthesis following the minus sign requires changing the signs of all of the following terms.

HOMEWORK 3 Factoring Sums or Differences of Cubes

Factor each polynomial.

(a) $x^3 + 27$ **(b)** $m^3 - 64n^3$ **(c)** $8q^6 + 125p^9$

Factoring by Substitution We introduce a new technique for factoring.

EXAMPLE 4 Factoring by Substitution

Factor each polynomial.

(a) $10(2a - 1)^2 - 19(2a - 1) - 15$ **(b)** $(2a - 1)^3 + 8$

(c) $6z^4 - 13z^2 - 5$

SOLUTION

(a) $10(2a - 1)^2 - 19(2a - 1) - 15$ Replace $2a - 1$ with u so that $(2a - 1)^2$ becomes u^2.

$= 10u^2 - 19u - 15$

$= (5u + 3)(2u - 5)$ Factor.

$= [5(2a - 1) + 3][2(2a - 1) - 5]$ Replace u with $2a - 1$.

Don't stop here. Replace u with $2a - 1$.

$= (10a - 5 + 3)(4a - 2 - 5)$ Distributive property

$= (10a - 2)(4a - 7)$ Simplify.

$= 2(5a - 1)(4a - 7)$ Factor out the common factor.

(b) $(2a - 1)^3 + 8 = u^3 + 8$ Replace $2a - 1$ with u.

$= u^3 + 2^3$ Write as a sum of cubes.

$= (u + 2)(u^2 - 2u + 4)$ Factor.

$= [(2a - 1) + 2][(2a - 1)^2 - 2(2a - 1) + 4]$ Replace u with $2a - 1$.

$= (2a + 1)(4a^2 - 4a + 1 - 4a + 2 + 4)$ Add, and then multiply.

$= (2a + 1)(4a^2 - 8a + 7)$ Combine like terms.

(c) $6z^4 - 13z^2 - 5 = 6u^2 - 13u - 5$ Replace z^2 with u.

$= (2u - 5)(3u + 1)$ Use FOIL to factor.

Remember to make the final substitution.

$= (2z^2 - 5)(3z^2 + 1)$ Replace u with z^2.

1.4 Exercises

Factor out the greatest common factor from each polynomial. **See Example 1 and Homework 1.**

1. $15r - 27$

2. $9z^4 + 81z$

3. $5h^2j + hj$

4. $-3z^5w^2 - 18z^3w^4$

5. $28r^4s^2 + 7r^3s - 35r^4s^3$

6. $6x(a + b) - 4y(a + b)$

7. $(4z - 5)(3z - 2) - (3z - 9)(3z - 2)$

8. $5(a + 3)^3 - 2(a + 3) + (a + 3)^2$

9. ***Concept Check*** Saad factored $16a^2 - 40a - 6a + 15$ by grouping and obtained $(8a - 3)(2a - 5)$. Kamal factored the same polynomial and gave an answer of $(3 - 8a)(5 - 2a)$. Which answer is correct?

Factor each polynomial by grouping. **See Homework 1.**

10. $10ab - 6b + 35a - 21$

11. $15 - 5m^2 - 3r^2 + m^2r^2$

12. $20z^2 - 8x + 5pz^2 - 2px$

Factor each trinomial, if possible. **See Example 2 and Homework 2.**

13. $8h^2 - 2h - 21$

14. $9y^2 - 18y + 8$

15. $9x^2 + 4x - 2$

16. $36x^3 + 18x^2 - 4x$

17. $14m^2 + 11mr - 15r^2$

18. $12s^2 + 11st - 5t^2$

19. $30a^2 + am - m^2$

20. $18x^5 + 15x^4z - 75x^3z^2$

21. $16p^2 - 40p + 25$

22. $20p^2 - 100pq + 125q^2$

23. $9m^2n^2 + 12mn + 4$

24. $(2p + q)^2 - 10(2p + q) + 25$

25. ***Concept Check*** Match each polynomial in Column I with its factored form in Column II.

I	II
(a) $8x^3 - 27$	**A.** $(3 - 2x)(9 + 6x + 4x^2)$
(b) $8x^3 + 27$	**B.** $(2x - 3)(4x^2 + 6x + 9)$
(c) $27 - 8x^3$	**C.** $(2x + 3)(4x^2 - 6x + 9)$

Factor each polynomial. **See Example 3 and Homework 3.**

26. $16q^2 - 25$

27. $y^4 - 81$

28. $36z^2 - 81y^4$

29. $(p - 2q)^2 - 100$

30. $m^4 - 1296$

31. $27 - r^3$

32. $8m^3 - 27n^3$

33. $27z^9 + 64y^{12}$

34. $(b + 3)^3 - 27$

35. $125 - (4a - b)^3$

36. ***Concept Check*** Which of the following is the correct factorization of $x^3 + 8$?

A. $(x + 2)^3$

B. $(x + 2)(x^2 + 2x + 4)$

C. $(x + 2)(x^2 - 2x + 4)$

D. $(x + 2)(x^2 - 4x + 4)$

Relating Concepts

For individual or collaborative investigation *(Exercises 37–39)*

The polynomial $x^6 - 1$ can be considered either a difference of squares or a difference of cubes. **Work Exercises 37–39 in order,** *to connect the results obtained when two different methods of factoring are used.*

37. Factor $x^6 - 1$ by first factoring as a difference of cubes, and then factor further by using the pattern for a difference of squares.

38. The polynomial $x^4 + x^2 + 1$ cannot be factored using the methods described in this section. However, there is a technique that enables us to factor it, as shown here. Supply the reason why each step is valid.

$$\begin{aligned} x^4 + x^2 + 1 &= x^4 + 2x^2 + 1 - x^2 && \underline{\hspace{4cm}} \\ &= (x^4 + 2x^2 + 1) - x^2 && \underline{\hspace{4cm}} \\ &= (x^2 + 1)^2 - x^2 && \underline{\hspace{4cm}} \\ &= (x^2 + 1 - x)(x^2 + 1 + x) && \underline{\hspace{4cm}} \\ &= (x^2 - x + 1)(x^2 + x + 1) && \underline{\hspace{4cm}} \end{aligned}$$

39. Factor $x^8 + x^4 + 1$ using the technique outlined in **Exercise 38.**

Factor each polynomial by substitution. ***See Example 4.***

40. $6(4z - 3)^2 + 7(4z - 3) - 3$

41. $4(5x + 7)^2 + 12(5x + 7) + 9$

42. $a^4 - 2a^2 - 48$

Factor by any method. ***See Examples 1–4 and Homework 1–3.***

43. $(2y - 1)^2 - 4(2y - 1) + 4$

44. $8r^2 - 3rs + 10s^2$

45. $36a^2 + 60a + 25$

46. $6p^4 + 7p^2 - 3$

47. $b^2 + 8b + 16 - a^2$

48. $q^2 + 6q + 9 - p^2$

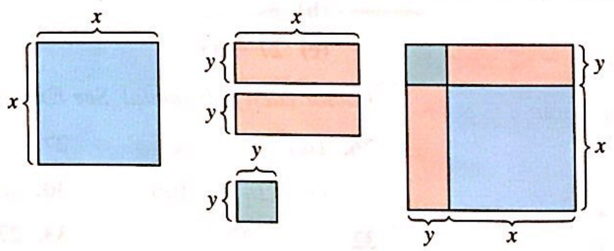
49. $216p^3 + 125q^3$

50. $100r^2 - 169s^2$

51. $(3a + 5)^2 - 18(3a + 5) + 81$

52. $4z^4 - 7z^2 - 15$

53. ***Geometric Modeling*** Explain how the figures give geometric interpretation to the formula $x^2 + 2xy + y^2 = (x + y)^2$.



Factor each polynomial over the set of rational number coefficients.

54. $81y^2 - \dfrac{1}{49}$

55. $\dfrac{121}{25}y^4 - 49x^2$

Concept Check *Find all values of b or c that will make the polynomial a perfect square trinomial.*

56. $9p^2 + bp + 25$

57. $49x^2 + 70x + c$

1.5 Rational Expressions

- Rational Expressions
- Lowest Terms of a Rational Expression
- Multiplication and Division
- Addition and Subtraction
- Complex Fractions

Rational Expressions The quotient of two polynomials P and Q, with $Q \neq 0$, is a **rational expression.**

$$\frac{x+6}{x+2}, \quad \frac{(x+6)(x+4)}{(x+2)(x+4)}, \quad \frac{2p^2+7p-4}{5p^2+20p} \qquad \text{Rational expressions}$$

The **domain** of a rational expression is the set of real numbers for which the expression is defined. Because the denominator of a fraction cannot be 0, the domain consists of all real numbers except those that make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving the resulting equation. For example, in the rational expression

$$\frac{x+6}{x+2},$$

the solution to the equation $x + 2 = 0$ is excluded from the domain. Since this solution is -2, the domain is the set of all real numbers x not equal to -2, or

$$\{x \mid x \neq -2\}. \qquad \text{Set-builder notation (Section 1.1)}$$

If the denominator of a rational expression contains a product, we determine the domain with the **zero-factor property**, which states that $ab = 0$ if and only if $a = 0$ or $b = 0$.

EXAMPLE 1 Finding the Domain

Find the domain of the rational expression.

$$\frac{(x+6)(x+4)}{(x+2)(x+4)}$$

SOLUTION

$$(x+2)(x+4) = 0 \qquad \text{Set the denominator equal to zero.}$$

$$x+2=0 \quad \text{or} \quad x+4=0 \qquad \text{Zero-factor property}$$

$$x=-2 \quad \text{or} \quad x=-4 \qquad \text{Solve each equation.}$$

The domain is the set of real numbers *not equal to* -2 or -4, written

$$\{x \mid x \neq -2, -4\}.$$

Lowest Terms of a Rational Expression A rational expression is written in **lowest terms** when the greatest common factor of its numerator and its denominator is 1. We use the following **fundamental principle of fractions.**

Fundamental Principle of Fractions

$$\frac{ac}{bc} = \frac{a}{b} \quad (b \neq 0, c \neq 0)$$

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HOMEWORK 1 **Writing Rational Expressions in Lowest Terms**

Write each rational expression in lowest terms.

(a) $\dfrac{2x^2+7x-4}{5x^2+20x}$ (b) $\dfrac{6-3x}{x^2-4}$

CAUTION The fundamental principle requires a pair of common *factors*, one in the numerator and one in the denominator. ***Only after a rational expression has been factored can any common factors be divided out.*** For example,

$$\frac{2x+4}{6}=\frac{2(x+2)}{2\cdot 3}=\frac{x+2}{3}. \quad \text{Factor first, and then divide.}$$

LOOKING AHEAD TO CALCULUS

A standard problem in calculus is investigating what value an expression such as $\frac{x^2-1}{x-1}$ approaches as x approaches 1. We cannot do this by simply substituting 1 for x in the expression since the result is the indeterminate form $\frac{0}{0}$. When we factor the numerator and write the expression in lowest terms, it becomes $x+1$. Then by substituting 1 for x, we get $1+1=2$, which is called the **limit** of $\frac{x^2-1}{x-1}$ as x approaches 1.

Multiplication and Division

We now multiply and divide fractions.

Multiplication and Division

For fractions $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0, d \neq 0)$, the following hold.

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd} \quad \text{and} \quad \frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c} \quad (c \neq 0)$$

That is, to find the product of two fractions, multiply their numerators to find the numerator of the product. Then multiply their denominators to find the denominator of the product. To divide two fractions, multiply the* dividend *(the first fraction) by the reciprocal of the* divisor *(the second fraction).

EXAMPLE 2 **Multiplying or Dividing Rational Expressions**

Multiply or divide, as indicated.

(a) $\dfrac{2y^2}{9}\cdot\dfrac{27}{8y^5}$ (b) $\dfrac{3m^2-2m-8}{3m^2+14m+8}\cdot\dfrac{3m+2}{3m+4}$

(c) $\dfrac{3p^2+11p-4}{24p^3-8p^2}\div\dfrac{9p+36}{24p^4-36p^3}$

(d) $\dfrac{x^3-y^3}{x^2-y^2}\cdot\dfrac{2x+2y+xz+yz}{2x^2+2y^2+zx^2+zy^2}$

SOLUTION

(a) $\dfrac{2y^2}{9}\cdot\dfrac{27}{8y^5}=\dfrac{2y^2\cdot 27}{9\cdot 8y^5}$ Multiply fractions.

$=\dfrac{2\cdot 9\cdot 3\cdot y^2}{9\cdot 2\cdot 4\cdot y^2\cdot y^3}$ Factor.

$=\dfrac{3}{4y^3}$ Fundamental principle

Although we usually factor first and then multiply the fractions (see parts (b)–(d)), we did the opposite here. Either order is acceptable.

(b) $\dfrac{3m^2 - 2m - 8}{3m^2 + 14m + 8} \cdot \dfrac{3m + 2}{3m + 4}$

$= \dfrac{(m-2)(3m+4)}{(m+4)(3m+2)} \cdot \dfrac{3m+2}{3m+4}$ Factor.

$= \dfrac{(m-2)(3m+4)(3m+2)}{(m+4)(3m+2)(3m+4)}$ Multiply fractions.

$= \dfrac{m-2}{m+4}$ Fundamental principle

(c) $\dfrac{3p^2 + 11p - 4}{24p^3 - 8p^2} \div \dfrac{9p + 36}{24p^4 - 36p^3}$

$= \dfrac{(p+4)(3p-1)}{8p^2(3p-1)} \div \dfrac{9(p+4)}{12p^3(2p-3)}$ Factor.

$= \dfrac{(p+4)(3p-1)}{8p^2(3p-1)} \cdot \dfrac{12p^3(2p-3)}{9(p+4)}$ Multiply by the reciprocal of the divisor.

$= \dfrac{12p^3(2p-3)}{9 \cdot 8p^2}$ Divide out common factors. Multiply fractions.

$= \dfrac{3 \cdot 4 \cdot p^2 \cdot p(2p-3)}{3 \cdot 3 \cdot 4 \cdot 2 \cdot p^2}$ Factor.

$= \dfrac{p(2p-3)}{6}$ Fundamental principle

(d) $\dfrac{x^3 - y^3}{x^2 - y^2} \cdot \dfrac{2x + 2y + xz + yz}{2x^2 + 2y^2 + zx^2 + zy^2}$

$= \dfrac{(x-y)(x^2+xy+y^2)}{(x+y)(x-y)} \cdot \dfrac{2(x+y) + z(x+y)}{2(x^2+y^2) + z(x^2+y^2)}$ Factor. Group terms and factor.

$= \dfrac{(x-y)(x^2+xy+y^2)}{(x+y)(x-y)} \cdot \dfrac{(2+z)(x+y)}{(2+z)(x^2+y^2)}$ Factor by grouping. **(Section 1.4)**

$= \dfrac{x^2+xy+y^2}{x^2+y^2}$ Multiply fractions; fundamental principle

Addition and Subtraction We now add and subtract fractions.

Addition and Subtraction

For fractions $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0, d \neq 0)$, the following hold.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

That is, to add (or subtract) two fractions in practice, find their least common denominator (LCD) and change each fraction to one with the LCD as denominator. The sum (or difference) of their numerators is the numerator of their sum (or difference), and the LCD is the denominator of their sum (or difference).

Finding the Least Common Denominator (LCD)

Step 1 Write each denominator as a product of prime factors.

Step 2 Form a product of all the different prime factors. Each factor should have as exponent the ***greatest*** exponent that appears on that factor.

HOMEWORK 2 **Adding or Subtracting Rational Expressions**

Add or subtract, as indicated.

(a) $\frac{5}{9x^2} + \frac{1}{6x}$ **(b)** $\frac{y}{y-2} + \frac{8}{2-y}$ **(c)** $\frac{3}{(x-1)(x+2)} - \frac{1}{(x+3)(x-4)}$

CAUTION ***When subtracting fractions where the second fraction has more than one term in the numerator, as in Homework 2(c), be sure to distribute the negative sign to each term.*** Use parentheses as in the second step to avoid an error.

Complex Fractions The quotient of two rational expressions is a **complex fraction.** There are two methods for simplifying a complex fraction.

EXAMPLE 3 **Simplifying Complex Fractions**

Simplify each complex fraction. In part (b), use two methods.

(a) $\dfrac{6-\frac{5}{k}}{1+\frac{5}{k}}$ **(b)** $\dfrac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}}$

SOLUTION

(a) Method 1 for simplifying uses the identity property for multiplication. We multiply both numerator and denominator by the LCD of all the fractions, k.

$$\frac{6-\frac{5}{k}}{1+\frac{5}{k}} = \frac{k\left(6-\frac{5}{k}\right)}{k\left(1+\frac{5}{k}\right)} = \frac{6k-k\left(\frac{5}{k}\right)}{k+k\left(\frac{5}{k}\right)} = \frac{6k-5}{k+5}$$

Distribute k to *all* terms within the parentheses.

$$\frac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}} = \frac{\left(\frac{a}{a+1}+\frac{1}{a}\right)a(a+1)}{\left(\frac{1}{a}+\frac{1}{a+1}\right)a(a+1)}$$

For Method 1, multiply both numerator and denominator by the LCD of all the fractions $a(a+1)$.

(Method 1)

$$= \frac{\frac{a}{a+1}(a)(a+1)+\frac{1}{a}(a)(a+1)}{\frac{1}{a}(a)(a+1)+\frac{1}{a+1}(a)(a+1)}$$

Distributive property

$$= \frac{a^2+(a+1)}{(a+1)+a}$$

Multiply.

$$= \frac{a^2+a+1}{2a+1}$$

Combine like terms.

$$\frac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}} = \frac{\frac{a^2+1(a+1)}{a(a+1)}}{\frac{1(a+1)+1(a)}{a(a+1)}}$$

For Method 2, find the LCD, and add terms in the numerator and denominator of the complex fraction.

(Method 2)

$$= \frac{\frac{a^2+a+1}{a(a+1)}}{\frac{2a+1}{a(a+1)}}$$

Combine terms in the numerator and denominator.

$$= \frac{a^2+a+1}{a(a+1)} \cdot \frac{a(a+1)}{2a+1}$$

Definition of division

The result is the same as in Method 1.

$$= \frac{a^2+a+1}{2a+1}$$

Multiply fractions, and write in lowest terms.

1.5 Exercises

Find the domain of each rational expression. ***See Example 1.***

1. $\frac{2x-4}{x+7}$
2. $\frac{9x+12}{(2x+3)(x-5)}$
3. $\frac{3}{x^2-5x-6}$
4. $\frac{x^2-25}{x-5}$
5. ***Concept Check*** Use specific values for x and y to show that in general, $\frac{1}{x}+\frac{1}{y}$ is not equivalent to $\frac{1}{x+y}$.

Write each rational expression in lowest terms. ***See Homework 1.***

6. $\frac{36y^2+72y}{9y^2}$
7. $\frac{-8(4-y)}{(y+2)(y-4)}$
8. $\frac{20r+10}{30r+15}$
9. $\frac{r^2-r-6}{r^2+r-12}$
10. $\frac{6y^2+11y+4}{3y^2+7y+4}$
11. $\frac{y^3-27}{y-3}$

Find each product or quotient. ***See Example 2.***

12. $\frac{8r^3}{6r} \div \frac{5r^2}{9r^3}$
13. $\frac{5m+25}{10} \div \frac{6m+30}{12}$
14. $\frac{y^3+y^2}{7} \cdot \frac{49}{y^4+y^3}$
15. $\frac{6r-18}{9r^2+6r-24} \div \frac{4r-12}{12r-16}$
16. $\frac{x^2+2x-15}{x^2+11x+30} \cdot \frac{x^2+2x-24}{x^2-8x+15}$
17. $\frac{y^2+y-2}{y^2+3y-4} \div \frac{y^2+3y+2}{y^2+4y+3}$
18. $\frac{x^2-y^2}{(x-y)^2} \cdot \frac{x^2-xy+y^2}{x^2-2xy+y^2} \div \frac{x^3+y^3}{(x-y)^4}$
19. $\frac{ac+ad+bc+bd}{a^2-b^2} \cdot \frac{a^3-b^3}{2a^2+2ab+2b^2}$
20. Explain how to find the least common denominator of several fractions.

Perform each addition or subtraction. ***See Homework 2.***

21. $\frac{8}{5p}+\frac{3}{4p}$
22. $\frac{8}{3p}+\frac{5}{4p}+\frac{9}{2p}$
23. $\frac{3}{z}+\frac{x}{z^2}$
24. $\frac{7}{18a^3b^2}-\frac{2}{9ab}$
25. $\frac{7x+8}{3x+2}-\frac{x+4}{3x+2}$
26. $\frac{m+1}{m-1}+\frac{m-1}{m+1}$

27. $\dfrac{4}{p-q} - \dfrac{2}{q-p}$

28. $\dfrac{m-4}{3m-4} - \dfrac{5m}{4-3m}$

29. $\dfrac{5}{x+2} + \dfrac{2}{x^2-2x+4} - \dfrac{60}{x^3+8}$

30. $\dfrac{p}{2p^2-9p-5} - \dfrac{2p}{6p^2-p-2}$

Simplify each expression. See Example 3.

31. $\dfrac{2-\frac{2}{y}}{2+\frac{2}{y}}$

32. $\dfrac{\frac{1}{y+3}-\frac{1}{y}}{\frac{1}{y}}$

33. $\dfrac{2+\frac{2}{1+x}}{2-\frac{2}{1-x}}$

34. $\dfrac{\frac{1}{x^3-y^3}}{\frac{1}{x^2-y^2}}$

35. $\dfrac{y+\frac{1}{y^2-9}}{\frac{1}{y+3}}$

36. $\dfrac{\frac{6}{x^2-25}+x}{\frac{1}{x-5}}$

37. $\dfrac{\frac{x+4}{x}-\frac{3}{x-2}}{\frac{x}{x-2}+\frac{1}{x}}$

38. $\dfrac{\frac{-2}{x+h}-\frac{-2}{x}}{h}$

39. $\dfrac{\frac{2}{(x+h)^2+16}-\frac{2}{x^2+16}}{h}$

1.6 Rational Exponents

- Negative Exponents and the Quotient Rule
- Rational Exponents
- Complex Fractions Revisited
- Radical Notation
- Simplified Radicals
- Operations with Radicals
- Rationalizing Denominators

Negative Exponents and the Quotient Rule

In **Section 1.3,** we justified the definition $a^0 = 1$ for $a \neq 0$ using the product rule for exponents. Suppose that n is a positive integer, and we wish to define a^{-n} to be consistent with the application of the product rule. Consider the product $a^n \cdot a^{-n}$, and apply the rule.

$$\begin{aligned} a^n \cdot a^{-n} &= a^{n+(-n)} && \text{Product rule} \\ &= a^0 && n \text{ and } -n \text{ are additive inverses.} \\ &= 1 && \text{Definition of } a^0 \end{aligned}$$

The expression a^{-n} acts as the *reciprocal* of a^n, which is written $\frac{1}{a^n}$. Thus, these two expressions must be equivalent.

Negative Exponent

Suppose that a is a nonzero real number and n is any integer.

$$a^{-n} = \frac{1}{a^n}$$

EXAMPLE 1 Using the Definition of a Negative Exponent

Evaluate each expression. In parts (d) and (e), write the expression without negative exponents. Assume all variables represent nonzero real numbers.

(a) 4^{-2} **(b)** -4^{-2} **(c)** $\left(\frac{2}{5}\right)^{-3}$ **(d)** $(xy)^{-3}$ **(e)** xy^{-3}

SOLUTION

(a) $4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$ **(b)** $-4^{-2} = -\dfrac{1}{4^2} = -\dfrac{1}{16}$

(c) $\left(\dfrac{2}{5}\right)^{-3} = \dfrac{1}{\left(\frac{2}{5}\right)^3} = \dfrac{1}{\frac{8}{125}} = \underbrace{1 \div \dfrac{8}{125} = 1 \cdot \dfrac{125}{8}}_{\text{Multiply by the reciprocal of the divisor.}} = \dfrac{125}{8}$

(d) $\underset{\substack{\uparrow \\ \text{Base is } xy.}}{(xy)^{-3}} = \dfrac{1}{(xy)^3}$, or $\dfrac{1}{x^3y^3}$ **(e)** $x\underset{\substack{\uparrow \\ \text{Base is } y.}}{y^{-3}} = x \cdot \dfrac{1}{y^3} = \dfrac{x}{y^3}$

CAUTION ***A negative exponent indicates a reciprocal, not a sign change of the expression.***

Example 1(c) showed the following.

$$\left(\frac{2}{5}\right)^{-3} = \frac{125}{8} = \left(\frac{5}{2}\right)^3$$

We can generalize this result. If $a \neq 0$ and $b \neq 0$, then for any integer n, the following is true.

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

The **quotient rule** for exponents follows from the definition of exponents.

Quotient Rule

Suppose that m and n are integers and a is a nonzero real number.

$$\frac{a^m}{a^n} = a^{m-n}$$

That is, when dividing powers of like bases, keep the same base and subtract the exponent of the denominator from the exponent of the numerator.

CAUTION When applying the quotient rule, be sure to subtract the exponents in the correct order. Be careful especially when the exponent in the denominator is negative, and avoid sign errors.

HOMEWORK 1 **Using the Quotient Rule**

Simplify each expression. Assume all variables represent nonzero real numbers.

(a) $\dfrac{12^5}{12^2}$ **(b)** $\dfrac{a^5}{a^{-8}}$ **(c)** $\dfrac{16m^{-9}}{12m^{11}}$ **(d)** $\dfrac{25r^7z^5}{10r^9z}$

The rules for exponents in **Section 1.3** were stated for positive integer exponents and for zero as an exponent. Those rules continue to apply in expressions involving negative exponents, as seen in the next example.

EXAMPLE 2 **Using the Rules for Exponents**

Simplify each expression. Write answers without negative exponents. Assume all variables represent nonzero real numbers.

(a) $3x^{-2}(4^{-1}x^{-5})^2$ **(b)** $\dfrac{12p^3q^{-1}}{8p^{-2}q}$ **(c)** $\dfrac{(3x^2)^{-1}(3x^5)^{-2}}{(3^{-1}x^{-2})^2}$

SOLUTION

(a)
$$
\begin{aligned}
3x^{-2}(4^{-1}x^{-5})^2 &= 3x^{-2}(4^{-2}x^{-10}) && \text{Power rules (Section 1.3)}\\
&= 3\cdot 4^{-2}\cdot x^{-2+(-10)} && \text{Rearrange factors; product rule (Section 1.3)}\\
&= 3\cdot 4^{-2}\cdot x^{-12} && \text{Simplify the exponent on } x.\\
&= \frac{3}{16x^{12}} && \text{Write with positive exponents.}
\end{aligned}
$$

(b)
$$
\begin{aligned}
\frac{12p^3q^{-1}}{8p^{-2}q} &= \frac{12}{8}\cdot\frac{p^3}{p^{-2}}\cdot\frac{q^{-1}}{q^1}\\
&= \frac{3}{2}\cdot p^{3-(-2)}q^{-1-1} && \text{Quotient rule}\\
&= \frac{3}{2}p^5q^{-2} && \text{Simplify the exponents.}\\
&= \frac{3p^5}{2q^2} && \text{Write with positive exponents.}
\end{aligned}
$$

(c)
$$
\begin{aligned}
\frac{(3x^2)^{-1}(3x^5)^{-2}}{(3^{-1}x^{-2})^2} &= \frac{3^{-1}x^{-2}3^{-2}x^{-10}}{3^{-2}x^{-4}} && \text{Power rules}\\
&= \frac{3^{-1+(-2)}x^{-2+(-10)}}{3^{-2}x^{-4}} = \frac{3^{-3}x^{-12}}{3^{-2}x^{-4}} && \text{Product rule}\\
&= 3^{-3-(-2)}x^{-12-(-4)} = 3^{-1}x^{-8} && \text{Quotient rule}\\
&= \frac{1}{3x^8} && \text{Write with positive exponents.}
\end{aligned}
$$

Be careful with signs.

CAUTION Notice the use of the power rule $(ab)^n = a^nb^n$ in **Example 2(c)**:

$$(3x^2)^{-1} = 3^{-1}(x^2)^{-1} = 3^{-1}x^{-2}.$$

***Remember to apply the exponent to the numerical coefficient* 3.**

Rational Exponents The definition of a^n can be extended to rational values of n by defining $a^{1/n}$ to be the nth root of a. By one of the power rules of exponents (extended to a rational exponent),

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a,$$

which suggests that $a^{1/n}$ is a number whose nth power is a.

The Expression $a^{1/n}$

$a^{1/n}$, n Even If n is an *even* positive integer, and if $a > 0$, then $a^{1/n}$ is the positive real number whose nth power is a. That is, $(a^{1/n})^n = a$. (In this case, $a^{1/n}$ is the principal nth root of a.)

$a^{1/n}$, n Odd If n is an *odd* positive integer, and a is *any nonzero real number*, then $a^{1/n}$ is the positive or negative real number whose nth power is a. That is, $(a^{1/n})^n = a$.

For all positive integers n, $0^{1/n} = 0$.

HOMEWORK 2 **Using the Definition of $a^{1/n}$**

Evaluate each expression.

(a) $36^{1/2}$ **(b)** $-100^{1/2}$ **(c)** $-(225)^{1/2}$ **(d)** $625^{1/4}$

(e) $(-1296)^{1/4}$ **(f)** $-1296^{1/4}$ **(g)** $(-27)^{1/3}$ **(h)** $-32^{1/5}$

The notation $a^{m/n}$ must be defined in such a way that all the previous rules for exponents still hold. For the power rule to hold, $(a^{1/n})^m$ must equal $a^{m/n}$. Therefore, $a^{m/n}$ is defined as follows.

The Expression $a^{m/n}$

Let m be any integer, n be any positive integer, and a be any real number for which $a^{1/n}$ is a real number.

$$a^{m/n} = (a^{1/n})^m$$

EXAMPLE 3 **Using the Definition of $a^{m/n}$**

Evaluate each expression.

(a) $125^{2/3}$ **(b)** $32^{7/5}$ **(c)** $-81^{3/2}$ **(d)** $(-27)^{2/3}$ **(e)** $16^{-3/4}$ **(f)** $(-4)^{5/2}$

SOLUTION

(a) $125^{2/3} = (125^{1/3})^2$

$= 5^2$, or 25

(b) $32^{7/5} = (32^{1/5})^7$

$= 2^7$, or 128

(c) $-81^{3/2} = -(81^{1/2})^3$

$= -9^3$, or -729

(d) $(-27)^{2/3} = [(-27)^{1/3}]^2$

$= (-3)^2$, or 9

(e) $16^{-3/4} = \dfrac{1}{16^{3/4}}$

$= \dfrac{1}{(16^{1/4})^3}$

$= \dfrac{1}{2^3}$, or $\dfrac{1}{8}$

(f) $(-4)^{5/2}$ is not a real number. This is because $(-4)^{1/2}$ is not a real number.

NOTE For all real numbers a, integers m, and positive integers n for which $a^{1/n}$ is a real number, $a^{m/n}$ can be interpreted as follows.

$$a^{m/n} = (a^{1/n})^m \quad \text{or} \quad a^{m/n} = (a^m)^{1/n}$$

So $a^{m/n}$ can be evaluated either as $(a^{1/n})^m$ or as $(a^m)^{1/n}$.

$$27^{4/3} = (27^{1/3})^4 = 3^4 = 81$$

or

$$27^{4/3} = (27^4)^{1/3} = 531{,}441^{1/3} = 81$$

The result is the same.

The earlier results for integer exponents also apply to rational exponents.

Definitions and Rules for Exponents

Suppose that r and s represent rational numbers. The results here are valid for all positive numbers a and b.

Product rule $a^r \cdot a^s = a^{r+s}$

Quotient rule $\dfrac{a^r}{a^s} = a^{r-s}$

Negative exponent $a^{-r} = \dfrac{1}{a^r}$

Power rules $(a^r)^s = a^{rs}$ $\quad (ab)^r = a^r b^r$ $\quad \left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$

HOMEWORK 3 Using the Rules for Exponents

Simplify each expression. Assume all variables represent positive real numbers

(a) $\dfrac{27^{1/3} \cdot 27^{5/3}}{27^3}$ **(b)** $81^{5/4} \cdot 4^{-3/2}$ **(c)** $6y^{2/3} \cdot 2y^{1/2}$

(d) $\left(\dfrac{3m^{5/6}}{y^{3/4}}\right)^2 \left(\dfrac{8y^3}{m^6}\right)^{2/3}$ **(e)** $m^{2/3}(m^{7/3} + 2m^{1/3})$

EXAMPLE 4 Factoring Expressions with Negative or Rational Exponents

Factor out the least power of the variable or variable expression. Assume all variables represent positive real numbers.

(a) $12x^{-2} - 8x^{-3}$ **(b)** $4m^{1/2} + 3m^{3/2}$ **(c)** $(y-2)^{-1/3} + (y-2)^{2/3}$

SOLUTION

(a) The least exponent on $12x^{-2} - 8x^{-3}$ is -3. Since 4 is a common numerical factor, factor out $4x^{-3}$.

$$12x^{-2} - 8x^{-3} = 4x^{-3}(3x^{-2-(-3)} - 2x^{-3-(-3)}) \quad \text{Factor.}$$
$$= 4x^{-3}(3x - 2) \quad \text{Simplify the exponents}$$

Check by multiplying on the right.

(b) $4m^{1/2} + 3m^{3/2} = m^{1/2}(4 + 3m)$ Factor out $m^{1/2}$.

To *check*, multiply $m^{1/2}$ by $4 + 3m$.

(c) $(y-2)^{-1/3} + (y-2)^{2/3} = (y-2)^{-1/3}[1 + (y-2)]$
$$= (y-2)^{-1/3}(y-1)$$

LOOKING AHEAD TO CALCULUS
The technique of **Example 4(c)** is used often in calculus.

Complex Fractions Revisited Negative exponents are sometimes used to write complex fractions. Recall that complex fractions are simplified either by first multiplying the numerator and denominator by the LCD of all the denominators, or by performing any indicated operations in the numerator and the denominator and then using the definition of division for fractions.

HOMEWORK 4 Simplifying a Fraction with Negative Exponents

Simplify $\dfrac{(x+y)^{-1}}{x^{-1}+y^{-1}}$. Write the result with only positive exponents.

CAUTION Remember that if $r \neq 1$, then $(x+y)^r \neq x^r + y^r$. In particular, this means that $(x+y)^{-1} \neq x^{-1} + y^{-1}$.

Radical Notation In **this section** we used rational exponents to express roots. An alternative notation for roots is **radical notation.**

Radical Notation for $a^{1/n}$

Suppose that a is a real number, n is a positive integer, and $a^{1/n}$ is a real number.

$$\sqrt[n]{a} = a^{1/n}$$

Radical Notation for $a^{m/n}$

Suppose that a is a real number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number.

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

In the radical $\sqrt[n]{a}$, the symbol $\sqrt[n]{}$ is a **radical symbol,** the number a is the **radicand,** and n is the **index.** We use the familiar notation $\sqrt{a}$ instead of $\sqrt[2]{a}$ for the square root.

For even values of n (square roots, fourth roots, and so on), when a is positive, there are two nth roots, one positive and one negative. In such cases, the notation $\sqrt[n]{a}$ represents the positive root, the **principal nth root.** We write the **negative root** as $-\sqrt[n]{a}$.

EXAMPLE 5 Evaluating Roots

Write each root using exponents and evaluate.

(a) $\sqrt[4]{16}$ (b) $-\sqrt[4]{16}$ (c) $\sqrt[5]{-32}$

(d) $\sqrt[3]{1000}$ (e) $\sqrt[6]{\dfrac{64}{729}}$ (f) $\sqrt[4]{-16}$

SOLUTION

(a) $\sqrt[4]{16} = 16^{1/4}$
$= 2$

(b) $-\sqrt[4]{16} = -16^{1/4}$
$= -2$

(c) $\sqrt[5]{-32} = (-32)^{1/5}$
$= -2$

(d) $\sqrt[3]{1000} = 1000^{1/3}$
$= 10$

(e) $\sqrt[6]{\dfrac{64}{729}} = \left(\dfrac{64}{729}\right)^{1/6}$
$= \dfrac{2}{3}$

(f) $\sqrt[4]{-16}$ is not a real number.

HOMEWORK 5 **Converting from Rational Exponents to Radicals**

Write in radical form and simplify. Assume all variable expressions represent positive real numbers.

(a) $8^{2/3}$ **(b)** $(-32)^{4/5}$ **(c)** $-16^{3/4}$ **(d)** $x^{5/6}$
(e) $3x^{2/3}$ **(f)** $2n^{1/2}$ **(g)** $(3a+b)^{1/4}$

LOOKING AHEAD TO CALCULUS
In calculus, the "power rule" for derivatives requires converting radicals to rational exponents.

CAUTION It is not possible to "distribute" exponents over a sum, so in **Homework 5(g),** $(3a+b)^{1/4}$ *cannot be written as* $(3a)^{1/4} + b^{1/4}$.

$$\sqrt[n]{x^n + y^n} \quad \textit{is not equivalent to} \quad x + y.$$

(For example, let $n = 2$, $x = 3$, and $y = 4$ to see this.)

EXAMPLE 6 **Converting from Radicals to Rational Exponents**

Write in exponential form. Assume all variable expressions represent positive real numbers.

(a) $\sqrt[4]{x^5}$ **(b)** $\sqrt{3y}$ **(c)** $10\left(\sqrt[5]{z}\right)^2$
(d) $5\sqrt[3]{(2x^4)^7}$ **(e)** $\sqrt{p^2+q}$

SOLUTION

(a) $\sqrt[4]{x^5} = x^{5/4}$ **(b)** $\sqrt{3y} = (3y)^{1/2}$ **(c)** $10\left(\sqrt[5]{z}\right)^2 = 10z^{2/5}$

(d) $5\sqrt[3]{(2x^4)^7} = 5(2x^4)^{7/3}$
$= 5 \cdot 2^{7/3}x^{28/3}$

(e) $\sqrt{p^2+q} = (p^2+q)^{1/2}$

We *cannot* simply write $\sqrt{x^2} = x$ for all real numbers x. For example, if $x = -5$, then

$$\sqrt{x^2} = \sqrt{(-5)^2} = \sqrt{25} = 5 \neq x.$$

To take care of the fact that a negative value of x can produce a positive result, we use absolute value. For any real number a, the following holds.

$$\sqrt{a^2} = |a|$$

For example, $\sqrt{(-9)^2} = |-9| = 9$ and $\sqrt{13^2} = |13| = 13$.

We can generalize this result to any *even* nth root.

Evaluating $\sqrt[n]{a^n}$

If n is an *even* positive integer, then $\sqrt[n]{a^n} = |a|$.

If n is an *odd* positive integer, then $\sqrt[n]{a^n} = a$.

HOMEWORK 6 **Using Absolute Value to Simplify Roots**

Simplify each expression.

(a) $\sqrt[2]{p^4}$ (b) $\sqrt[4]{p^4}$ (c) $\sqrt{16m^8r^6}$

(d) $\sqrt[6]{(-2)^6}$ (e) $\sqrt[5]{m^5}$ (f) $\sqrt{(2k+3)^2}$

(g) $\sqrt{x^2 - 4x + 4}$

NOTE When working with variable radicands, we will *usually* assume that all variables in radicands represent only nonnegative real numbers.

The following rules for working with radicals are simply the power rules for exponents written in radical notation.

Rules for Radicals

Suppose that a and b represent real numbers, and m and n represent positive integers for which the indicated roots are real numbers.

Rule	Description
Product rule $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	The product of two roots is the root of the product.
Quotient rule $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (b \neq 0)$	The root of a quotient is the quotient of the roots.
Power rule $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	The index of the root of a root is the product of their indexes.

EXAMPLE 7 **Simplifying Radical Expressions**

Simplify. Assume all variable expressions represent positive real numbers.

(a) $\sqrt{6} \cdot \sqrt{54}$ (b) $\sqrt[3]{m} \cdot \sqrt[3]{m^2}$ (c) $\sqrt{\frac{7}{64}}$

(d) $\sqrt[4]{\frac{a}{b^4}}$ (e) $\sqrt[7]{\sqrt[3]{2}}$ (f) $\sqrt[4]{\sqrt{3}}$

SOLUTION

(a) $\sqrt{6} \cdot \sqrt{54} = \sqrt{6 \cdot 54}$ Product rule

$= \sqrt{324}$, or 18

(b) $\sqrt[3]{m} \cdot \sqrt[3]{m^2} = \sqrt[3]{m^3}$

$= m$

(c) $\sqrt{\dfrac{7}{64}} = \dfrac{\sqrt{7}}{\sqrt{64}}$ Quotient rule

$= \dfrac{\sqrt{7}}{8}$

(d) $\sqrt[4]{\dfrac{a}{b^4}} = \dfrac{\sqrt[4]{a}}{\sqrt[4]{b^4}}$

$= \dfrac{\sqrt[4]{a}}{b}$

(e) $\sqrt[7]{\sqrt[3]{2}} = \sqrt[21]{2}$ Power rule

(f) $\sqrt[4]{\sqrt{3}} = \sqrt[4\cdot 2]{3} = \sqrt[8]{3}$

NOTE Converting to rational exponents shows why these rules work.

$$\sqrt[7]{\sqrt[3]{2}} = (2^{1/3})^{1/7} = 2^{(1/3)(1/7)} = 2^{1/21} = \sqrt[21]{2} \quad \text{Example 7(e)}$$

Simplified Radicals In working with numbers, we prefer to write a number in its simplest form. For example, $\frac{10}{2}$ is written as 5, and $-\frac{9}{6}$ is written as $-\frac{3}{2}$. Similarly, expressions with radicals can be written in their simplest forms.

Simplified Radicals

An expression with radicals is simplified when all of the following conditions are satisfied.

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions.
3. No denominator contains a radical.
4. Exponents in the radicand and the index of the radical have greatest common factor 1.
5. All indicated operations have been performed (if possible).

HOMEWORK 7 Simplifying Radicals

Simplify each radical.

(a) $\sqrt{175}$ (b) $-3\sqrt[5]{32}$ (c) $\sqrt[3]{81x^5y^7z^6}$

Operations with Radicals Radicals with the same radicand and the same index, such as $3\sqrt[4]{11pq}$ and $-7\sqrt[4]{11pq}$, are **like radicals.** On the other hand examples of **unlike radicals** are as follows.

$2\sqrt{5}$ and $2\sqrt{3}$ Radicands are different.

$2\sqrt{3}$ and $2\sqrt[3]{3}$ Indexes are different.

We add or subtract like radicals by using the distributive property. ***Only like radicals can be combined.*** Sometimes we need to simplify radicals before adding or subtracting.

EXAMPLE 8 Adding and Subtracting Radicals

Add or subtract, as indicated. Assume all variables represent positive real numbers.

(a) $3\sqrt[4]{11pq} + \left(-7\sqrt[4]{11pq}\right)$

(b) $\sqrt{98x^3y} + 3x\sqrt{32xy}$

(c) $\sqrt[3]{64m^4n^5} - \sqrt[3]{-27m^{10}n^{14}}$

SOLUTION

(a) $3\sqrt[4]{11pq} + \left(-7\sqrt[4]{11pq}\right) = -4\sqrt[4]{11pq}$

(b) $\sqrt{98x^3y} + 3x\sqrt{32xy} = \sqrt{49 \cdot 2 \cdot x^2 \cdot x \cdot y} + 3x\sqrt{16 \cdot 2 \cdot x \cdot y}$ Factor.

$= 7x\sqrt{2xy} + 3x(4)\sqrt{2xy}$ Remove all perfect squares from the radicals.

$= 7x\sqrt{2xy} + 12x\sqrt{2xy}$ Multiply.

$= (7x + 12x)\sqrt{2xy}$ Distributive property (Section 1.2)

$= 19x\sqrt{2xy}$ Add.

(c) $\sqrt[3]{64m^4n^5} - \sqrt[3]{-27m^{10}n^{14}} = \sqrt[3]{(64m^3n^3)(mn^2)} - \sqrt[3]{(-27m^9n^{12})(mn^2)}$

$= 4mn\sqrt[3]{mn^2} - (-3)m^3n^4\sqrt[3]{mn^2}$

$= 4mn\sqrt[3]{mn^2} + 3m^3n^4\sqrt[3]{mn^2}$

$= (4mn + 3m^3n^4)\sqrt[3]{mn^2}$

This *cannot* be simplified further.

If the index of the radical and an exponent in the radicand have a common factor, we can simplify the radical by first writing it in exponential form. We simplify the rational exponent, and then write the result as a radical again, as shown in **Homework 8** on the next page.

HOMEWORK 8 Simplifying Radicals

Simplify each radical. Assume all variables represent positive real numbers.

(a) $\sqrt[6]{3^2}$ **(b)** $\sqrt[6]{x^{12}y^3}$ **(c)** $\sqrt[9]{\sqrt{6^3}}$

In **Homework 8(a),** we simplified $\sqrt[6]{3^2}$ as $\sqrt[3]{3}$. However, to simplify $\left(\sqrt[6]{x}\right)^2$, the variable x must represent a nonnegative number. For example, consider the statement

$$(-8)^{2/6} = \left[(-8)^{1/6}\right]^2.$$

This result is not a real number, since $(-8)^{1/6}$ is not a real number. On the other hand,

$$(-8)^{1/3} = -2.$$

Here, even though $\frac{2}{6} = \frac{1}{3}$,

$$\left(\sqrt[6]{x}\right)^2 \neq \sqrt[3]{x}.$$

If a is nonnegative, then it is always true that $a^{m/n} = a^{mp/(np)}$. Simplifying rational exponents on negative bases should be considered case by case.

EXAMPLE 9 Multiplying Radical Expressions

Find each product.

(a) $\left(\sqrt{7} - \sqrt{10}\right)\left(\sqrt{7} + \sqrt{10}\right)$ **(b)** $\left(\sqrt{2} + 3\right)\left(\sqrt{8} - 5\right)$

SOLUTION

(a) $(\sqrt{7}-\sqrt{10})(\sqrt{7}+\sqrt{10}) = (\sqrt{7})^2 - (\sqrt{10})^2$ — Product of the sum and difference of two terms (Section 1.3)

$= 7 - 10$ — $(\sqrt{a})^2 = a$

$= -3$ — Subtract.

(b) $(\sqrt{2}+3)(\sqrt{8}-5) = \sqrt{2}(\sqrt{8}) - \sqrt{2}(5) + 3\sqrt{8} - 3(5)$ — FOIL (Section 1.3)

$= \sqrt{16} - 5\sqrt{2} + 3(2\sqrt{2}) - 15$ — Multiply; $\sqrt{8} = 2\sqrt{2}$.

$= 4 - 5\sqrt{2} + 6\sqrt{2} - 15$ — Simplify.

$= -11 + \sqrt{2}$ — Combine like terms.

Rationalizing Denominators The third condition for a simplified radical requires that no denominator contain a radical. We achieve this by **rationalizing the denominator**—that is, multiplying by a form of 1.

HOMEWORK 9 Rationalizing Denominators

Rationalize each denominator. →

(a) $\dfrac{4}{\sqrt{3}}$ **(b)** $\sqrt[4]{\dfrac{3}{5}}$

EXAMPLE 10 Simplifying Radical Expressions with Fractions

Simplify each expression. Assume all variables represent positive real numbers.

(a) $\dfrac{\sqrt[4]{xy^3}}{\sqrt[4]{x^3y^2}}$ **(b)** $\sqrt[3]{\dfrac{5}{x^6}} - \sqrt[3]{\dfrac{4}{x^9}}$

SOLUTION

(a) $\dfrac{\sqrt[4]{xy^3}}{\sqrt[4]{x^3y^2}} = \sqrt[4]{\dfrac{xy^3}{x^3y^2}}$ — Quotient rule

$= \sqrt[4]{\dfrac{y}{x^2}}$ — Simplify the radicand.

$= \dfrac{\sqrt[4]{y}}{\sqrt[4]{x^2}}$ — Quotient rule

$= \dfrac{\sqrt[4]{y}}{\sqrt[4]{x^2}} \cdot \dfrac{\sqrt[4]{x^2}}{\sqrt[4]{x^2}}$ — Rationalize the denominator.

$= \dfrac{\sqrt[4]{x^2y}}{x}$ — $\sqrt[4]{x^2} \cdot \sqrt[4]{x^2} = \sqrt[4]{x^4} = x$

LOOKING AHEAD TO CALCULUS

Another standard problem in calculus is investigating the value that an expression such as $\frac{\sqrt{x^2+9}-3}{x^2}$ approaches as x approaches 0. This cannot be done by simply substituting 0 for x, since the result is $\frac{0}{0}$. However, by **rationalizing the numerator,** we can show that for $x \neq 0$ the expression is equivalent to $\frac{1}{\sqrt{x^2+9}+3}$. Then, by substituting 0 for x, we find that the original expression approaches $\frac{1}{6}$ as x approaches 0.

(b) $\sqrt[3]{\frac{5}{x^6}} - \sqrt[3]{\frac{4}{x^9}} = \frac{\sqrt[3]{5}}{\sqrt[3]{x^6}} - \frac{\sqrt[3]{4}}{\sqrt[3]{x^9}}$ Quotient rule

$= \frac{\sqrt[3]{5}}{x^2} - \frac{\sqrt[3]{4}}{x^3}$ Simplify the denominators.

$= \frac{x\sqrt[3]{5}}{x^3} - \frac{\sqrt[3]{4}}{x^3}$ Write with a common denominator. (Section 1.5)

$= \frac{x\sqrt[3]{5} - \sqrt[3]{4}}{x^3}$ Subtract the numerators.

In **Example 9(a),** we saw that the product

$(\sqrt{7}-\sqrt{10})(\sqrt{7}+\sqrt{10})$ equals -3, a rational number.

This suggests a way to rationalize a denominator that is a binomial in which one or both terms is a square root radical. The expressions $a - b$ and $a + b$ are **conjugates.**

HOMEWORK 10 Rationalizing a Binomial Denominator

Rationalize the denominator of $\frac{1}{1-\sqrt{2}}$.

1.6 Exercises

Concept Check *In Exercise 1, match each expression in Column I with its equivalent expression in Column II. Choices may be used once, more than once, or not at all.*

I	II
1. (a) 5^{-3}	**A.** 125
(b) -5^{-3}	**B.** -125
(c) $(-5)^{-3}$	**C.** $\frac{1}{125}$
(d) $-(-5)^{-3}$	**D.** $-\frac{1}{125}$

Write each expression with only positive exponents and evaluate if possible. Assume all variables represent nonzero real numbers. **See Example 1.**

2. $(-5)^{-2}$ **3.** -7^{-2} **4.** $\left(\frac{4}{3}\right)^{-3}$

5. $(5t)^{-3}$ **6.** $5t^{-3}$ **7.** $-b^{-4}$

Perform the indicated operations. Write each answer using only positive exponents.
Assume all variables represent nonzero real numbers. **See Homework 1 and Example 2.**

8. $\frac{5^9}{5^7}$ **9.** $\frac{y^{14}}{y^{10}}$ **10.** $\frac{y^8}{y^{12}}$ **11.** $\frac{7^5}{7^{-3}}$

12. $\frac{15s^{-4}}{5s^{-8}}$ **13.** $\frac{15a^{-5}b^{-1}}{25a^{-2}b^4}$ **14.** $-2m^{-1}(m^3)^2$ **15.** $(3p^{-4})^2(p^3)^{-1}$

16. $\dfrac{(m^4)^0}{9m^{-3}}$ **17.** $\dfrac{(-8xy)y^3}{4x^5y^4}$ **18.** $\dfrac{12k^{-2}(k^{-3})^{-4}}{6k^5}$ **19.** $\dfrac{(8y^2)^{-4}(8y^5)^{-2}}{(8^{-3}y^{-4})^2}$

Evaluate each expression. **See Homework 2.**

20. $121^{1/2}$ **21.** $625^{1/4}$ **22.** $\left(-\dfrac{8}{27}\right)^{1/3}$ **23.** $(-64)^{1/4}$

Concept Check *In Exercise 24, match each expression from Column I with its equivalent expression from Column II. Choices may be used once, more than once, or not at all.*

I	II	
24. (a) $\left(\dfrac{8}{27}\right)^{2/3}$	**A.** $\dfrac{9}{4}$	**B.** $-\dfrac{9}{4}$
(b) $\left(\dfrac{8}{27}\right)^{-2/3}$	**C.** $-\dfrac{4}{9}$	**D.** $\dfrac{4}{9}$
(c) $-\left(\dfrac{27}{8}\right)^{2/3}$	**E.** $\dfrac{8}{27}$	**F.** $-\dfrac{27}{8}$
(d) $-\left(\dfrac{27}{8}\right)^{-2/3}$	**G.** $\dfrac{27}{8}$	**H.** $-\dfrac{8}{27}$

Perform the indicated operations. Write each answer using only positive exponents. Assume all variables represent positive real numbers. **See Example 3 and Homework 3.**

25. $27^{4/3}$ **26.** $64^{3/2}$ **27.** $(-32)^{-4/5}$

28. $\left(\dfrac{121}{100}\right)^{-3/2}$ **29.** $6^{4/3} \cdot 6^{2/3}$ **30.** $\dfrac{125^{7/3}}{125^{5/3}}$

31. $r^{-8/9} \cdot r^{17/9}$ **32.** $\dfrac{z^{3/4}}{z^{5/4} \cdot z^{-2}}$ **33.** $\dfrac{(r^{1/5}s^{2/3})^{15}}{r^2}$

34. $\dfrac{(p^3)^{1/4}}{(p^{5/4})^2}$ **35.** $\left(\dfrac{25^4a^3}{b^2}\right)^{1/8}\left(\dfrac{4^2b^{-5}}{a^2}\right)^{1/4}$ **36.** $\dfrac{z^{1/3}z^{-2/3}z^{1/6}}{(z^{-1/6})^3}$

Find each product. Assume all variables represent positive real numbers. **See Homework 3(e) in this section and Example 4 in Section 1.3.**

37. $p^{11/5}(3p^{4/5} + 9p^{19/5})$ **38.** $-5y(3y^{9/10} + 4y^{3/10})$

39. $(2z^{1/2} + z)(z^{1/2} - z)$ **40.** $(p^{1/2} - p^{-1/2})(p^{1/2} + p^{-1/2})$

Factor, using the given common factor. Assume all variables represent positive real numbers. **See Example 4.**

41. $y^{-5} - 3y^{-3}$; y^{-5} **42.** $5r^{-6} - 10r^{-8}$; $5r^{-8}$

43. $3m^{2/3} - 4m^{-1/3}$; $m^{-1/3}$ **44.** $6r^{-2/3} - 5r^{-5/3}$; $r^{-5/3}$

45. $-3p^{-3/4} - 30p^{-7/4}$; $-3p^{-7/4}$

46. $(3r + 1)^{-2/3} + (3r + 1)^{1/3} + (3r + 1)^{4/3}$; $(3r + 1)^{-2/3}$

47. $7(5t + 3)^{-5/3} + 14(5t + 3)^{-2/3} - 21(5t + 3)^{1/3}$; $7(5t + 3)^{-5/3}$

48. $6y^3(4y - 1)^{-3/7} - 8y^2(4y - 1)^{4/7} + 16y(4y - 1)^{11/7}$; $2y(4y - 1)^{-3/7}$

Perform all indicated operations and write each answer with positive integer exponents. ***See Homework 4.***

49. $\dfrac{p^{-1} - q^{-1}}{(pq)^{-1}}$

50. $\dfrac{x^{-2} + y^{-2}}{x^{-2} - y^{-2}} \cdot \dfrac{x + y}{x - y}$

51. $\dfrac{a - 16b^{-1}}{(a + 4b^{-1})(a - 4b^{-1})}$

Simplify each rational expression. Use factoring, and refer to ***Section 1.5*** *as needed. Assume all variable expressions represent positive real numbers.*

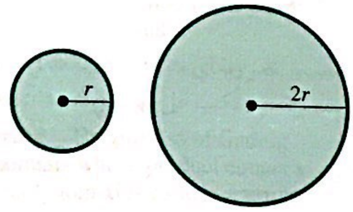
52. $\dfrac{(y^2 + 2)^5(3y) - y^3(6)(y^2 + 2)^4(3y)}{(y^2 + 2)^7}$

53. $\dfrac{10(4x^2 - 9)^2 - 25x(4x^2 - 9)^3}{15(4x^2 - 9)^6}$

54. $\dfrac{7(3t + 1)^{1/4} - (t - 1)(3t + 1)^{-3/4}}{(3t + 1)^{3/4}}$

Concept Check *Answer the question.*

55. If the radius of a circle is doubled, by what factor will the area change?



Concept Check *Calculate each value mentally.*

56. $0.1^{3/2} \cdot 90^{3/2}$ **57.** $\dfrac{20^{3/2}}{5^{3/2}}$

Write each root using exponents and evaluate. ***See Example 5.***

58. $\sqrt[3]{216}$ **59.** $\sqrt[4]{256}$ **60.** $\sqrt[3]{-343}$

61. $\sqrt[4]{-256}$ **62.** $\sqrt[7]{128}$ **63.** $-\sqrt[3]{-343}$

Concept Check *In Exercise 64, match the rational exponent expression in Column I with the equivalent radical expression in Column II. Assume that x is not 0.* ***See Homework 5.***

I	II	
64. (a) $-3x^{1/3}$	**A.** $\dfrac{3}{\sqrt[3]{x}}$	**B.** $-3\sqrt[3]{x}$
(b) $-3x^{-1/3}$	**C.** $\dfrac{1}{\sqrt[3]{3x}}$	**D.** $\dfrac{-3}{\sqrt[3]{x}}$
(c) $3x^{-1/3}$	**E.** $3\sqrt[3]{x}$	**F.** $\sqrt[3]{-3x}$
(d) $3x^{1/3}$	**G.** $\sqrt[3]{3x}$	**H.** $\dfrac{1}{\sqrt[3]{-3x}}$

If the expression is in exponential form, write it in radical form. If it is in radical form, write it in exponential form. Assume all variables represent positive real numbers. ***See Homework 5 and Example 6.***

65. $p^{5/4}$ **66.** $(5r + 3t)^{4/7}$ **67.** $\sqrt[4]{z^5}$ **68.** $-m\sqrt{2y^5}$

Concept Check *Answer each question.*

69. For what positive integers n greater than or equal to 2 is $\sqrt[n]{a^n} = a$ always a true statement?

70. Which of the following expressions is *not* simplified? Give the simplified form.

A. $\sqrt[3]{2y}$ **B.** $\dfrac{\sqrt{5}}{2}$ **C.** $\sqrt[4]{m^3}$ **D.** $\sqrt{\dfrac{3}{4}}$

Simplify each expression. ***See Homework 6.***

71. $\sqrt[6]{x^6}$ 72. $\sqrt[4]{81p^{12}q^4}$ 73. $\sqrt[4]{(5+2m)^4}$

Simplify each expression. Assume all variables represent positive real numbers. See ***Examples 5, 7, 9, and 10, and Homework 6–9.***

74. $\sqrt[3]{250}$ 75. $-\sqrt[4]{243}$ 76. $\sqrt{7}\cdot\sqrt{5xt}$

77. $\sqrt[3]{9x}\cdot\sqrt[3]{4y}$ 78. $-\sqrt{\frac{16}{49}}$ 79. $-\sqrt[4]{\frac{3}{16}}$

80. $\sqrt[6]{\frac{r}{s^6}}$ 81. $5\sqrt[3]{-343}$ 82. $\sqrt[3]{25(-3)^4(5)^3}$

83. $\sqrt{24m^6n^5}$ 84. $\sqrt[3]{27+a^3}$ 85. $\sqrt{\frac{5}{3p}}$

86. $\sqrt{\frac{g^3h^5}{r^3}}$ 87. $\sqrt[3]{\frac{9}{16p^4}}$ 88. $\sqrt[4]{\frac{32x^5}{y^5}}$

89. $\sqrt[9]{5^3}$ 90. $\sqrt[4]{\sqrt{25}}$ 91. $\sqrt[5]{\sqrt[3]{9}}$

Simplify each expression. Assume all variables represent positive real numbers. See ***Examples 8, 9, and 10.***

92. $4\sqrt{18k}-\sqrt{72k}+\sqrt{50k}$

93. $\sqrt[3]{32}-5\sqrt[3]{4}+2\sqrt[3]{108}$

94. $\sqrt[4]{256x^5y^6}+\sqrt[4]{625x^9y^2}$

95. $3\sqrt{11}-5\sqrt{13}$

96. $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})$

97. $(\sqrt[3]{7}+3)(\sqrt[3]{7^2}-3\sqrt[3]{7}+9)$

98. $(\sqrt{5}+\sqrt{10})^2$

99. $(4\sqrt{5}+\sqrt{2})(3\sqrt{2}-\sqrt{5})$

100. $\dfrac{\sqrt[3]{8m^2n^3}\cdot\sqrt[3]{2m^2}}{\sqrt[3]{32m^4n^3}}$

101. $\sqrt[4]{\frac{7}{t^{12}}}+\sqrt[4]{\frac{9}{t^4}}$

102. $\dfrac{2}{\sqrt{12}}-\dfrac{1}{\sqrt{27}}-\dfrac{5}{\sqrt{48}}$

103. $\dfrac{5}{\sqrt[3]{2}}-\dfrac{2}{\sqrt[3]{16}}+\dfrac{1}{\sqrt[3]{54}}$

Rationalize the denominator of each radical expression. Assume all variables represent nonnegative numbers and that no denominators are 0. ***See Homework 10.***

104. $\dfrac{\sqrt{7}}{\sqrt{3}-\sqrt{7}}$

105. $\dfrac{1+\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$

106. $\dfrac{9-r}{3-\sqrt{r}}$

107. $\dfrac{a}{\sqrt{a+b}-1}$

108. ***Concept Check*** What should the numerator and denominator of

$$\frac{1}{\sqrt[3]{3}-\sqrt[3]{5}}$$

be multiplied by in order to rationalize the denominator? Write this fraction with a rationalized denominator.

Concept Check *Simplify each expression mentally.*

109. $\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}}$ 110. $\sqrt{0.1}\cdot\sqrt{40}$ 111. $\sqrt[6]{2}\cdot\sqrt[6]{4}\cdot\sqrt[6]{8}$

Glossary

absolute value The absolute value of a real number is the distance between 0 and the number on the number line.

القيمة المطلقة القيمة المطلقة لأي عدد حقيقي هي المسافة بين الصفر والعدد على الخط العددي.

algebraic expression Any collection of numbers or variables joined by the basic operations of addition, subtraction, multiplication, or division (except by 0), or by the operations of raising to powers or taking roots, formed according to the rules of algebra, is an algebraic expression.

التعبير الجبري أي مجموعة أعداد أو متغيِّرات تعد تعبيرًا جبريا إذا تم ربطها عن طريق العمليات الأساسية المتمثلة في الإضافة والطرح والضرب والقسمة (ما عدا الصفر)، أو عن طريق عمليات رفع القوى الأسية أو كسر الجذور، وتكونت حسب قواعد الجبر.

base of an exponential The base is the number that is a repeated factor in exponential notation. In the expression a^n, a is the base.

قاعدة أس القاعدة هي العدد الذي يشكل عاملًا متكررًا في الترميز الأسّي. في التعبير a^n، تكون a هي القاعدة.

binomial A binomial is a polynomial containing exactly two terms.

ثنائي الحد أي ثنائي حد هو متعدد الحدود يحتوي على عنصرين اثنين بالتحديد.

coefficient (numerical coefficient) The real number factor in a term of an algebraic expression is the coefficient of the other factors.

المعامل (المعامل العددي) عامل العدد الحقيقي في أي عنصر من أي تعبير جبري هو المعامل الخاص بالعوامل الأخرى.

complement of a set The set of all elements in the universal set U that do not belong to set A is the complement of A, written A'.

تكملة مجموعة مجموعة جميع العناصر في المجموعة العامة U التي لا تنتمي إلى مجموعة A هي تكملة A، وتكتب A'.

complex fraction A complex fraction is a quotient of two rational expressions.

المركب أي كسر مركب هو حاصل تعبيرين جذريين.

conjugates The expressions $a - b$ and $a + b$ are conjugates.

المترافقات التعبيرات مثل $a - b$ و$a + b$ تعد مترافقات.

coordinate (on a number line) A number that corresponds to a particular point on a number line is the coordinate of the point.

الإحداث (على خط عددي) أي عدد يوازي نقطة محددة على خط عددي هو إحداث النقطة.

coordinate system (on a number line) The correspondence between points on a number line and the real numbers is a coordinate system.

النظام الإحداثي (على خط عددي) النظام الإحداثي هو التوازي بين النقاط على أي خط عددي والأعداد الحقيقية.

degree of a polynomial The greatest degree of any term in a polynomial is the degree of the polynomial.

درجة متعدد الحدود هي أكبر درجة لأي عنصر من عناصر أي متعدد حدود.

degree of a term The degree of a term is the sum of the exponents on the variable factors in the term.

درجة العنصر درجة العنصر هي مجموع الأسس في عوامل متغيرة في العنصر.

disjoint sets Two sets that have no elements in common are disjoint sets.

المجموعات المنفصلة أي مجموعتين لا توجد بينهما عناصر مشتركة تسمى مجموعات منفصلة.

domain of a rational expression The domain of a rational expression is the set of real numbers for which the expression is defined.

نطاق التعبير الجذري نطاق التعبير الجذري هو مجموعة من الأعداد الحقيقية التي يتم تعريف التعبير بها.

elements (members) The objects that belong to a set are the elements (members) of the set.

العناصر (الأعضاء) هي الكائنات التي تنتمي إلى مجموعة تشكل عناصر (أعضاء) المجموعة.

empty set (null set) The empty set or null set, written $\emptyset$ or $\{ \}$, is the set containing no elements.

مجموعة فارغة (مجموعة خالية) المجموعة الفارغة أو المجموعة الخالية، تكتب $\emptyset$ أو $\{ \}$، هي المجموعة التي لا تحتوي على أي عناصر.

exponent In the expression a^n, the exponent n indicates the number of times that the base a is used as a factor.

الأس في التعبير a^n، فإن الأس n يشير إلى أعداد المرات التي تستخدم فيها القاعدة a على أنها معامل.

factored completely A polynomial is factored completely when it is written as a product of prime polynomials.

مضروب بالكامل أي متعدد حدود يكون مضروبًا بالكامل حين يكتب على أنه ناتج لمتعددات حدود أولية.

factored form A polynomial is in factored form when it is written as a product of polynomials.

الصيغة المضروبة أي متعددة حدود تعد في صيغة مضروبة حين تُكتب على أنها ناتج لمتعددات الحدود.

factoring The process of finding polynomials whose product equals a given polynomial is called factoring.

التحليل يقصد بعملية إيجاد متعددات الحدود التي يساوي ناتجها متعدد حدود معين.

factoring by grouping Factoring by grouping is a method of grouping the terms of a polynomial in such a way that the polynomial can be factored even though the greatest common factor of its terms is 1.

التحليل بالتجميع التحليل بالتجميع هو طريقة لتجميع عناصر أي متعدد حدود بحيث عن طريقها يمكن ضرب متعدد الحدود حتى لو كان أكبر عامل مشترك لعناصره 1.

finite set A finite set is a set that has a limited number of elements.

المجموعة النهائية أي مجموعة نهائية هي أي مجموعة لديها أعداد محدودة من العناصر.

index of a radical In a radical of the form $\sqrt[n]{a}$, n is the index.

مؤشر العلامة الجذرية أي جذر في الصيغة $\sqrt[n]{a}$ تكون n هي المؤشر.

inequality An inequality says that one expression is greater than, greater than or equal to, less than, or less than or equal to, another.

المتباينة المتباينة هي الشيء الذي يشير إلى أن أي تعبير أكبر من الآخر، أكبر منه أو يساويه، أقل منه، أقل منه أو يساويه.

infinite set (This is an informal definition.) An infinite set is a set that has an unending list of distinct elements.

مجموعة لانهائية (هذا تعريف غير رسمي). أي مجموعة لانهائية هي أي مجموعة لديها قائمة بلا نهاية من العناصر المتباينة.

integers The set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

الأعداد الصحيحة مجموعة الأعداد الصحيحة هي $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

intersection The intersection of sets A and B, written $A \cap B$, is the set of elements that belong to both A and B.

التقاطع تقاطع المجموعتين A و B، تكتب $A \cap B$، هو مجموعة العناصر التي تنتمي لكل من A و B.

irrational numbers Real numbers that cannot be represented as quotients of integers are irrational numbers.

الأعداد الصماء الأعداد الحقيقية التي لا يمكن تمثيلها على أنها حاصل الأعداد الصحيحة للأعداد الصماء.

like radicals Radicals with the same radicand and the same index are like radicals.

الجذريات المتشابهة الجذريات المتشابهة هي الجذريات التي لديها نفس الجذر ونفس المؤشر.

like terms Terms with the same variables each raised to the same powers are like terms.

العناصر المتشابهة العناصر المتشابهة هي العناصر ذات نفس المتغيرات ويرفع كل عنصر إلى نفس الأسس.

lowest terms A rational expression is in lowest terms when the greatest common factor of its numerator and its denominator is 1.

العناصر الدنيا أي تعبير جذري يكون في صورة عناصر دنيا حين يكون المعامل المشترك الأكبر لصورة كسره ومقام كسره هو 1.

monomial A monomial is a polynomial containing exactly one term.

أحاديّ الحد أحاديّ الحد هو أي متعدد الحدود يحتوي على عنصر واحد بالتمام.

natural numbers (counting numbers) The natural numbers, or counting numbers, form the set of numbers $\{1, 2, 3, 4, \ldots\}$.

الأعداد الطبيعية (أعداد العدّ) الأعداد الطبيعية أو أعداد العدّ يكونون مجموعة الأعداد $\{1, 2, 3, 4, \ldots\}$.

polynomial A polynomial is a term or a finite sum of terms, with only positive or zero integer exponents present on the variables.

متعدد الحدود أي متعدد الحدود هو عنصر أو مجموع نهائي من العناصر، لا يوجد فيه إس عدد صحيح موجب أو صفري ظاهر على المتغيرات.

power (exponential expression, exponential) An expression of the form a^n is called a power, an exponential expression, or an exponential.

الأس (التعبير الأسيّ، الأسيّ) أي تعبير بصيغة a^n يطلق عليه أس، أو تعبير أسيّ.

prime polynomial A polynomial with variable terms that cannot be written as a product of two polynomials of lesser degree is a prime polynomial.

متعدد الحدود الأولي أي متعدد حدود بعناصر متغيرة لا يمكن كتابته على أنه ناتج لمتعدديّ حدود اثنين من درجة أقل هو متعدد حدود أولي.

principal *n*th root For even values of n (square roots, fourth roots, and so on), when a is positive, there are two real nth roots, one positive and one negative. In such cases, the notation $\sqrt[n]{a}$ represents the positive root, or principal nth root.

الجذر الرئيس بالنسبة للقيم الزوجية لـ n (الجذور التربيعية، الجذور الرباعية وما إلى غير ذلك) حين تكون a موجبة، يوجد جذران عدديان حقيقيان n أحدهما موجب والآخر سالب. في مثل هذه الحالات، يمثل الترميز $\sqrt[n]{a}$ جذرًا موجبًا أو الجذر الرئيسي العددي.

radicand The number or expression under a radical symbol is the radicand.

المجذور العدد أو التعبير تحت أي علامة جذرية هو المجذور.

rational expression The quotient of two polynomials P and Q, with $Q \neq 0$, is a rational expression.

التعبير الجذري ناتج أي متعدديّ حدود اثنين P و Q مع $Q \neq 0$ يمثل التعبير الجذري.

rationalizing the denominator Rationalizing a denominator is the process of writing a radical expression so that there are no radicals in the denominator.

حذف جذور مقام الكسر حذف مقام كسر هي عملية كتابة التعبير الجذري بحيث لا توجد جذور في مقام الكسر.

rational numbers The rational numbers are the set of numbers $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

الأعداد الجذرية الأعداد الجذرية هي مجموعة الأعداد $\frac{p}{q}$ حيث تكون p و q أعدادًا صحيحة و $q \neq 0$.

real numbers The set of all numbers that correspond to points on a number line is the real numbers.

الأعداد الحقيقية مجموعة الأعداد التي تتوافق مع نقاط على خط عددي هي الأعداد الحقيقية.

set A set is a collection of objects.

المجموعة هي مجموعة كائنات.

set-builder notation Set-builder notation uses the form $\{x \mid x$ has a certain property$\}$ to describe a set without having to list all of its elements.

تدوين بناء المجموعات يستخدم تدوين بناء المجموعات الصيغة $\{x \mid x$ حيث x لها خاصية محددة$\}$ لوصف مجموعة دون الحاجة إلى سرد كافة عناصر هذه المجموعة.

set operations The processes of finding the complement of a set, the intersection of two sets, and the union of two sets are set operations.

عمليات المجموعة يقصد بعمليات المجموعة عملية إيجاد متمِّم المجموعة، وتقاطع مجموعتين واتحادهما.

subset If every element of set A is also an element of set B, then A is a subset of B, written $A \subseteq B$.

المجموعة الجزئية إذا كان كل عنصر من المجموعة A موجودًا ضمن عناصر المجموعة B، إذن، تكون المجموعة A مجموعة جزئية من المجموعة B، وتكتب كما يلي $A \subseteq B$.

term The product of a real number and one or more variables raised to powers is a term.

العنصر حاصل ضرب عدد حقيقي في واحد أو أكثر من المتغيرات المرفوعة إلى قوة أسية.

trinomial A trinomial is a polynomial containing exactly three terms.

ثلاثي الحدود هو متعدد حدود يحتوي على ثلاثة عناصر.

union The union of sets A and B, written $A \cup B$, is the set of all elements that belong to set A or set B (or both).

الاتحاد اتحاد المجموعتين A و B وتكتب كما يلي $A \cup B$، هو مجموعة تتضمن كافة عناصر المجموعة A أو المجموعة B (أو كليهما).

universal set The universal set, written U, contains all the elements under discussion in a particular situation.

المجموعة العامة تكتب كما يلي U تتضمن كافة العناصر التي محل دراسة في موضع معيّن.

Venn diagram A Venn diagram is a diagram used to illustrate relationships among sets or probability concepts.

شكل فن هو شكل يستخدم لتوضيح العلاقة بين المجموعات أو المفاهيم الاحتمالية.

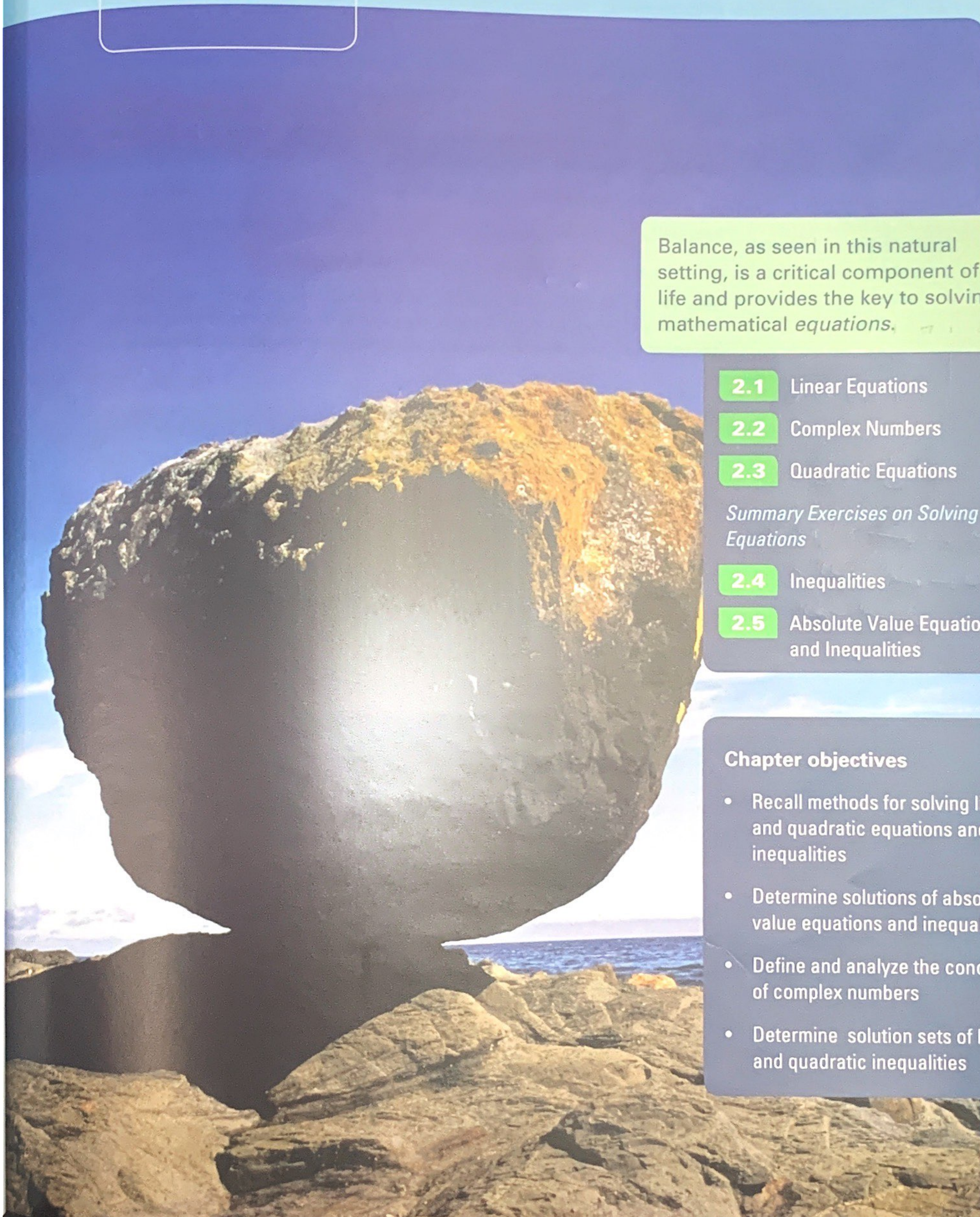
whole numbers The set of whole numbers $\{0, 1, 2, 3, 4, \ldots\}$ is the union of the set of natural numbers and $\{0\}$.

الأعداد الصحيحة مجموعة الأعداد الصحية $\{0, 1, 2, 3, 4, \ldots\}$ هي اتحاد مجموعة الأعداد الطبيعية و $\{0\}$.

zero-factor property The zero-factor property states that if the product of two (or more) complex numbers is 0, then at least one of the numbers must be 0.

خاصية العامل الصفري تنص خاصية العامل الصفري على أنه إذا كان حاصل ضرب عددين مركبين (أو أكثر) هو صفر، إذن يجب أن يكون أقل هذه الأعداد صفرًا.

2 Equations and Inequalities



Balance, as seen in this natural setting, is a critical component of life and provides the key to solving mathematical *equations*.

Chapter objectives

- Recall methods for solving linear and quadratic equations and inequalities
- Determine solutions of absolute value equations and inequalities
- Define and analyze the concept of complex numbers
- Determine solution sets of linear and quadratic inequalities

2.1 Linear Equations → معادلات خطية

- Basic Terminology of Equations
- Solving Linear Equations
- Identities, Conditional Equations, and Contradictions
- Solving for a Specified Variable (Literal Equations)

$a=b$ → $a+c=b+c$
→ $a-c=b-c$
→ $a.c=b.c$
→ $\frac{a}{c}=\frac{b}{c}$

* يعني أي شي تسويه في الطرف اليمين تسويه في الطرف اليسار.

Basic Terminology of Equations An **equation** is a statement that two expressions are equal.

$$x+2=9, \quad 11x=5x+6x, \quad x^2-2x-1=0 \quad \text{Equations}$$

To *solve* an equation means to find all numbers that make the equation a true statement. These numbers are the **solutions,** or **roots,** of the equation. A number that is a solution of an equation is said to *satisfy* the equation, and the solutions of an equation make up its **solution set.** Equations with the same solution set are **equivalent equations.** For example,

$$x=4, \quad x+1=5, \quad \text{and} \quad 6x+3=27 \quad \text{are equivalent equations}$$

because they have the same solution set, $\{4\}$. However, the equations

$$x^2=9 \quad \text{and} \quad x=3 \quad \text{are } not \text{ equivalent,}$$

$x = \pm 3$

since the first has solution set $\{-3, 3\}$ while the solution set of the second is $\{3\}$.

One way to solve an equation is to rewrite it as a series of simpler equivalent equations using the **addition and multiplication properties of equality.**

Addition and Multiplication Properties of Equality

Let a, b, and c represent real numbers.

If $a=b$, then $a+c=b+c$.

That is, the same number may be added to each side of an equation without changing the solution set.

If $a=b$ and $c \neq 0$, then $ac=bc$.

That is, each side of an equation may be multiplied by the same nonzero number without changing the solution set. (Multiplying each side by zero leads to $0=0$.)

These properties can be extended: The same number may be subtracted from each side of an equation, and each side may be divided by the same nonzero number, without changing the solution set.

Solving Linear Equations We use the properties of equality to solve *linear equations.*

Linear Equation in One Variable

A **linear equation in one variable** is an equation that can be written in the form

$$ax+b=0,$$

where a and b are real numbers with $a \neq 0$.

A linear equation is a **first-degree equation** since the greatest degree of the variable is 1.

$$3x + \sqrt{2} = 0, \quad \frac{3}{4}x = 12, \quad 0.5(x + 3) = 2x - 6 \qquad \text{Linear equations}$$

$$\sqrt{x} + 2 = 5, \quad \frac{1}{x} = -8, \quad x^2 + 3x + 0.2 = 0 \qquad \text{Nonlinear equations}$$

EXAMPLE 1 **Solving a Linear Equation**

Solve $3(2x - 4) = 7 - (x + 5)$.

SOLUTION

$$3(2x - 4) = 7 - (x + 5) \qquad \text{Be careful with signs.}$$
$$6x - 12 = 7 - x - 5 \qquad \text{Distributive property (Section 1.2)}$$
$$6x - 12 = 2 - x \qquad \text{Combine like terms. (Section 1.3)}$$
$$6x - 12 + x = 2 - x + x \qquad \text{Add } x \text{ to each side.}$$
$$7x - 12 = 2 \qquad \text{Combine like terms.}$$
$$7x - 12 + 12 = 2 + 12 \qquad \text{Add 12 to each side.}$$
$$7x = 14 \qquad \text{Combine like terms.}$$
$$\frac{7x}{7} = \frac{14}{7} \qquad \text{Divide each side by 7.}$$
$$x = 2$$

Since replacing x with 2 results in a true statement, 2 is a solution of the given equation. The solution set is $\{2\}$.

Very important ←

HOMEWORK 1 **Solving a Linear Equation with Fractions**

Solve $\frac{2x + 4}{3} + \frac{1}{2}x = \frac{1}{4}x - \frac{7}{3}$.

إذا كان المطلوب حل معادله تحتوي على مقامات رقمه فنقوم بضرب طرفي المعادله في حاصل ضرب هذه المقامات الرقمية.

Identities, Conditional Equations, and Contradictions An equation satisfied by every number that is a meaningful replacement for the variable is an **identity**.

$$3(x + 1) = 3x + 3 \qquad \text{Identity}$$

→ All real number = مجموعة الحل

An equation that is satisfied by some numbers but not others is a **conditional equation.**

$$2x = 4 \qquad \text{Conditional equation}$$

The equations in **Example 1 and Homework 1** are conditional equations. An equation that has no solution is a **contradiction.**

$$x = x + 1 \qquad \text{Contradiction}$$

→ $\emptyset$ = مجموعة الحل

Homework 1 :

$$\frac{2x+4}{3} + \frac{1}{2}x = \frac{1}{4}x - \frac{7}{3} \quad \cdot 3.2.4$$

$$= \frac{2.\cancel{3}.4(2x+4)}{\cancel{3}} + \frac{\cancel{2}.3.4}{\cancel{2}}x = \frac{2.3.\cancel{4}}{\cancel{4}}x - \frac{2.\cancel{3}.4.7}{\cancel{3}}$$

$$= 8(2x+4) + 12x = 6x - 56$$

$$= 16x + 32 + 12x = 6x - 56$$

$$= 28x + 32 = 6x - 56$$

$$= 22x = -88$$

$$x = -4$$

$\{-4\}$.

Very important

EXAMPLE 2 Identifying Types of Equations

Determine whether each equation is an *identity*, a *conditional equation*, or a *contradiction*. Give the solution set.

(a) $-2(x+4)+3x=x-8$ **(b)** $5x-4=11$ **(c)** $3(3x-1)=9x+7$

SOLUTION

(a) $-2(x+4)+3x=x-8$

$-2x-8+3x=x-8$ Distributive property

$x-8=x-8$ Combine like terms.

$0=0$ Subtract x. Add 8.

When a *true* statement such as $0=0$ results, the equation is an identity, and the solution set is **{all real numbers}.**

(b) $5x-4=11$

$5x=15$ Add 4 to each side.

$x=3$ Divide each side by 5.

This is a conditional equation, and its solution set is $\{3\}$.

(c) $3(3x-1)=9x+7$

$9x-3=9x+7$ Distributive property

$-3=7$ Subtract $9x$.

When a *false* statement such as $-3=7$ results, the equation is a contradiction, and the solution set is the **empty set,** or **null set,** symbolized **$\emptyset$.**

Identifying Types of Linear Equations → Read it

1. If solving a linear equation leads to a true statement such as $0=0$, the equation is an **identity.** Its solution set is **{all real numbers}. (See Example 2(a).)**
2. If solving a linear equation leads to a single solution such as $x=3$, the equation is **conditional.** Its solution set consists of a single element. **(See Example 2(b).)**
3. If solving a linear equation leads to a false statement such as $-3=7$, the equation is a **contradiction.** Its solution set is **$\emptyset$. (See Example 2(c).)**

الحل بالنسبة لمتغير واحد.

Solving for a Specified Variable (Literal Equations) A formula is an example of a **literal equation** (an equation involving letters).

Very important

HOMEWORK 2 Solving for a Specified Variable

Solve for the specified variable.

(a) $I=Prt,$ for t $= t = \frac{I}{Pr}$

(b) $A-P=Prt,$ for P

$A = P + Prt$

(c) $3(2x-5a)+4b=4x-2,$ for x

$6x-15a+4b=4x-2$

$2x=15a-4b-2$

$x=\frac{15}{a}-2b-1$

EXAMPLE 3 **Applying the Simple Interest Formula**

Atif borrowed \$5240 for new furniture. He will pay it off in 11 months at an annual simple interest rate of 4.5%. How much interest will he pay?

SOLUTION Use the simple interest formula $I = Prt$.

$$I = Prt = 5240(0.045)\left(\frac{11}{12}\right) = \$216.15$$

$P = 5240$, $r = 0.045$, and $t = \frac{11}{12}$ (year)

He will pay \$216.15 interest on his purchase.

2.1 Exercises

Concept Check In Exercises 1–4, decide whether each statement is true *or* false.

1. The solution set of $2x + 5 = x - 3$ is $\{-8\}$.
2. The equation $5(x - 8) = 5x - 40$ is an example of an identity.
3. The equations $x^2 = 4$ and $x + 2 = 4$ are equivalent equations.
4. It is possible for a linear equation to have exactly two solutions.
5. Explain the difference between an identity and a conditional equation.
6. Make a complete list of the steps needed to solve a linear equation. (Some equations will not require every step.)
7. *Concept Check* Which one is not a linear equation?

   A. $5x + 7(x - 1) = -3x$ B. $9x^2 - 4x + 3 = 0$

   C. $7x + 8x = 13x$ D. $0.04x - 0.08x = 0.40$
8. In solving the equation $3(2x - 8) = 6x - 24$, a student obtains the result $0 = 0$ and gives the solution set $\{0\}$. Is this correct? Explain.

Solve each equation. ***See Example 1 and Homework 1.***

9. $5x + 4 = 3x - 4$
10. $9x + 11 = 7x + 1$
11. $6(3x - 1) = 8 - (10x - 14)$
12. $4(-2x + 1) = 6 - (2x - 4)$
13. $\frac{5}{6}x - 2x + \frac{4}{3} = \frac{5}{3}$
14. $\frac{7}{4} + \frac{1}{5}x - \frac{3}{2} = \frac{4}{5}x$
15. $3x + 5 - 5(x + 1) = 6x + 7$
16. $5(x + 3) + 4x - 3 = -(2x - 4) + 2$
17. $2[x - (4 + 2x) + 3] = 2x + 2$
18. $4[2x - (3 - x) + 5] = -6x - 28$
19. $\frac{1}{14}(3x - 2) = \frac{x + 10}{10}$
20. $\frac{1}{15}(2x + 5) = \frac{x + 2}{9}$
21. $0.2x - 0.5 = 0.1x + 7$
22. $0.01x + 3.1 = 2.03x - 2.96$
23. $-4(2x - 6) + 8x = 5x + 24 + x$
24. $-8(3x + 4) + 6x = 4(x - 8) + 4x$
25. $0.5x + \frac{4}{3}x = x + 10$
26. $\frac{2}{3}x + 0.25x = x + 2$
27. $0.08x + 0.06(x + 12) = 7.72$
28. $0.04(x - 12) + 0.06x = 1.52$

2.2 Complex Numbers

Basic Concepts of Complex Numbers The set of real numbers does not include all the numbers needed in algebra. For example, there is no real number solution of the equation

$$x^2 = -1,$$

since no real number, when squared, gives -1. To extend the real number system to include solutions of equations of this type, the number i is defined to have the following property.

The Imaginary Unit i

$$i = \sqrt{-1}, \quad \text{and therefore,} \quad i^2 = -1.$$

(Note that $-i$ is also a square root of -1.)

Square roots of negative numbers were not incorporated into an integrated number system until the 16th century. They were then used as solutions of equations and later (in the 18th century) in surveying. Today, such numbers are used extensively in science and engineering.

Complex numbers are formed by adding real numbers and multiples of i.

Complex Number

If a and b are real numbers, then any number of the form $a + bi$ is a **complex number.** In the complex number $a + bi$, a is the **real part** and b is the **imaginary part.***

Two complex numbers $a + bi$ and $c + di$ are equal provided that their real parts are equal and their imaginary parts are equal; that is, they are equal if and only if $a = c$ and $b = d$.

For a complex number $a + bi$, if $b = 0$, then $a + bi = a$, which is a real number. Thus, the set of real numbers is a subset of the set of complex numbers. If $a = 0$ and $b \neq 0$, the complex number is said to be a **pure imaginary number.** For example, $3i$ is a pure imaginary number. A pure imaginary number, or a number such as $7 + 2i$ with $a \neq 0$ and $b \neq 0$, is a **nonreal complex number.** A complex number written in the form $a + bi$ (or $a + ib$) is in **standard form.** (The form $a + ib$ is used to write expressions such as $i\sqrt{5}$, since $\sqrt{5}i$ could be mistaken for $\sqrt{5i}$.)

The relationships among the subsets of the complex numbers are shown in **Figure 1.**

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* In some texts, the term bi is defined to be the imaginary part.

Complex Numbers $a + bi$, for a and b Real

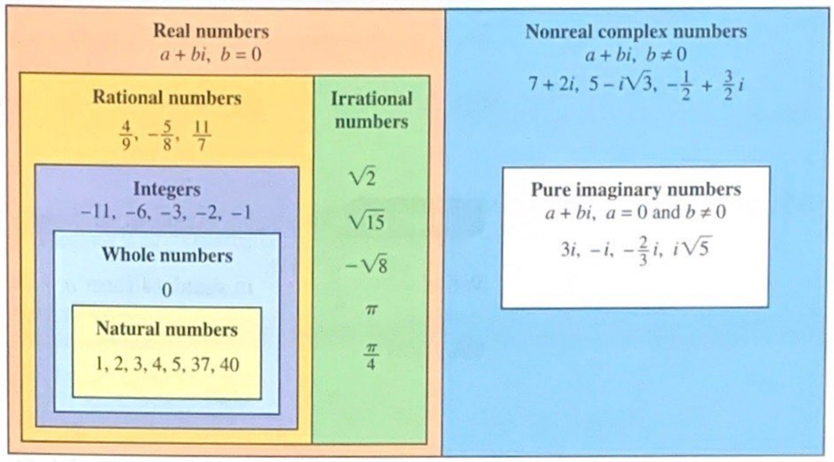


Figure 1

For a positive real number a, the expression $\sqrt{-a}$ is defined as follows.

The Expression $\sqrt{-a}$

If $a > 0$, then $\quad \sqrt{-a} = i\sqrt{a}.$

EXAMPLE 1 Writing $\sqrt{-a}$ as $i\sqrt{a}$

Write as the product of a real number and i, using the definition of $\sqrt{-a}$.

(a) $\sqrt{-16}$ **(b)** $\sqrt{-70}$ **(c)** $\sqrt{-48}$

SOLUTION

(a) $\sqrt{-16} = i\sqrt{16} = 4i$ **(b)** $\sqrt{-70} = i\sqrt{70}$

(c) $\sqrt{-48} = i\sqrt{48} = i\sqrt{16 \cdot 3} = 4i\sqrt{3}$ Product rule for radicals (Section 1.6)

Operations on Complex Numbers Products or quotients with negative radicands are simplified by first rewriting $\sqrt{-a}$ as $i\sqrt{a}$ for a positive number a. Then the properties of real numbers and the fact that $i^2 = -1$ are applied.

CAUTION ***When working with negative radicands, use the definition $\sqrt{-a} = i\sqrt{a}$ before using any of the other rules for radicals.*** In particular, the rule $\sqrt{c} \cdot \sqrt{d} = \sqrt{cd}$ is valid only when c and d are *not* both negative. For example,

$$\sqrt{-4} \cdot \sqrt{-9} = 2i \cdot 3i = 6i^2 = -6 \quad \text{is correct,}$$

while $$\sqrt{-4} \cdot \sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6 \quad \text{is incorrect.}$$

HOMEWORK 1 Finding Products and Quotients Involving $\sqrt{-a}$

Multiply or divide, as indicated. Simplify each answer.

(a) $\sqrt{-7} \cdot \sqrt{-7}$ **(b)** $\sqrt{-6} \cdot \sqrt{-10}$ **(c)** $\dfrac{\sqrt{-20}}{\sqrt{-2}}$ **(d)** $\dfrac{\sqrt{-48}}{\sqrt{24}}$

EXAMPLE 2 Simplifying a Quotient Involving $\sqrt{-a}$

Write $\dfrac{-8+\sqrt{-128}}{4}$ in standard form $a + bi$.

SOLUTION

$$\frac{-8+\sqrt{-128}}{4} = \frac{-8+\sqrt{-64 \cdot 2}}{4}$$

$$= \frac{-8+8i\sqrt{2}}{4} \qquad \sqrt{-64} = 8i$$

Be sure to factor before simplifying.

$$= \frac{4(-2+2i\sqrt{2})}{4} \qquad \text{Factor. (Section 1.4)}$$

$$= -2+2i\sqrt{2} \qquad \text{Lowest terms (Section 1.5)}$$

With the definitions $i^2 = -1$ and $\sqrt{-a} = i\sqrt{a}$ for $a > 0$, all properties of real numbers are extended to complex numbers. As a result, complex numbers are added, subtracted, multiplied, and divided using real number properties and the definitions on the following pages.

Addition and Subtraction of Complex Numbers

For complex numbers $a + bi$ and $c + di$,

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

and

$$(a+bi) - (c+di) = (a-c) + (b-d)i.$$

That is, to add or subtract complex numbers, add or subtract the real parts and add or subtract the imaginary parts.

HOMEWORK 2 Adding and Subtracting Complex Numbers

Find each sum or difference.

(a) $(3-4i) + (-2+6i)$ **(b)** $(-4+3i) - (6-7i)$

The product of two complex numbers is found by multiplying as though the numbers were binomials and using the fact that $i^2 = -1$, as follows.

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bic + bidi && \text{FOIL (Section 1.3)}\\ &= ac + adi + bci + bdi^2 && \text{Associative property}\\ &= ac + (ad+bc)i + bd(-1) && \text{Distributive property; } i^2 = -1\\ &= (ac-bd) + (ad+bc)i && \text{Group like terms.}\end{aligned}$$

Multiplication of Complex Numbers

For complex numbers $a + bi$ and $c + di$,

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

This definition is not practical in routine calculations. To find a given product, it is easier just to multiply as with binomials.

EXAMPLE 3 Multiplying Complex Numbers

Find each product.

(a) $(2 - 3i)(3 + 4i)$ **(b)** $(4 + 3i)^2$ **(c)** $(6 + 5i)(6 - 5i)$

SOLUTION

(a) $(2 - 3i)(3 + 4i) = 2(3) + 2(4i) - 3i(3) - 3i(4i)$ FOIL

$= 6 + 8i - 9i - 12i^2$ Multiply.

$= 6 - i - 12(-1)$ Combine like terms; $i^2 = -1$

$= 18 - i$ Standard form

(b) $(4 + 3i)^2 = 4^2 + 2(4)(3i) + (3i)^2$ Square of a binomial (Section 1.3)

Remember to add twice the product of the two terms.

$= 16 + 24i + 9i^2$ Multiply.

$= 16 + 24i + 9(-1)$ $i^2 = -1$

$= 7 + 24i$ Standard form

(c) $(6 + 5i)(6 - 5i) = 6^2 - (5i)^2$ Product of the sum and difference of two terms (Section 1.3)

$= 36 - 25(-1)$ Square 6 and 5; $i^2 = -1$.

$= 36 + 25$ Multiply.

$= 61$, or $61 + 0i$ Standard form

Example 3(c) showed that $(6 + 5i)(6 - 5i) = 61$. The numbers $6 + 5i$ and $6 - 5i$ differ only in the sign of their imaginary parts and are called **complex conjugates.** ***The product of a complex number and its conjugate is always a real number.*** This product is the sum of the squares of the real and imaginary parts.

Property of Complex Conjugates

For real numbers a and b,

$$(a + bi)(a - bi) = a^2 + b^2.$$

To find the quotient of two complex numbers in standard form, we multiply both the numerator and the denominator by the complex conjugate of the denominator.

HOMEWORK 3 Dividing Complex Numbers

Write each quotient in standard form $a + bi$.

(a) $\dfrac{3 + 2i}{5 - i}$ **(b)** $\dfrac{3}{i}$

Powers of i can be simplified using the facts

$$i^2 = -1 \quad \text{and} \quad i^4 = (i^2)^2 = (-1)^2 = 1.$$

Consider the following powers of i.

$$i^1 = i \qquad i^5 = i^4 \cdot i = 1 \cdot i = i$$
$$i^2 = -1 \qquad i^6 = i^4 \cdot i^2 = 1(-1) = -1$$
$$i^3 = i^2 \cdot i = (-1) \cdot i = -i \qquad i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \qquad i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1 \quad \text{and so on.}$$

Powers of i cycle through the same four outcomes (i, -1, $-i$, and 1) since i^4 has the same multiplicative property as 1. Also, any power of i with an exponent that is a multiple of 4 has value 1. As with real numbers, $i^0 = 1$.

EXAMPLE 4 Simplifying Powers of *i*

Simplify each power of i.

(a) i^{15} **(b)** i^{-3}

SOLUTION

(a) Since $i^4 = 1$, write the given power as a product involving i^4.

$$i^{15} = i^{12} \cdot i^3 = (i^4)^3 \cdot i^3 = 1^3(-i) = -i$$

(b) Multiply i^{-3} by 1 in the form of i^4 to create the least positive exponent for i.

$$i^{-3} = i^{-3} \cdot 1 = i^{-3} \cdot i^4 = i \qquad i^4 = 1$$

2.2 Exercises

Concept Check *Determine whether each statement is* true *or* false. *If it is false, tell why.*

1. Every real number is a complex number.
2. No real number is a pure imaginary number.
3. Every pure imaginary number is a complex number.
4. A number can be both real and complex.
5. There is no real number that is a complex number.
6. A complex number might not be a pure imaginary number.

Identify each number as real, complex, pure imaginary, *or* nonreal complex. *(More than one of these descriptions will apply.)*

7. -4 **8.** $13i$ **9.** $5 + i$ **10.** π **11.** $\sqrt{-25}$

Write each number as the product of a real number and i. **See Example 1.**

12. $\sqrt{-25}$ **13.** $\sqrt{-10}$ **14.** $\sqrt{-288}$ **15.** $-\sqrt{-18}$

Multiply or divide, as indicated. Simplify each answer. **See Homework 1.**

16. $\sqrt{-13} \cdot \sqrt{-13}$
17. $\sqrt{-3} \cdot \sqrt{-8}$
18. $\dfrac{\sqrt{-30}}{\sqrt{-10}}$
19. $\dfrac{\sqrt{-24}}{\sqrt{8}}$
20. $\dfrac{\sqrt{-10}}{\sqrt{-40}}$
21. $\dfrac{\sqrt{-6} \cdot \sqrt{-2}}{\sqrt{3}}$

Write each number in standard form $a + bi$. **See Example 2.**

22. $\dfrac{-6 - \sqrt{-24}}{2}$
23. $\dfrac{10 + \sqrt{-200}}{5}$
24. $\dfrac{-3 + \sqrt{-18}}{24}$

Find each sum or difference. Write the answer in standard form. **See Homework 2.**

25. $(3 + 2i) + (9 - 3i)$
26. $(-2 + 4i) - (-4 + 4i)$
27. $(2 - 5i) - (3 + 4i) - (-2 + i)$
28. $-i\sqrt{2} - 2 - \left(6 - 4i\sqrt{2}\right) - \left(5 - i\sqrt{2}\right)$

Find each product. Write the answer in standard form. **See Example 3.**

29. $(2 + i)(3 - 2i)$
30. $(2 + 4i)(-1 + 3i)$
31. $(3 - 2i)^2$
32. $(3 + i)(3 - i)$
33. $(-2 - 3i)(-2 + 3i)$
34. $\left(\sqrt{6} + i\right)\left(\sqrt{6} - i\right)$
35. $i(3 - 4i)(3 + 4i)$
36. $3i(2 - i)^2$
37. $(2 + i)(2 - i)(4 + 3i)$

Find each quotient. Write the answer in standard form $a + bi$. **See Homework 3.**

38. $\dfrac{6 + 2i}{1 + 2i}$
39. $\dfrac{2 - i}{2 + i}$
40. $\dfrac{1 - 3i}{1 + i}$
41. $\dfrac{-5}{i}$
42. $\dfrac{8}{-i}$
43. $\dfrac{2}{3i}$

(Modeling) Alternating Current *Complex numbers are used to describe current I, voltage E, and impedance Z (the opposition to current). These three quantities are related by the equation*

$$E = IZ, \quad \text{which is known as } \textbf{Ohm's Law}.$$

Thus, if any two of these quantities are known, the third can be found. In each exercise, solve the equation $E = IZ$ for the remaining value.

44. $I = 5 + 7i, Z = 6 + 4i$
45. $I = 10 + 4i, E = 88 + 128i$

Simplify each power of i. **See Example 4.**

46. i^{25}
47. i^{22}
48. i^{23}
49. i^{32}
50. i^{-13}
51. $\dfrac{1}{i^{-11}}$

52. Suppose that your friend tells you that she has discovered a method of simplifying a positive power of i. "Just divide the exponent by 2. Your answer is then the simplified form of i^2 raised to the quotient times i raised to the remainder." Explain why her method works.

53. Show that $\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$ is a square root of i.

54. Show that $-2 + i$ is a solution of the equation $x^2 + 4x + 5 = 0$.

2.3 Quadratic Equations

A *quadratic equation* is defined as follows.

Quadratic Equation in One Variable

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers with $a \neq 0$, is a **quadratic equation.** The given form is called **standard form.**

A quadratic equation is a **second-degree equation**—that is, an equation with a squared variable term and no terms of greater degree.

$$x^2 = 25, \quad 4x^2 + 4x - 5 = 0, \quad 3x^2 = 4x - 8 \qquad \text{Quadratic equations}$$

Solving a Quadratic Equation The factoring method of solving a quadratic equation depends on the **zero-factor property.**

Zero-Factor Property

If a and b are complex numbers with $ab = 0$, then $a = 0$ or $b = 0$ or both equal zero.

EXAMPLE 1 Using the Zero-Factor Property

Solve $6x^2 + 7x = 3$.

SOLUTION

Don't factor out x here.

$$6x^2 + 7x = 3$$

$$6x^2 + 7x - 3 = 0 \qquad \text{Standard form}$$

$$(3x - 1)(2x + 3) = 0 \qquad \text{Factor. (Section 1.4)}$$

$$3x - 1 = 0 \quad \text{or} \quad 2x + 3 = 0 \qquad \text{Zero-factor property}$$

$$3x = 1 \quad \text{or} \quad 2x = -3 \qquad \text{Solve each equation. (Section 2.1)}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -\frac{3}{2}$$

CHECK $6x^2 + 7x = 3$ Original equation

$$6\left(\frac{1}{3}\right)^2 + 7\left(\frac{1}{3}\right) \stackrel{?}{=} 3 \quad \text{Let } x = \tfrac{1}{3}. \qquad 6\left(-\frac{3}{2}\right)^2 + 7\left(-\frac{3}{2}\right) \stackrel{?}{=} 3 \quad \text{Let } x = -\tfrac{3}{2}.$$

$$\frac{6}{9} + \frac{7}{3} \stackrel{?}{=} 3 \qquad \frac{54}{4} - \frac{21}{2} \stackrel{?}{=} 3$$

$$3 = 3 \checkmark \text{ True} \qquad 3 = 3 \checkmark \text{ True}$$

Both values check, since true statements result. The solution set is $\left\{\frac{1}{3}, -\frac{3}{2}\right\}$.

A quadratic equation of the form $x^2 = k$ can also be solved by factoring.

$$x^2 = k$$

$$x^2 - k = 0 \quad \text{Subtract } k.$$

$$(x - \sqrt{k})(x + \sqrt{k}) = 0 \quad \text{Factor.}$$

$$x - \sqrt{k} = 0 \quad \text{or} \quad x + \sqrt{k} = 0 \quad \text{Zero-factor property}$$

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{Solve each equation.}$$

This proves the **square root property.**

Square Root Property

If $x^2 = k$, then $x = \sqrt{k}$ **or** $x = -\sqrt{k}$.

That is, the solution set of $x^2 = k$ is

$$\left\{\sqrt{k}, -\sqrt{k}\right\}, \quad \textit{which may be abbreviated} \quad \left\{\pm\sqrt{k}\right\}.$$

Both solutions $\sqrt{k}$ and $-\sqrt{k}$ are real if $k > 0$, and both are pure imaginary if $k < 0$. If $k < 0$, then we write the solution set as

$$\left\{\pm i\sqrt{|k|}\right\}.$$

If $k = 0$, then there is only one distinct solution, 0, sometimes called a **double solution.**

HOMEWORK 1 **Using the Square Root Property**

Solve each quadratic equation.

(a) $x^2 = 17$ **(b)** $x^2 = -25$ **(c)** $(x - 4)^2 = 12$

Completing the Square Any quadratic equation can be solved by the method of **completing the square,** summarized as follows.

Solving a Quadratic Equation by Completing the Square

To solve $ax^2 + bx + c = 0$, where $a \neq 0$, by completing the square, use these steps.

Step 1 If $a \neq 1$, divide both sides of the equation by a.

Step 2 Rewrite the equation so that the constant term is alone on one side of the equality symbol.

Step 3 Square half the coefficient of x, and add this square to each side of the equation.

Step 4 Factor the resulting trinomial as a perfect square and combine like terms on the other side.

Step 5 Use the square root property to complete the solution.

EXAMPLE 2 **Using Completing the Square ($a = 1$)**

Solve $x^2 - 4x - 14 = 0$.

SOLUTION $x^2 - 4x - 14 = 0$

Step 1 This step is not necessary since $a = 1$.

Step 2	$x^2 - 4x = 14$	Add 14 to each side.
Step 3	$x^2 - 4x + 4 = 14 + 4$	$[\frac{1}{2}(-4)]^2 = 4$; Add 4 to each side.
Step 4	$(x - 2)^2 = 18$	Factor. (Section 1.4) Combine like terms.
Step 5	$x - 2 = \pm\sqrt{18}$	Square root property
Take *both* roots.	$x = 2 \pm \sqrt{18}$	Add 2 to each side.
	$x = 2 \pm 3\sqrt{2}$	Simplify the radical.

The solution set is $\{2 \pm 3\sqrt{2}\}$.

HOMEWORK 2 **Using Completing the Square ($a \neq 1$)**

Solve $9x^2 - 12x + 9 = 0$.

The Quadratic Formula If we start with the equation, $ax^2 + bx + c = 0$, for $a > 0$, and complete the square to solve for x in terms of the constants a, b, and c, the result is a general formula for solving any quadratic equation.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{Divide each side by } a. \text{ (Step 1)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Subtract } \tfrac{c}{a} \text{ from each side. (Step 2)}$$

Square half the coefficient of x: $\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{Add } \tfrac{b^2}{4a^2} \text{ to each side. (Step 3)}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}+\frac{-c}{a}$$ Factor. Use the commutative property. (Step 4)

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}+\frac{-4ac}{4a^2}$$ Write fractions with a common denominator. (Section 1.5)

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$ Add fractions. (Section 1.5)

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$ Square root property (Step 5)

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$ Since $a>0$, $\sqrt{4a^2}=2a$. (Section 1.6)

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$ Subtract $\frac{b}{2a}$ from each side.

Quadratic Formula This result is also true for $a<0$. $\longrightarrow$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ Combine terms on the right.

Quadratic Formula

The solutions of the quadratic equation $ax^2+bx+c=0$, where $a\neq 0$, are given by the quadratic formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

EXAMPLE 3 Using the Quadratic Formula (Real Solutions)

Solve $x^2-4x=-2$.

SOLUTION

$$x^2-4x+2=0$$ Write in standard form. Here $a=1$, $b=-4$, and $c=2$.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ Quadratic formula

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2(1)}$$ Substitute $a=1$, $b=-4$, and $c=2$.

The fraction bar extends *under* $-b$.

$$x=\frac{4\pm\sqrt{16-8}}{2}$$ Simplify.

$$x=\frac{4\pm 2\sqrt{2}}{2}$$ $\sqrt{16-8}=\sqrt{8}=\sqrt{4\cdot 2}=2\sqrt{2}$ (Section 1.6)

$$x=\frac{2\left(2\pm\sqrt{2}\right)}{2}$$ Factor out 2 in the numerator. (Section 1.5)

Factor first, then divide.

$$x=2\pm\sqrt{2}$$ Lowest terms

The solution set is $\left\{2\pm\sqrt{2}\right\}$.

CAUTION ***Remember to extend the fraction bar in the quadratic formula under the $-b$ term in the numerator.***

Throughout this text, unless otherwise specified, we use the set of complex numbers as the domain when solving equations of degree 2 or greater.

HOMEWORK 3 **Using the Quadratic Formula (Nonreal Complex Solutions)**

Solve $2x^2 = x - 4$.

The equation $x^3 + 8 = 0$ is a **cubic equation** because the greatest degree of the terms is 3.

EXAMPLE 4 **Solving a Cubic Equation**

Solve $x^3 + 8 = 0$ using factoring and the quadratic formula.

SOLUTION $x^3 + 8 = 0$

$(x + 2)(x^2 - 2x + 4) = 0$ Factor as a sum of cubes. (Section 1.4)

$x + 2 = 0$ or $x^2 - 2x + 4 = 0$ Zero-factor property

$$x = -2 \quad \text{or} \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$ Quadratic formula with $a = 1, b = -2, c = 4$

$$x = \frac{2 \pm \sqrt{-12}}{2}$$ Simplify.

$$x = \frac{2 \pm 2i\sqrt{3}}{2}$$ Simplify the radical.

$$x = \frac{2\left(1 \pm i\sqrt{3}\right)}{2}$$ Factor out 2 in the numerator.

$$x = 1 \pm i\sqrt{3}$$ Lowest terms

The solution set is $\left\{-2, 1 \pm i\sqrt{3}\right\}$.

Solving for a Specified Variable To solve a quadratic equation for a specified variable, we usually apply the square root property or the quadratic formula.

HOMEWORK 4 **Solving for a Quadratic Variable in a Formula**

Solve for the specified variable. Use $\pm$ when taking square roots.

(a) $\mathscr{A} = \dfrac{\pi d^2}{4}$, for d **(b)** $rt^2 - st = k \ (r \neq 0)$, for t

NOTE In **Homework 4,** we took both positive and negative square roots. However, if the variable represents time or length in an application, we consider only the *positive* square root.

The Discriminant The quantity under the radical in the quadratic formula, $b^2 - 4ac$, is called the **discriminant**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \leftarrow \text{Discriminant}$$

When the numbers a, b, and c are *integers* (but not necessarily otherwise), the value of the discriminant can be used to determine whether the solutions of a quadratic equation are rational, irrational, or nonreal complex numbers. The number and type of solutions based on the value of the discriminant are shown in the following table.

Discriminant	Number of Solutions	Type of Solutions
Positive, perfect square	Two	Rational
Positive, but not a perfect square	Two	Irrational ← As seen in **Example 3**
Zero	One (a double solution)	Rational
Negative	Two	Nonreal complex ← As seen in **Homework 3**

CAUTION ***The restriction on a, b, and c is important.*** For example,

$$x^2 - \sqrt{5}x - 1 = 0 \quad \text{has discriminant} \quad b^2 - 4ac = 5 + 4 = 9,$$

which would indicate two rational solutions *if the coefficients were integers.* By the quadratic formula, the two solutions $\frac{\sqrt{5} \pm 3}{2}$ are *irrational* numbers.

EXAMPLE 5 Using the Discriminant

Determine the number of distinct solutions, and tell whether they are *rational, irrational,* or *nonreal complex* numbers.

(a) $5x^2 + 2x - 4 = 0$ **(b)** $x^2 - 10x = -25$ **(c)** $2x^2 - x + 1 = 0$

SOLUTION

(a) For $5x^2 + 2x - 4 = 0$, use $a = 5$, $b = 2$, and $c = -4$.

$$b^2 - 4ac = 2^2 - 4(5)(-4) = 84 \leftarrow \text{Discriminant}$$

The discriminant 84 is positive and not a perfect square, so there are two distinct irrational solutions.

(b) First, write the equation in standard form as $x^2 - 10x + 25 = 0$. Thus, $a = 1$, $b = -10$, and $c = 25$.

$$b^2 - 4ac = (-10)^2 - 4(1)(25) = 0 \leftarrow \text{Discriminant}$$

There is one distinct rational solution, a double solution.

(c) For $2x^2 - x + 1 = 0$, use $a = 2$, $b = -1$, and $c = 1$.

$$b^2 - 4ac = (-1)^2 - 4(2)(1) = -7 \leftarrow \text{Discriminant}$$

There are two distinct nonreal complex solutions. (They are complex conjugates.)

2.3 Exercises

Concept Check Match the equation in Column I with its solution(s) in Column II.

I	II	
1. $x^2 = 25$	**A.** $\pm 5i$	**B.** $\pm 2\sqrt{5}$
2. $x^2 + 5 = 0$	**C.** $\pm i\sqrt{5}$	**D.** 5
3. $x^2 = -20$	**E.** $\pm\sqrt{5}$	**F.** -5
4. $x - 5 = 0$	**G.** ± 5	**H.** $\pm 2i\sqrt{5}$

Concept Check *Use Choices A–D to answer each question in Exercises 5–6.*

A. $3x^2 - 17x - 6 = 0$ **B.** $(2x + 5)^2 = 7$

C. $x^2 + x = 12$ **D.** $(3x - 1)(x - 7) = 0$

5. Which equation is set up for direct use of the zero-factor property? Solve it.

6. Only one of the equations does not require Step 1 of the method for completing the square described in this section. Which one is it? Solve it.

Solve each equation by the zero-factor property. ***See Example 1.***

7. $x^2 - 5x + 6 = 0$ **8.** $5x^2 - 3x - 2 = 0$ **9.** $-4x^2 + x = -3$

10. $x^2 - 100 = 0$ **11.** $4x^2 - 4x + 1 = 0$ **12.** $25x^2 + 30x + 9 = 0$

Solve each equation by the square root property. ***See Homework 1.***

13. $x^2 = 16$ **14.** $27 - x^2 = 0$ **15.** $x^2 = -81$

16. $(3x - 1)^2 = 12$ **17.** $(x + 5)^2 = -3$ **18.** $(5x - 3)^2 = -3$

Solve each equation by completing the square. ***See Example 2 and Homework 2.***

19. $x^2 - 4x + 3 = 0$ **20.** $2x^2 - x - 28 = 0$ **21.** $x^2 - 2x - 2 = 0$

22. $2x^2 + x = 10$ **23.** $-2x^2 + 4x + 3 = 0$ **24.** $-4x^2 + 8x = 7$

25. *Concept Check* Faisal claimed that the equation $x^2 - 8x = 0$ cannot be solved by the quadratic formula since there is no value for c. Is he correct?

Solve each equation using the quadratic formula. ***See Example 3 and Homework 3.***

26. $x^2 - x - 1 = 0$ **27.** $x^2 - 6x = -7$

28. $x^2 = 2x - 5$ **29.** $-4x^2 = -12x + 11$

30. $\frac{1}{2}x^2 + \frac{1}{4}x - 3 = 0$ **31.** $0.2x^2 + 0.4x - 0.3 = 0$

32. $(4x - 1)(x + 2) = 4x$ **33.** $(x - 9)(x - 1) = -16$

Solve each cubic equation using factoring and the quadratic formula. ***See Example 4.***

34. $x^3 - 8 = 0$ **35.** $x^3 + 27 = 0$

Solve each equation for the indicated variable. Assume no denominators are 0. ***See Homework 4.***

36. $s = \frac{1}{2}gt^2$, for t **37.** $F = \frac{kMv^2}{r}$, for v

38. $r = r_0 + \frac{1}{2}at^2$, for t **39.** $h = -16t^2 + v_0 t + s_0$, for t

For each equation, **(a)** *solve for x in terms of y, and* **(b)** *solve for y in terms of x.* ***See Homework 4.***

40. $4x^2 - 2xy + 3y^2 = 2$ **41.** $2x^2 + 4xy - 3y^2 = 2$

Evaluate the discriminant for each equation. Then use it to predict the number of distinct solutions, and whether they are rational, irrational, *or* nonreal complex. *Do not solve the equation.* ***See Example 5.***

42. $x^2 - 8x + 16 = 0$ **43.** $3x^2 + 5x + 2 = 0$ **44.** $4x^2 = -6x + 3$

45. $9x^2 + 11x + 4 = 0$ **46.** $8x^2 - 72 = 0$

47. Is it possible for the solution set of a quadratic equation with integer coefficients to consist of a single irrational number? Explain.

Find the values of a, b, and c for which the quadratic equation

$$ax^2 + bx + c = 0$$

has the given numbers as solutions. (Hint: Use the zero-factor property in reverse.)

48. 4, 5

49. $1 + \sqrt{2}, 1 - \sqrt{2}$

Summary Exercises on Solving Equations

This section of miscellaneous equations provides practice in solving all the types introduced in this chapter so far. Solve each equation.

1. $4x - 3 = 2x + 3$

2. $5 - (6x + 3) = 2(2 - 2x)$

3. $x(x + 6) = 9$

4. $x^2 = 8x - 12$

5. $\sqrt{x + 2} + 5 = \sqrt{x + 15}$

6. $\dfrac{5}{x + 3} - \dfrac{6}{x - 2} = \dfrac{3}{x^2 + x - 6}$

7. $\dfrac{3x + 4}{3} - \dfrac{2x}{x - 3} = x$

8. $\dfrac{x}{2} + \dfrac{4}{3}x = x + 5$

9. $5 - \dfrac{2}{x} + \dfrac{1}{x^2} = 0$

10. $(2x + 1)^2 = 9$

11. $x^{-2/5} - 2x^{-1/5} - 15 = 0$

12. $\sqrt{x + 2} + 1 = \sqrt{2x + 6}$

13. $x^4 - 3x^2 - 4 = 0$

14. $1.2x + 0.3 = 0.7x - 0.9$

15. $\sqrt[6]{2x + 1} = \sqrt[6]{9}$

16. $3x^2 - 2x = -1$

17. $3[2x - (6 - 2x) + 1] = 5x$

18. $\sqrt{x} + 1 = \sqrt{11 - \sqrt{x}}$

19. $(14 - 2x)^{2/3} = 4$

20. $2x^{-1} - x^{-2} = 1$

21. $\dfrac{3}{x - 3} = \dfrac{3}{x - 3}$

22. $a^2 + b^2 = c^2$, for a

2.4 Inequalities

- Linear Inequalities
- Three-Part Inequalities
- Quadratic Inequalities
- Rational Inequalities

An **inequality** says that one expression is greater than, greater than or equal to, less than, or less than or equal to another (**Section 1.2**). As with equations, a value of the variable for which the inequality is true is a solution of the inequality, and the set of all solutions is the solution set of the inequality. Two inequalities with the same solution set are equivalent.

Inequalities are solved with the properties of inequality, which are similar to the properties of equality in **Section 2.1.**

Properties of Inequality

Let a, b, and c represent real numbers.

1. If $a < b$, then $a + c < b + c$.

2. If $a < b$ and if $c > 0$, then $ac < bc$.

3. If $a < b$ and if $c < 0$, then $ac > bc$.

Replacing $<$ with $>$, $\le$, or $\ge$ results in similar properties. (Restrictions on c remain the same.)

NOTE Multiplication may be replaced by division in Properties 2 and 3. ***Always remember to reverse the direction of the inequality symbol when multiplying or dividing by a negative number.***

Linear Inequalities The definition of a *linear inequality* is similar to the definition of a linear equation.

Linear Inequality in One Variable

A **linear inequality in one variable** is an inequality that can be written in the form

$$ax + b > 0,$$

where a and b are real numbers, with $a \neq 0$. (Any of the symbols $\ge$, $<$, and $\le$ may also be used.)

EXAMPLE 1 Solving a Linear Inequality

Solve $-3x + 5 > -7$.

SOLUTION

$$-3x + 5 > -7$$

$$-3x + 5 - 5 > -7 - 5 \qquad \text{Subtract 5.}$$

$$-3x > -12 \qquad \text{Combine like terms.}$$

Don't forget to reverse the inequality symbol here.

$$\frac{-3x}{-3} < \frac{-12}{-3} \qquad \text{Divide by } -3\text{. Reverse the direction of the inequality symbol when multiplying or dividing by a negative number.}$$

$$x < 4$$

Thus, the original inequality $-3x + 5 > -7$ is satisfied by any real number less than 4. The solution set can be written $\{x \mid x < 4\}$. A graph of the solution set is shown in **Figure 2**, where the parenthesis is used to show that 4 itself does not belong to the solution set. Note that testing values from the solution set in the original inequality will produce true statements, while testing values outside the solution set produces false statements.

Figure 2

The solution set of the inequality,

$$\{x \mid x < 4\}, \qquad \text{Set builder notation (Section 1.1)}$$

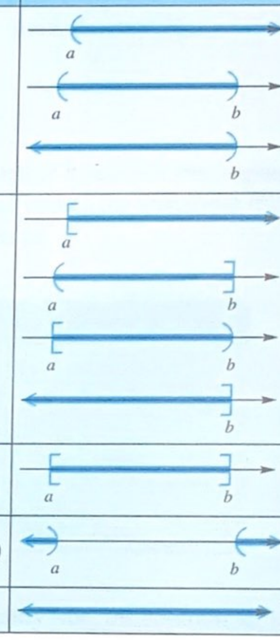
is an example of an **interval.** We use a simplified notation, called **interval notation,** to write intervals. With this notation, we write the interval as

$$(-\infty, 4). \qquad \text{Interval notation}$$

The symbol $-\infty$ does not represent an actual number. Rather it is used to show that the interval includes all real numbers less than 4. The interval $(-\infty, 4)$ is an example of an **open interval,** since the endpoint, 4, is not part of the interval. A **closed interval** includes both endpoints. A square bracket is used to show that a number *is* part of the graph, and a parenthesis is used to indicate that a number *is not* part of the graph.

In the table that follows, we assume that $a < b$.

Type of Interval	Set	Interval Notation	Graph
Open interval	$\{x \mid x > a\}$	(a, ∞)	
	$\{x \mid a < x < b\}$	(a, b)	
	$\{x \mid x < b\}$	$(-\infty, b)$	
Other intervals	$\{x \mid x \geq a\}$	$[a, \infty)$	
	$\{x \mid a < x \leq b\}$	$(a, b]$	
	$\{x \mid a \leq x < b\}$	$[a, b)$	
	$\{x \mid x \leq b\}$	$(-\infty, b]$	
Closed interval	$\{x \mid a \leq x \leq b\}$	$[a, b]$	
Disjoint interval	$\{x \mid x < a \text{ or } x > b\}$	$(-\infty, a) \cup (b, \infty)$	
All real numbers	$\{x \mid x \text{ is a real number}\}$	$(-\infty, \infty)$	



HOMEWORK 1 **Solving a Linear Inequality**

Solve $4 - 3x \leq 7 + 2x$. Give the solution set in interval notation.

A product will break even, or begin to produce a profit, only if the revenue from selling the product at least equals the cost of producing it. If R represents revenue and C is cost, then the **break-even point** is the point where $R = C$.

EXAMPLE 2 **Finding the Break-Even Point**

If the revenue and cost of a certain product are given by

$$R = 4x \quad \text{and} \quad C = 2x + 1000,$$

where x is the number of units produced and sold, at what production level does *R at least equal C*?

SOLUTION Set $R \geq C$ and solve for x.

At least equal to translates as $\geq$.

$$R \geq C$$

$$4x \geq 2x + 1000 \quad \text{Substitute.}$$

$$2x \geq 1000 \quad \text{Subtract } 2x.$$

$$x \geq 500 \quad \text{Divide by 2.}$$

The break-even point is at $x = 500$. This product will at least break even if the number of units produced and sold is in the interval $[500, \infty)$.

Three-Part Inequalities The inequality $-2 < 5 + 3x < 20$ says that $5 + 3x$ is *between* -2 and 20. This inequality is solved using an extension of the properties of inequality given earlier, working with all three expressions at the same time.

HOMEWORK 2 **Solving a Three-Part Inequality**

Solve $-2 < 5 + 3x < 20$.

Quadratic Inequalities Solving *quadratic inequalities* is more complicated than solving linear inequalities and depends on finding solutions of quadratic equations.

Quadratic Inequality

A **quadratic inequality** is an inequality that can be written in the form

$$ax^2 + bx + c < 0,$$

for real numbers a, b, and c, with $a \neq 0$. (The symbol $<$ can be replaced with $>$, $\leq$, or $\geq$.)

One method of solving a quadratic inequality involves finding the solutions of the corresponding quadratic equation and then testing values in the intervals on a number line determined by those solutions.

Solving a Quadratic Inequality

Step 1 Solve the corresponding quadratic equation.

Step 2 Identify the intervals determined by the solutions of the equation.

Step 3 Use a test value from each interval to determine which intervals form the solution set.

EXAMPLE 3 **Solving a Quadratic Inequality**

Solve $x^2 - x - 12 < 0$.

SOLUTION

Step 1 Find the values of x that satisfy $x^2 - x - 12 = 0$.

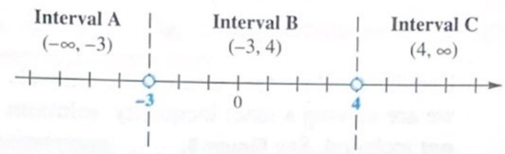
$$x^2 - x - 12 = 0 \quad \text{Corresponding quadratic equation}$$

$$(x + 3)(x - 4) = 0 \quad \text{Factor. (Section 1.4)}$$

$$x + 3 = 0 \quad \text{or} \quad x - 4 = 0 \quad \text{Zero-factor property (Section 2.3)}$$

$$x = -3 \quad \text{or} \quad x = 4 \quad \text{Solve each equation.}$$

Step 2 The two numbers -3 and 4 cause the expression $x^2 - x - 12$ to *equal* zero and can be used to divide the number line into three intervals, as shown in **Figure 3.** The expression $x^2 - x - 12$ will take on a value that is either ***less than*** zero or ***greater than*** zero on each of these intervals. Since we are looking for x-values that make the expression ***less than*** zero, use open circles at -3 and 4 to indicate that they are not included in the solution set.



Use open circles since the inequality symbol does not include *equality*. -3 and 4 do not satisfy the *inequality*.

Figure 3

Step 3 Choose a test value in each interval to see whether it satisfies the original inequality, $x^2 - x - 12 < 0$. If the test value makes the statement true, then the entire interval belongs to the solution set.

Interval	Test Value	Is $x^2 - x - 12 < 0$ True or False?
A: $(-\infty, -3)$	-4	$(-4)^2 - (-4) - 12 \overset{?}{<} 0$ $8 < 0$ False
B: $(-3, 4)$	0	$0^2 - 0 - 12 \overset{?}{<} 0$ $-12 < 0$ True
C: $(4, \infty)$	5	$5^2 - 5 - 12 \overset{?}{<} 0$ $8 < 0$ False

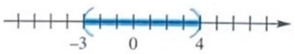


Figure 4

Since the values in Interval B make the inequality true, the solution set is $(-3, 4)$. See **Figure 4.**

HOMEWORK 3 **Solving a Quadratic Inequality**

Solve $2x^2 + 5x - 12 \geq 0$.

NOTE Inequalities that use the symbols $<$ and $>$ are **strict inequalities,** while $\leq$ and $\geq$ are used in **nonstrict inequalities.** The solutions of the equation in **Example 3** were not included in the solution set since the inequality was a *strict* inequality. In **Homework 3,** the solutions of the equation *were* included in the solution set because of the nonstrict inequality.

EXAMPLE 4 **Finding Projectile Height**

If a projectile is launched from ground level with an initial velocity of 96 ft per sec, its height s in feet t seconds after launching is given by the following equation.

$$s = -16t^2 + 96t$$

When will the projectile be greater than 80 ft above ground level?

SOLUTION

$$-16t^2 + 96t > 80 \quad \text{Set } s \text{ greater than 80.}$$

$$-16t^2 + 96t - 80 > 0 \quad \text{Subtract 80.}$$

Reverse the direction of the inequality symbol.

$$t^2 - 6t + 5 < 0 \quad \text{Divide by } -16.$$

Now solve the corresponding *equation*.

$$t^2 - 6t + 5 = 0$$

$$(t-1)(t-5) = 0 \quad \text{Factor.}$$

$$t - 1 = 0 \quad \text{or} \quad t - 5 = 0 \quad \text{Zero-factor property}$$

$$t = 1 \quad \text{or} \quad t = 5 \quad \text{Solve each equation.}$$

Use these values to determine the intervals $(-\infty, 1)$, $(1, 5)$, and $(5, \infty)$. Since we are solving a strict inequality, solutions of the equation $t^2 - 6t + 5 = 0$ are ***not*** included. See **Figure 5.**

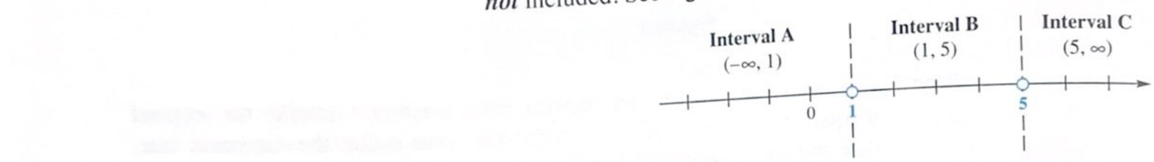


Figure 5

Choose a test value from each interval and use the procedure of **Example 3** and **Homework 3** to determine that values in Interval B, $(1, 5)$, satisfy the inequality. The projectile is greater than 80 ft above ground level between 1 and 5 sec after it is launched.

Rational Inequalities Inequalities involving one or more rational expressions are **rational inequalities.**

$$\frac{5}{x+4} \geq 1 \quad \text{and} \quad \frac{2x-1}{3x+4} < 5 \quad \text{Rational inequalities}$$

Solving a Rational Inequality

Step 1 Rewrite the inequality, if necessary, so that 0 is on one side and there is a single fraction on the other side.

Step 2 Determine the values that will cause either the numerator or the denominator of the rational expression to equal 0. These values determine the intervals on the number line to consider.

Step 3 Use a test value from each interval to determine which intervals form the solution set.

A value causing a denominator to equal zero will never be included in the solution set. If the inequality is strict, any value causing the numerator to equal zero will be excluded. If the inequality is nonstrict, any such value will be included.

CAUTION Solving a rational inequality such as

$$\frac{5}{x+4} \geq 1$$

by multiplying each side by $x + 4$ to obtain $5 \geq x + 4$ requires considering *two cases*, since the sign of $x + 4$ depends on the value of x. If $x + 4$ is negative, then the inequality symbol must be reversed. The procedure described in the preceding box and used in the next two examples eliminates the need for considering separate cases.

HOMEWORK 4 **Solving a Rational Inequality**

Solve $\dfrac{5}{x+4} \geq 1.$

CAUTION ***Be careful with the endpoints of the intervals when solving rational inequalities.***

EXAMPLE 5 **Solving a Rational Inequality**

Solve $\dfrac{2x-1}{3x+4} < 5.$

SOLUTION

$$\frac{2x-1}{3x+4} - 5 < 0 \quad \text{Subtract 5.}$$

$$\frac{2x-1}{3x+4} - \frac{5(3x+4)}{3x+4} < 0 \quad \text{Common denominator is } 3x+4.$$

$$\frac{2x-1-5(3x+4)}{3x+4} < 0 \quad \text{Write as a single fraction.}$$

Be careful with signs.

$$\frac{2x-1-15x-20}{3x+4} < 0 \quad \text{Distributive property (Section 1.2)}$$

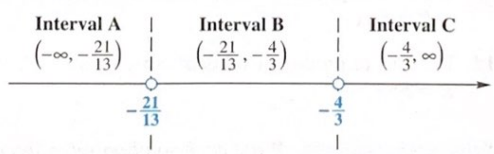
$$\frac{-13x-21}{3x+4} < 0 \quad \text{Combine like terms in the numerator.}$$

Set the numerator and denominator equal to 0 and solve the resulting equations to find the values of x where sign changes may occur.

$$-13x - 21 = 0 \quad \text{or} \quad 3x + 4 = 0$$

$$x = -\frac{21}{13} \quad \text{or} \quad x = -\frac{4}{3}$$

Use these values to form intervals on the number line, as seen in **Figure 5.**



Use an open circle at $-\frac{21}{13}$ because of the strict inequality, and use an open circle at $-\frac{4}{3}$ since it causes the denominator to equal 0.

Figure 5

Now choose test values from the intervals in **Figure 5.** Verify that

-2 from Interval A makes the inequality true.

-1.5 from Interval B makes the inequality false.

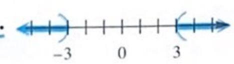
0 from Interval C makes the inequality true.

Because of the $<$ symbol, neither endpoint satisfies the inequality, so the solution set is

$$\left(-\infty, -\frac{21}{13}\right) \cup \left(-\frac{4}{3}, \infty\right).$$

2.4 Exercises

Concept Check *Match the inequality in each exercise in Column I with its equivalent interval notation in Column II.*

I	II
1. $x < -6$	A. $(-2, 6]$
2. $-2 < x \le 6$	B. $[-2, 6)$
3. $x \ge -6$	C. $(-\infty, -6]$
4. 	D. $[6, \infty)$
5. 	E. $(-\infty, -3) \cup (3, \infty)$
	F. $(-\infty, -6)$
	G. $(0, 8)$
	H. $(-\infty, \infty)$
	I. $[-6, \infty)$
	J. $(-\infty, 6]$

6. Explain how to determine whether to use a parenthesis or a square bracket when graphing the solution set of a linear inequality.

Solve each inequality. Write each solution set in interval notation. ***See Example 1 and Homework 1.***

7. $-2x + 8 \le 16$
8. $-2x - 2 \le 1 + x$
9. $3(x + 5) + 1 \ge 5 + 3x$
10. $8x - 3x + 2 < 2(x + 7)$
11. $\dfrac{4x + 7}{-3} \le 2x + 5$
12. $\dfrac{1}{3}x + \dfrac{2}{5}x - \dfrac{1}{2}(x + 3) \le \dfrac{1}{10}$

Break-Even Interval *Find all intervals where each product will at least break even.* ***See Example 2.***

13. The cost to produce x units of picture frames is $C = 50x + 5000$, while the revenue is $R = 60x$.

14. The cost to produce x units of coffee cups is $C = 105x + 900$, while the revenue is $R = 85x$.

Solve each inequality. Write each solution set in interval notation. ***See Homework 2.***

15. $-5 < 5 + 2x < 11$
16. $10 \le 2x + 4 \le 16$
17. $-11 > -3x + 1 > -17$
18. $-4 \le \dfrac{x + 1}{2} \le 5$
19. $-3 \le \dfrac{x - 4}{-5} < 4$

Solve each quadratic inequality. Write each solution set in interval notation. ***See Example 3 and Homework 3.***

20. $x^2 - x - 6 > 0$

21. $2x^2 - 9x \leq 18$

22. $-x^2 - 4x - 6 \leq -3$

23. $x(x - 1) \leq 6$

24. $x^2 \leq 9$

25. $x^2 + 5x + 7 < 0$

26. $x^2 - 2x \leq 1$

27. ***Concept Check*** Which one of the following inequalities has solution set $(-\infty, \infty)$?

A. $(x - 3)^2 \geq 0$

B. $(5x - 6)^2 \leq 0$

C. $(6x + 4)^2 > 0$

D. $(8x + 7)^2 < 0$

Relating Concepts

For individual or collaborative investigation *(Exercises 28–29)*

Inequalities that involve more than two factors, such as

$$(3x - 4)(x + 2)(x + 6) \leq 0,$$

can be solved using an extension of the method shown in ***Example 3 and Homework 3. Work Exercises 28–29 in order,*** *to see how the method is extended.*

28. Use the zero-factor property to solve $(3x - 4)(x + 2)(x + 6) = 0$.

29. Plot the three solutions from Exercise 28 on a number line, using closed circles because of the nonstrict inequality, $\leq$. The number line should show four intervals formed by the three points. For each interval, choose a number from the interval and decide whether it satisfies the original inequality.

Use the technique described in ***Relating Concepts Exercises 28–29*** *to solve each inequality. Write each solution set in interval notation.*

30. $(2x - 3)(x + 2)(x - 3) \geq 0$

31. $4x - x^3 \geq 0$

32. $(x + 1)^2 (x - 3) < 0$

33. $x^3 + 4x^2 - 9x \geq 36$

34. $x^2(x + 4)^2 \geq 0$

Solve each rational inequality. Write each solution set in interval notation. ***See Homework 4 and Example 5.***

35. $\dfrac{x - 3}{x + 5} \leq 0$

36. $\dfrac{1 - x}{x + 2} < -1$

37. $\dfrac{3}{x - 6} \leq 2$

38. $\dfrac{-4}{1 - x} < 5$

39. $\dfrac{10}{3 + 2x} \leq 5$

40. $\dfrac{7}{x + 2} \geq \dfrac{1}{x + 2}$

41. $\dfrac{3}{2x - 1} > \dfrac{-4}{x}$

42. $\dfrac{4}{2 - x} \geq \dfrac{3}{1 - x}$

43. $\dfrac{x + 3}{x - 5} \leq 1$

Solve each rational inequality. Write each solution set in interval notation.

44. $\dfrac{2x - 3}{x^2 + 1} \geq 0$

45. $\dfrac{(5 - 3x)^2}{(2x - 5)^3} > 0$

46. $\dfrac{(2x - 3)(3x + 8)}{(x - 6)^3} \geq 0$

2.5 Absolute Value Equations and Inequalities

- Basic Concepts
- Absolute Value Equations
- Absolute Value Inequalities
- Special Cases
- Absolute Value Models for Distance and Tolerance

Basic Concepts Recall from **Section 1.2** that the **absolute value** of a number a, written $|a|$, gives the distance from a to 0 on a number line. By this definition, the equation $|x| = 3$ can be solved by finding all real numbers at a distance of 3 units from 0. As shown in **Figure 6,** two numbers satisfy this equation, 3 and -3, so the solution set is $\{-3, 3\}$.

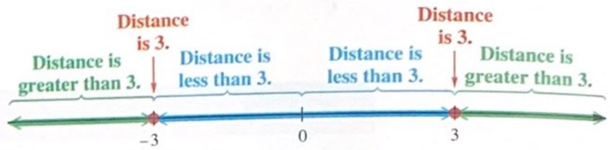


Figure 6

Similarly, $|x| < 3$ is satisfied by all real numbers whose distances from 0 are less than 3—that is, the interval

$$-3 < x < 3, \quad \text{or} \quad (-3, 3).$$

See **Figure 6.** Finally, $|x| > 3$ is satisfied by all real numbers whose distances from 0 are greater than 3. These numbers are less than -3 or greater than 3, so the solution set is

$$(-\infty, -3) \cup (3, \infty).$$

Notice in **Figure 6** that the union of the solution sets of $|x| = 3$, $|x| < 3$, and $|x| > 3$ is the set of real numbers.

These observations support the cases for solving absolute value equations and inequalities summarized in the table below. If the equation or inequality fits the form of Case 1, 2, or 3, change it to its equivalent form and solve. The solution set and its graph will look similar to those shown.

LOOKING AHEAD TO CALCULUS

The precise definition of a **limit** in calculus requires writing absolute value inequalities.

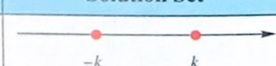
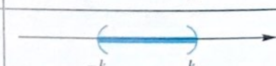
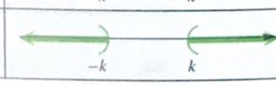
A standard problem in calculus is to find the "interval of convergence" of something called a **power series,** by solving an inequality of the form

$$|x - a| < r.$$

This inequality says that x can be any number within r units of a on the number line, so its solution set is indeed an interval—namely the interval $(a - r, a + r)$.

For each equation or inequality in Cases 1–3 in the table, assume that $k > 0$.

Solving Absolute Value Equations and Inequalities

Absolute Value Equation or Inequality	Equivalent Form	Graph of the Solution Set	Solution Set
Case 1: $\lvert x\rvert = k$	$x = k$ or $x = -k$		$\{-k, k\}$
Case 2: $\lvert x\rvert < k$	$-k < x < k$		$(-k, k)$
Case 3: $\lvert x\rvert > k$	$x < -k$ or $x > k$		$(-\infty, -k) \cup (k, \infty)$

In Cases 2 and 3, the strict inequality may be replaced by its nonstrict form. Additionally, if an absolute value equation takes the form $|a| = |b|$, then a and b must be equal in value or opposite in value.

Thus, the equivalent form of $|a| = |b|$ ***is*** $a = b$ **or** $a = -b$.

Absolute Value Equations Because absolute value represents distance from 0 on a number line, solving an absolute value equation requires solving two possibilities, as shown in the examples that follow.

EXAMPLE 1 **Solving Absolute Value Equations (Case 1 and the Special Case $|a| = |b|$)**

Solve each equation.

(a) $|5 - 3x| = 12$ **(b)** $|4x - 3| = |x + 6|$

SOLUTION

(a) For the given expression $5 - 3x$ to have absolute value 12, it must represent either 12 or -12. This equation fits the form of Case 1.

$$|5 - 3x| = 12$$

Don't forget this second possibility.

$$5 - 3x = 12 \quad \text{or} \quad 5 - 3x = -12 \qquad \text{Case 1}$$

$$-3x = 7 \quad \text{or} \quad -3x = -17 \qquad \text{Subtract 5.}$$

$$x = -\frac{7}{3} \quad \text{or} \quad x = \frac{17}{3} \qquad \text{Divide by } -3.$$

Check the solutions $-\frac{7}{3}$ and $\frac{17}{3}$ by substituting them in the original absolute value equation. The solution set is $\left\{-\frac{7}{3}, \frac{17}{3}\right\}$.

(b) If the absolute values of two expressions are equal, then those expressions are either equal in value or opposite in value.

$$|4x - 3| = |x + 6|$$

$$4x - 3 = x + 6 \quad \text{or} \quad 4x - 3 = -(x + 6) \qquad \text{Consider both possibilities.}$$

$$3x = 9 \quad \text{or} \quad 4x - 3 = -x - 6 \qquad \text{Solve each linear equation. (Section 2.1)}$$

$$x = 3 \quad \text{or} \quad 5x = -3$$

$$x = -\frac{3}{5}$$

Absolute Value Inequalities

HOMEWORK 1 **Solving Absolute Value Inequalities (Cases 2 and 3)**

Solve each inequality.

(a) $|2x + 1| < 7$ **(b)** $|2x + 1| > 7$

Cases 1, 2, and 3 require that the absolute value expression be isolated on one side of the equation or inequality.

EXAMPLE 2 **Solving an Absolute Value Inequality (Case 3)**

Solve $|2 - 7x| - 1 > 4$.

SOLUTION

$$|2 - 7x| - 1 > 4$$

$$|2 - 7x| > 5 \qquad \text{Add 1 to each side.}$$

$$2 - 7x < -5 \quad \text{or} \quad 2 - 7x > 5 \qquad \text{Case 3}$$

$$-7x < -7 \quad \text{or} \quad -7x > 3 \qquad \text{Subtract 2.}$$

$$x > 1 \quad \text{or} \quad x < -\frac{3}{7} \qquad \text{Divide by } -7\text{. Reverse the direction of each inequality.}$$

(Section 2.4)

The solution set is $(-\infty, -\frac{3}{7}) \cup (1, \infty)$.

Special Cases The three cases given in this section require the constant k to be positive. ***When $k \leq 0$, use the fact that the absolute value of any expression must be nonnegative, and consider the truth of the statement.***

HOMEWORK 2 **Solving Special Cases**

Solve each equation or inequality.

(a) $|2 - 5x| \geq -4$ **(b)** $|4x - 7| < -3$ **(c)** $|5x + 15| = 0$.

Absolute Value Models for Distance and Tolerance If a and b represent two real numbers, then the absolute value of their difference,

$$\text{either} \quad |a - b| \quad \text{or} \quad |b - a|, \quad \text{(Section 1.2)}$$

represents the distance between them.

EXAMPLE 3 **Using Absolute Value Inequalities to Describe Distances**

Write each statement using an absolute value inequality.

(a) k is no less than 5 units from 8. **(b)** n is within 0.001 unit of 6.

SOLUTION

(a) Since the distance from k to 8, written $|k - 8|$ or $|8 - k|$, is no less than 5, the distance is greater than or equal to 5. This can be written as

$$|k - 8| \geq 5, \quad \text{or, equivalently,} \quad |8 - k| \geq 5. \qquad \text{Either form is acceptable.}$$

(b) This statement indicates that the distance between n and 6 is less than 0.001.

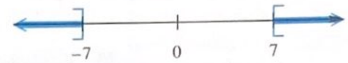
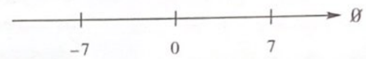
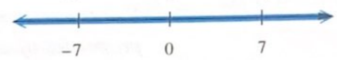
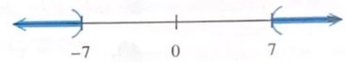
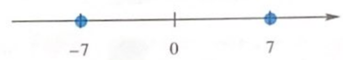
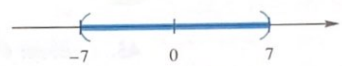
$$|n - 6| < 0.001, \quad \text{or, equivalently,} \quad |6 - n| < 0.001$$

HOMEWORK 4 **Using Absolute Value to Model Tolerance**

In quality control and other applications, we often wish to keep the difference between two quantities within some predetermined amount, called the **tolerance.** Suppose $y = 2x + 1$ and we want y to be within 0.01 unit of 4. For what values of x will this be true?

2.5 Exercises

Concept Check *Match each equation or inequality in Column I with the graph of its solution set in Column II.*

I	II
1. $\lvert x\rvert = 7$	A. 
2. $\lvert x\rvert = -7$	B. 
3. $\lvert x\rvert > -7$	C. 
4. $\lvert x\rvert > 7$	D. 
5. $\lvert x\rvert < 7$	E. 
6. $\lvert x\rvert \ge 7$	F. 
7. $\lvert x\rvert \le 7$	G. 
8. $\lvert x\rvert \ne 7$	H. 

Solve each equation. ***See Example 1.***

9. $\lvert 3x - 1\rvert = 2$ **10.** $\lvert 5 - 3x\rvert = 3$ **11.** $\left\lvert \frac{x-4}{2} \right\rvert = 5$

12. $\left\lvert \frac{5}{x-3} \right\rvert = 10$ **13.** $\left\lvert \frac{6x+1}{x-1} \right\rvert = 3$ **14.** $\lvert 2x - 3\rvert = \lvert 5x + 4\rvert$

15. $\lvert 4 - 3x\rvert = \lvert 2 - 3x\rvert$ **16.** $\lvert 5x - 2\rvert = \lvert 2 - 5x\rvert$

17. The equation $\lvert 7x + 3\rvert = -5x$ cannot have a positive solution. Why?

Solve each inequality. Give the solution set using interval notation. ***See Homework 1.***

18. $\lvert 2x + 5\rvert < 3$ **19.** $\lvert 2x + 5\rvert \ge 3$ **20.** $\left\lvert \frac{1}{2} - x \right\rvert < 2$

21. $4\lvert x - 3\rvert > 12$ **22.** $\lvert 5 - 3x\rvert > 7$ **23.** $\lvert 5 - 3x\rvert \le 7$

24. $\left\lvert \frac{2}{3}x + \frac{1}{2} \right\rvert \le \frac{1}{6}$ **25.** $\lvert 0.01x + 1\rvert < 0.01$

Solve each equation or inequality. ***See Example 2 and Homework 2.***

26. $\lvert 4x + 3\rvert - 2 = -1$ **27.** $\lvert 6 - 2x\rvert + 1 = 3$ **28.** $\lvert 3x + 1\rvert - 1 < 2$

29. $\left\lvert 5x + \frac{1}{2} \right\rvert - 2 < 5$ **30.** $\lvert 10 - 4x\rvert + 1 \ge 5$ **31.** $\lvert 3x - 7\rvert + 1 < -2$

Solve each equation or inequality. ***See Homework 2.***

32. $\lvert 10 - 4x\rvert \ge -4$ **33.** $\lvert 6 - 3x\rvert < -11$ **34.** $\lvert 8x + 5\rvert = 0$

35. $\lvert 4.3x + 9.8\rvert < 0$ **36.** $\lvert 2x + 1\rvert \le 0$ **37.** $\lvert 3x + 2\rvert > 0$

Relating Concepts

For individual or collaborative investigation *(Exercises 38–39)*

To see how to solve an equation that involves the absolute value of a quadratic polynomial, such as $|x^2 - x| = 6$, ***work Exercises 38–39 in order.***

38. For $x^2 - x$ to have an absolute value equal to 6, what are the two possible values that it may be? (*Hint:* One is positive and the other is negative.)

39. Write an equation stating that $x^2 - x$ is equal to the negative value you found in **Exercise 38,** and solve it using the quadratic formula. (*Hint:* The solutions are not real numbers.)

Use the method described in ***Relating Concepts Exercises 38–39,*** *if applicable, and properties of absolute value to solve each equation or inequality. (Hint: Exercise 43 can be solved by inspection.)*

40. $|3x^2 + x| = 14$

41. $|4x^2 - 23x - 6| = 0$

42. $|x^2 + 1| - |2x| = 0$

43. $|x^4 + 2x^2 + 1| < 0$

44. $\left|\dfrac{x-4}{3x+1}\right| \geq 0$

45. ***Concept Check*** Write an equation involving absolute value that says the distance between p and q is 2 units.

Write each statement as an absolute value equation or inequality. ***See Example 3.***

46. m is no more than 2 units from 7.

47. p is within 0.0001 unit of 9.

48. r is no less than 1 unit from 29.

49. ***Tolerance*** Suppose that $y = 5x + 1$ and we want y to be within 0.002 unit of 6. For what values of x will this be true?

(Modeling) *Solve each problem.* ***See Homework 3.***

50. ***Weights of Babies*** Dr. Aziz has found that, over the years, 95% of the babies he has delivered weighed x pounds, where

$$|x - 8.2| \leq 1.5.$$

What range of weights corresponds to this inequality?

51. ***Conversion of Methanol to Gasoline*** The industrial process that is used to convert methanol to gasoline is carried out at a temperature range of 680°F to 780°F. Using F as the variable, write an absolute value inequality that corresponds to this range.

(Modeling) Carbon Dioxide Emissions When humans breathe, carbon dioxide is emitted. In one study, the emission rates of carbon dioxide by college students were measured during both lectures and exams. The average individual rate R_L (in grams per hour) during a lecture class satisfied the inequality

$$|R_L - 26.75| \leq 1.42,$$

whereas during an exam the rate R_E satisfied the inequality

$$|R_E - 38.75| \leq 2.17.$$

(*Source:* Wang, T. C., *ASHRAE Trans.*, 81 (Part 1), 32.)

Use this information in Exercise 52.

52. Find the range of values for R_L and R_E.

Glossary

absolute value The absolute value of a real number is the distance between 0 and the number on the number line.

القيمة المطلقة القيمة المطلقة لأي عدد حقيقي هي المسافة بين الصفر والعدد على الخط العددي.

break-even point The break-even point is the point where the revenue from selling a product is equal to the cost of producing it.

نقطة التعادل هي النقطة التي يتساوى فيها الإيراد الوارد من بيع منتج مع تكلفة إنتاجه.

closed interval A closed interval is an interval that includes both of its endpoints.

الفترة المغلقة هي أي فترة تتضمن كلا من نقطتيها الحديتين.

completing the square The process of adding to a binomial the expression that makes it a perfect square trinomial is called completing the square.

إكمال المربع يطلق على عملية إضافة تعبير إلى أي ثنائي حد وهو التعبير الذي من شأنه أن يخرج مربعًا تامًا ثلاثي الأبعاد؛ عملية إكمال المربع.

complex number A complex number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

العدد المركب أي عدد المركب بصيغة $a + bi$ بحيث تكون a و b أعدادًا حقيقية و $i = \sqrt{-1}$.

conditional equation An equation that is satisfied by some numbers but not by others is a conditional equation.

المعادلة الشرطية أي معادلة تكتفي ببعض الأعداد لكن ليس بآخرين تعد معادلة شرطية.

contradiction An equation that has no solution is a contradiction.

المضادة أي معادلة لا حل لها تعد مضادة.

discriminant The quantity under the radical in the quadratic formula, $b^2 - 4ac$, is the discriminant of the expression $ax^2 + bx + c$.

المميزة الكمية تحت جذر الصيغة التربيعية، $b^2 - 4ac$ هي مميزة (الدالة) التعبير $ax^2 + bx + c$.

empty set (null set) The empty set or null set, written $\emptyset$ or $\{ \}$, is the set containing no elements.

مجموعة فارغة (مجموعة خالية) المجموعة الفارغة أو المجموعة الخالية، تكتب $\emptyset$ أو $\{ \}$، هي المجموعة التي لا تحتوي على أي عناصر.

equation An equation is a statement that two expressions are equal.

المعادلة أي معادلة هي بيان يفيد بتساوي تعبيرين اثنين.

equivalent equations Equations with the same solution set are equivalent equations.

معادلات مكافئة المعادلات ذات مجموعة الحلول نفسها تعد معادلات مكافئة.

identity An equation satisfied by every number that is a meaningful replacement for the variable is an identity.

المتطابقة أي معادلة تكتفي بكل عدد يمثل إبدالًا مهمًا للمتغير يطلق عليها متطابقة.

imaginary part In the complex number $a + bi$, b is the imaginary part.

الجزء التخيلي في الأعداد المركبة بشكل $a + bi$، b الجزء التخيلي.

inequality An inequality says that one expression is greater than, greater than or equal to, less than, or less than or equal to, another.

المتباينة المتباينة هي الشيء الذي يشير إلى أن أي تعبير أكبر من الآخر، أكبر منه أو يساويه، أقل منه، أقل منه أو يساويه.

interval An interval is a portion of the real number line, which may or may not include its endpoint(s).

الفترة أي فترة تعد جزءًا من خط الأعداد الحقيقي، الذي قد يشتمل على نقطته (النقاط) الحدية أو لا.

interval notation Interval notation is a simplified notation for writing intervals. It uses parentheses and brackets to show whether the endpoints are included.

ترميز الفترة ترميز الفترة يعد ترميزًا ميسرًا لكتابة الفترات. تستخدم علامات الحصر والأقواس لإظهار أن الحدود الحدية مشمولة بداخلها.

linear equation (first-degree equation) in n unknowns Any equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, for real numbers $a_1, a_2, \ldots, a_n$ (not all of which are 0) and b, is a linear equation, or a first-degree equation, in n unknowns.

المعادلة الخطية (معادلة من الدرجة الأولى) في n أي معادلة من صيغة $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ وللأعداد الحقيقية $a_1, a_2, \cdots, a_n$ (ليس كلها صفرًا) و b تعد معادلة خطية، أو معادلة من الدرجة الأولى في n المجهول.

linear equation (first-degree equation) in one variable A linear equation in one variable is an equation that can be written in the form $ax + b = 0$, where a and b are real numbers with $a \neq 0$.

معادلة خطية (معادلة من الدرجة الأولى) في متغير واحد أي معادلة خطية في متغير واحد تعد معادلة يمكن أن تكتب بصيغة $ax + b = 0$، حيث إن a و b تكون أعدادًا حقيقية مع $a \neq 0$.

linear inequality in one variable A linear inequality in one variable is an inequality that can be written in the form $ax + b > 0$, where a and b are real numbers with $a \neq 0$. (Any of the symbols $<$, $\geq$, and $\leq$ may also be used.)

المتباينة الخطية في متغير واحد أي متباينة خطية في متغير واحد تعد متباينة يمكن كتابتها بصيغة $ax + b > 0$ بحيث تكون a و b أعدادًا حقيقية مع $a \neq 0$. (أي يمكن أيضا استخدام رموز $<$، $\geq$ و $\leq$).

literal equation A literal equation is an equation that relates two or more variables (letters).

المعادلة الحرفية المعادلة الحرفية هي معادلة تربط بين متغيرين أو أكثر (حروف).

nonreal complex number A complex number $a + bi$ in which $b \neq 0$ is a nonreal complex number.

العدد التركيبي غير الحقيقي أي عدد تركيبي مثل $a + bi$ بحيث تكون فيه $b \neq 0$ فهو عدد تركيبي غير حقيقي.

nonstrict inequality An inequality in which the symbol is either $\leq$ or $\geq$ is a nonstrict inequality.

المتباينة غير الحادة أي متباينة يكون الرمز فيها إما $\leq$ or $\geq$ تعد متباينة غير حادة.

open interval An open interval is an interval that does not include its endpoint(s).

فترة مفتوحة أي فترة مفتوحة هي أي فترة لا تشتمل على نقطتها (نقاطها) الحديّة.

pure imaginary number A complex number $a + bi$ in which $a = 0$ and $b \neq 0$ is a pure imaginary number.

العدد التخيلي الصافي العدد المركب $a + bi$ بحيث تكون فيه $a = 0$ و $b \neq 0$ هي عدد تخيلي صافي.

quadratic equation (second-degree equation) An equation that can be written in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers with $a \neq 0$, is a quadratic equation.

معادلة تربيعية (معادلة من الدرجة الثانية) أي معادلة يمكن كتابتها بصيغة $ax^2 + bx + c = 0$ حيث تكون a و b و c أعدادًا حقيقية مع $a \neq 0$، فهي معادلة تربيعية.

quadratic inequality A quadratic inequality is an inequality that can be written in the form $ax^2 + bx + c < 0$, for real numbers a, b, and c, with $a \neq 0$. (The symbol $<$ can be replaced with $>$, $\leq$, and $\geq$.)

المتباينة التربيعية أي معادلة تربيعية هي متباينة يمكن كتابتها بصيغة $ax^2 + bx + c < 0$، وللأعداد الحقيقية بهذه الصيغة a وb وc مع $a \neq 0$. (يمكن إبدال الرمز $<$ بـ $>$ و$\leq$ و$\geq$.)

rational inequality A rational inequality is an inequality in which one or both sides contain rational expressions.

المتباينة الجذرية المتباينة الجذرية هي أي متباينة يكون واحد من جوانبها أو اثنان يحتوي على تعبيرات جذرية.

real numbers The set of all numbers that correspond to points on a number line is the real numbers.

الأعداد الحقيقية مجموعة الأعداد التي تتوافق مع نقاط على خط عددي هي الأعداد الحقيقية.

real part In the complex number $a + bi$, a is the real part.

الجزء الحقيقي في العدد المركب $a + bi$، a هي الجزء الحقيقي.

solution (root) A solution or root of an equation is a number that makes the equation a true statement.

الحل (الجذر) حل أو جذر معادلة هو اعدد الذي يجعل المعادلة عبارة صحيحة.

solution set The solution set of an equation is the set of all numbers that satisfy the equation.

مجموعة الحلول مجموعة حلول المعادلة هي مجموعة كل الأعداد التي تتوافق مع المعادلة.

standard form of a complex number A complex number written in the form $a + bi$ (or $a + ib$) is in standard form.

الصيغة النمطية للعدد المركب هي عدد مركّب مكتوبة بالصيغة $a + bi$ (أو $a + ib$).

standard form of a linear equation The form $Ax + By = C$ is the standard form of a linear equation.

الصيغة النمطية للمعادلة الخطية الصيغة $Ax + By = C$ هي الصيغة النمطية للمعادلة الخطية.

strict inequality An inequality in which the symbol is either $<$ or $>$ is a strict inequality.

المتباينة الحادة هي متباينة يكون فيها الرمز إما $<$ أو $>$.

zero-factor property The zero-factor property states that if the product of two (or more) complex numbers is 0, then at least one of the numbers must be 0.

خاصية العامل الصفري تنص خاصية العامل الصفري على أنه إذا كان حاصل ضرب عددين مركبين (أو أكثر) هو صفر، إذن يجب أن يكون أقل هذه الأعداد صفرًا.

3.1 Functions → الدوال.

- Relations and Functions
- Domain and Range
- Determining Whether Relations Are Functions
- Function Notation
- Increasing, Decreasing, and Constant Functions

Relations and Functions Recall that we can describe one quantity in terms of another.

- The letter grade you receive in a mathematics course depends on your numerical scores.
- The amount you pay (in dollars) for gas at the gas station depends on the number of gallons pumped.
- The dollars spent on household expenses depends on the category.

We used ordered pairs to represent these corresponding quantities. For example, $(3, \$10.50)$ indicates that you pay \$10.50 for 3 gallons of gas. Since the amount you pay *depends* on the number of gallons pumped, the amount (in dollars) is called the *dependent variable*, and the number of gallons pumped is called the *independent variable*.

Generalizing, if the value of the second component y depends on the value of the first component x, then y is the **dependent variable** and x is the **independent variable.**

Independent variable ↓ ↓ Dependent variable

$$(x, y)$$

A set of ordered pairs such as $\{(3, 10.50), (8, 28.00), (10, 35.00)\}$ is called a *relation*. A special kind of relation called a *function* is very important in mathematics and its applications.

Relation and Function

A **relation** is a set of ordered pairs. A **function** is a relation in which, for each distinct value of the first component of the ordered pairs, there is *exactly one* value of the second component.

NOTE The relation from the beginning of this section representing the number of gallons of gasoline and the corresponding cost is a function since each x-value is paired with exactly one y-value.

EXAMPLE 1 Deciding Whether Relations Define Functions

Decide whether each relation defines a function.

$$F = \{(1, 2), (-2, 4), (3, 4)\}$$
$$G = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$
$$H = \{(-4, 1), (-2, 1), (-2, 0)\}$$

SOLUTION Relation F is a function, because for each different x-value there is exactly one y-value. We can show this correspondence as follows.

$$\{1, -2, 3\} \quad x\text{-values of } F$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$\{2, 4, 4\} \quad y\text{-values of } F$$

As the correspondence below shows, relation G is not a function because one first component corresponds to *more than one* second component.

$\{1, 2\}$ x-values of G

$\{1, 2, 3\}$ y-values of G

In relation H the last two ordered pairs have the same x-value paired with two different y-values (-2 is paired with both 1 and 0), so H is a relation but not a function. ***In a function, no two ordered pairs can have the same first component and different second components.***

Different y-values

$$H = \{(-4, 1), (-2, 1), (-2, 0)\} \quad \text{Not a function}$$

Same x-value

لازم كل عنصر في (x) يخرج منه سهم واحد بس

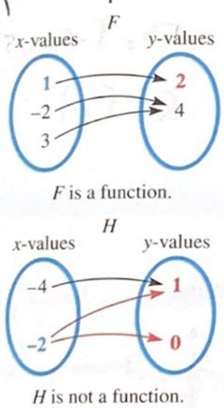
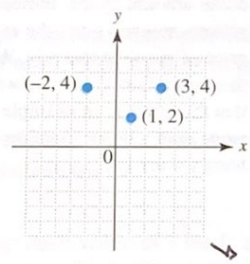


Figure 1

Relations and functions can also be expressed as a correspondence or *mapping* from one set to another, as shown in **Figure 1** for function F and relation H from **Example 1.** The arrow from 1 to 2 indicates that the ordered pair $(1, 2)$ belongs to F—each first component is paired with exactly one second component. In the mapping for relation H, which is not a function, the first component -2 is paired with two different second components, 1 and 0.

x	y
1	2
-2	4
3	4

لأنه إحداثي الـ x لا يتكرر



Graph of F

Figure 2

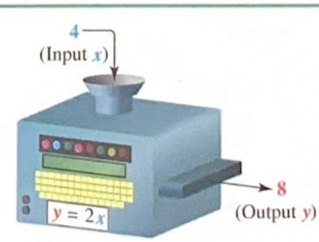
باختبار الخط الرأسي

Since relations and functions are sets of ordered pairs, we can represent them using tables and graphs. A table and graph for function F are shown in **Figure 2.**

Finally, we can describe a relation or function using a rule that tells how to determine the dependent variable for a specific value of the independent variable. The rule may be given in words: for instance, "the dependent variable is twice the independent variable." Usually the rule is an equation, such as the one below.

$$\text{Dependent variable} \rightarrow y = 2x \leftarrow \text{Independent variable}$$

NOTE Another way to think of a function relationship is to think of the independent variable as an input and the dependent variable as an output. This is illustrated by the input-output (function) machine for the function defined by $y = 2x$.



Function machine

In a function, there is exactly one value of the dependent variable, the second component, for each value of the independent variable, the first component.

On this particular day, an *input* of pumping 7.870 gallons of gasoline led to an *output* of \$29.58 from the purchaser's wallet. This is an example of a function whose domain consists of numbers of gallons pumped, and whose range consists of amounts from the purchaser's wallet. Dividing the dollar amount by the number of gallons pumped gives the exact price of gasoline that day. Was this pump fair? (Later we will see that this price is an example of the slope m of a linear function of the form $y = mx$.)

Domain and Range For every relation there are two important sets of elements called the *domain* and *range*.

Domain and Range

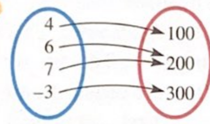
In a relation consisting of ordered pairs (x, y), the set of all values of the independent variable (x) is the **domain.** The set of all values of the dependent variable (y) is the **range.**

HOMEWORK 1 **Finding Domains and Ranges of Relations**

Give the domain and range of each relation. Tell whether the relation defines a function.

(a) $\{(3, -1), (4, 2), (4, 5), (6, 8)\}$ → D = {3,4,6} R = {-1,2,5,8}

(b)



D={4,6,7,-3}
R = {100, 200, 300}

(c)

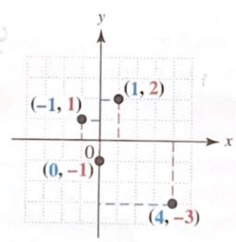
x	y
-5	2
0	2
5	2

D = {-5, 0, 5}
R = {2}

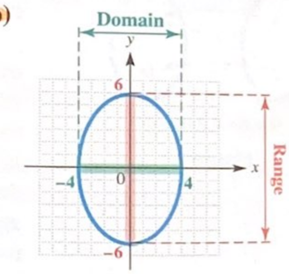
EXAMPLE 2 **Finding Domains and Ranges from Graphs**

Give the domain and range of each relation.

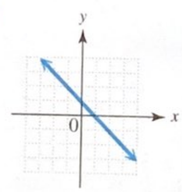
(a)



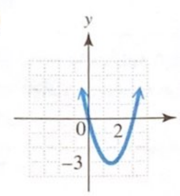
(b)



(c)



(d)



SOLUTION

(a) The domain is the set of x-values, $\{-1, 0, 1, 4\}$. The range is the set of y-values, $\{-3, -1, 1, 2\}$.

(b) The x-values of the points on the graph include all numbers between -4 and 4, inclusive. The y-values include all numbers between -6 and 6, inclusive.

The domain is $[-4, 4]$. Use interval notation.

The range is $[-6, 6]$. (Section 2.4)

(c) The arrowheads indicate that the line extends indefinitely left and right, as well as up and down. Therefore, both the domain and the range include all real numbers, which is written $(-\infty, \infty)$.

(d) The arrowheads indicate that the graph extends indefinitely left and right, as well as upward. The domain is $(-\infty, \infty)$. Because there is a least y-value, -3, the range includes all numbers greater than or equal to -3, written $[-3, \infty)$.

Since relations are often defined by equations, such as $y = 2x + 3$ and $y^2 = x$, we must sometimes determine the domain of a relation from its equation. In this book, we assume the following agreement on the domain of a relation.

Agreement on Domain → القيم المقبولة في الـ Domain

Unless specified otherwise, the domain of a relation is assumed to be all real numbers that produce real numbers when substituted for the independent variable.

To illustrate this agreement, since any real number can be used as a replacement for x in $y = 2x + 3$, the domain of this function is the set of all real numbers. As another example, the function defined by $y = \frac{1}{x}$ has all real numbers *except* 0 as domain, since y is undefined if $x = 0$.

In general, the domain of a function defined by an algebraic expression is all real numbers, except those numbers that lead to division by 0 or to an even root of a negative number.

(There are also exceptions for logarithmic and trigonometric functions. They are covered in further treatment of precalculus mathematics.)

* إذا كان الـ Polynomial ← odd
الـ Range ← $\mathbb{R}$

* إذا كان الـ Polynomial ← even
راح يكون لها ~~قيم~~ أكبر أو أصغر
حالة

① $f(x) = \text{Poly}$
$D = \mathbb{R}$

② $f(x) = \sqrt{\bigcirc}$
$\bigcirc \geq 0$

③ $f(x) = \dfrac{\text{Poly}}{\sqrt{}}$
> 0

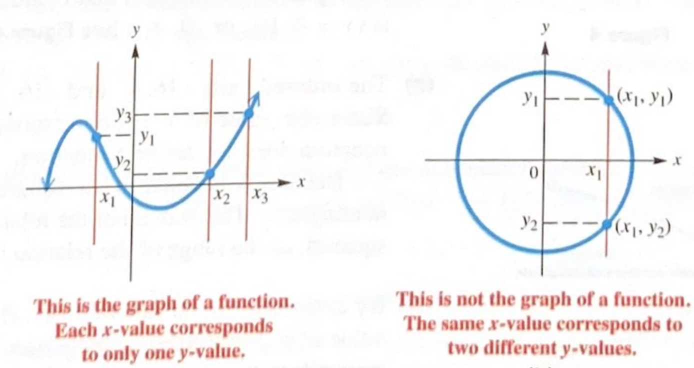
④ $f = \dfrac{\text{Poly}}{\text{Poly}}$
$D = \mathbb{R} - \{$أصفار المقام$\}$

Determining Whether Relations Are Functions Since each value of x leads to only one value of y in a function, any vertical line must intersect the graph in at most one point. This is the **vertical line test** for a function.

Vertical Line Test → Very important

If every vertical line intersects the graph of a relation in no more than one point, then the relation is a function.

The graph in **Figure 3(a)** represents a function because each vertical line intersects the graph in no more than one point. The graph in **Figure 3(b)** is not the graph of a function since a vertical line intersects the graph in more than one point.



This is the graph of a function. Each x-value corresponds to only one y-value.
(a)

This is not the graph of a function. The same x-value corresponds to two different y-values.
(b)

Figure 3

HOMEWORK 2 **Using the Vertical Line Test**

Use the vertical line test to determine whether each relation graphed in **Example 2** is a function.

The vertical line test is a simple method for identifying a function defined by a graph. Deciding whether a relation defined by an equation or an inequality is a function, as well as determining the domain and range, is more difficult. The next example gives some hints that may help.

Very important ←

EXAMPLE 3 **Identifying Functions, Domains, and Ranges**

Decide whether each relation defines a function and give the domain and range.

(a) $y = x + 4$ **(b)** $y = \sqrt{2x - 1}$ **(c)** $y^2 = x$

(d) $y \leq x - 1$ **(e)** $y = \dfrac{5}{x - 1}$

* أي متباينة تكون not function

SOLUTION

(a) In the defining equation (or rule), $y = x + 4$, y is always found by adding 4 to x. Thus, each value of x corresponds to just one value of y, and the relation defines a function. The variable x can represent any real number, so the domain is

$$\{x \mid x \text{ is a real number}\}, \quad \text{or} \quad (-\infty, \infty).$$

Since y is always 4 more than x, y also may be any real number, and so the range is $(-\infty, \infty)$.

(b) For any choice of x in the domain of $y = \sqrt{2x - 1}$, there is exactly one corresponding value for y (the radical is a nonnegative number), so this equation defines a function. Since the equation involves a square root, the quantity under the radical sign cannot be negative.

$$2x - 1 \geq 0 \quad \text{Solve the inequality. (Section 2.4)}$$

$$2x \geq 1 \quad \text{Add 1.}$$

$$x \geq \frac{1}{2} \quad \text{Divide by 2.}$$

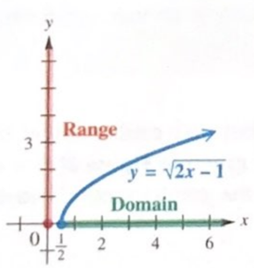


Figure 4

The domain of the function is $\left[\frac{1}{2}, \infty\right)$. Because the radical must represent a nonnegative number, as x takes values greater than or equal to $\frac{1}{2}$, the range is $\{y \mid y \geq 0\}$, or $[0, \infty)$. See **Figure 4.**

(c) The ordered pairs $(16, 4)$ and $(16, -4)$ both satisfy the equation $y^2 = x$. Since one value of x, 16, corresponds to two values of y, 4 and -4, this equation does not define a function.

Because x is equal to the square of y, the values of x must always be nonnegative. The domain of the relation is $[0, \infty)$. Any real number can be squared, so the range of the relation is $(-\infty, \infty)$. See **Figure 5.**

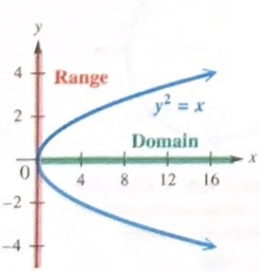


Figure 5

(d) By definition, y is a function of x if every value of x leads to exactly one value of y. Substituting a particular value of x, say 1, into $y \leq x - 1$ corresponds to many values of y. The ordered pairs

$$(1, 0), \quad (1, -1), \quad (1, -2), \quad (1, -3), \quad \text{and so on}$$

all satisfy the inequality, so y is not a function of x here. Any number can be used for x or for y, so the domain and the range of this relation are both the set of real numbers, $(-\infty, \infty)$.

(e) Given any value of x in the domain of

$$y = \frac{5}{x-1},$$

we find y by subtracting 1 from x, and then dividing the result into 5. This process produces exactly one value of y for each value in the domain, so this equation defines a function.

The domain of $y = \frac{5}{x-1}$ includes all real numbers except those that make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving for x.

$$x - 1 = 0$$

$$x = 1 \quad \text{Add 1. (Section 2.1)}$$

Thus, the domain includes all real numbers except 1, written as the interval $(-\infty, 1) \cup (1, \infty)$. Values of y can be positive or negative, but never 0, because a fraction cannot equal 0 unless its numerator is 0. Therefore, the range is the interval $(-\infty, 0) \cup (0, \infty)$, as shown in **Figure 6.**

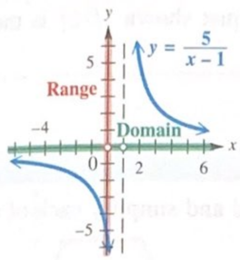


Figure 6

Variations of the Definition of Function

1. A **function** is a relation in which, for each distinct value of the first component of the ordered pairs, there is exactly one value of the second component.
2. A **function** is a set of ordered pairs in which no first component is repeated.
3. A **function** is a rule or correspondence that assigns exactly one range value to each distinct domain value.

LOOKING AHEAD TO CALCULUS

One of the most important concepts in calculus, that of the **limit of a function,** is defined using function notation:

$$\lim_{x \to a} f(x) = L$$

(read "the limit of $f(x)$ as x approaches a is equal to L") means that the values of $f(x)$ become as close as we wish to L when we choose values of x sufficiently close to a.

Function Notation When a function f is defined with a rule or an equation using x and y for the independent and dependent variables, we say, "y is a function of x" to emphasize that y *depends on* x. We use the notation

$$y = f(x),$$

called **function notation,** to express this and read $f(x)$ as **"f of x."** The letter f is the name given to this function.

For example, if $y = 3x - 5$, we can name the function f and write

$$f(x) = 3x - 5.$$

Note that $f(x)$ is just another name for the dependent variable y. For example, if $y = f(x) = 3x - 5$ and $x = 2$, then we find y, or $f(2)$, by replacing x with 2.

$$f(2) = 3 \cdot 2 - 5 \quad \text{Let } x = 2.$$

$$f(2) = 1 \quad \text{Multiply, and then subtract.}$$

The statement "In the function f, if $x = 2$, then $y = 1$" represents the ordered pair $(2, 1)$ and is abbreviated with function notation as follows.

$$f(2) = 1$$

The symbol $f(2)$ is read "f of 2" or "f at 2."

These ideas can be illustrated as follows.

Name of the function

Defining expression

$$y = f(x) = \overbrace{3x - 5}$$

Value of the function

Name of the independent variable

CAUTION ***The symbol $f(x)$ does not indicate "f times x,"*** but represents the y-value for the indicated x-value. As just shown, $f(2)$ is the y-value that corresponds to the x-value 2.

---

a) $f(x) = -4 + 10 - 3 = 3$

b) $f(q) = -q^2 + 5q - 3$

c) $g(a+1) = \cancel{g}\ 2(a+1) + 3$
$= 2a + 2 + 3$
$= 2a + 5$

HOMEWORK 3 **Using Function Notation**

Let $f(x) = -x^2 + 5x - 3$ and $g(x) = 2x + 3$. Find and simplify each of the following.

(a) $f(2)$ **(b)** $f(q)$ **(c)** $g(a + 1)$

Functions can be evaluated in a variety of ways, as shown in **Example 4.**

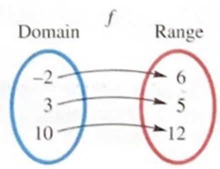
EXAMPLE 4 **Using Function Notation**

For each function, find $f(3)$.

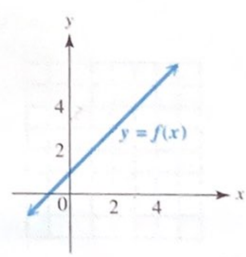
(a) $f(x) = 3x - 7$ $= 9 - 7 = 2$

(b) $f = \{(-3, 5), (0, 3), (3, 1), (6, -1)\}$

(c)



(d)



SOLUTION

(a) $f(x) = 3x - 7$

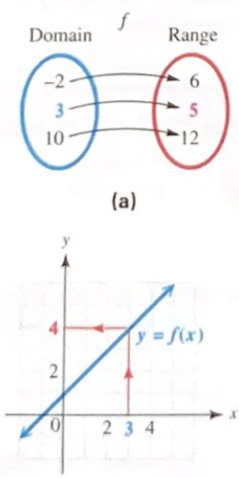
$f(3) = 3(3) - 7$ Replace x with 3.

$f(3) = 2$ Simplify.

(b) For $f = \{(-3, 5), (0, 3), (3, 1), (6, -1)\}$, we want $f(3)$, the y-value of the ordered pair where $x = 3$. As indicated by the ordered pair $(3, 1)$, when $x = 3$, $y = 1$, so $f(3) = 1$.

(c) In the mapping, repeated in **Figure 7(a),** the domain element 3 is paired with 5 in the range, so $f(3) = 5$.

(d) To evaluate $f(3)$ using the graph, find 3 on the x-axis. See **Figure 7(b).** Then move up until the graph of f is reached. Moving horizontally to the y-axis gives 4 for the corresponding y-value. Thus, $f(3) = 4$.



(a)

(b)

Figure 7

If a function f is defined by an equation with x and y (and not with function notation), use the following steps to find $f(x)$.

Finding an Expression for $f(x)$

Consider an equation involving x and y. Assume that y can be expressed as a function f of x. To find an expression for $f(x)$, use the following steps.

Step 1 Solve the equation for y.

Step 2 Replace y with $f(x)$.

HOMEWORK 4 **Writing Equations Using Function Notation**

Assume that y is a function f of x. Rewrite each equation using function notation. Then find $f(-2)$ and $f(a)$.

(a) $y = x^2 + 1$ **(b)** $x - 4y = 5$

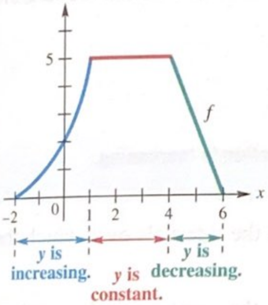


Figure 8

Increasing, Decreasing, and Constant Functions Informally speaking, a function *increases* on an interval of its domain if its graph rises from left to right on the interval. It *decreases* on an interval of its domain if its graph falls from left to right on the interval. It is *constant* on an interval of its domain if its graph is horizontal on the interval.

For example, consider **Figure 8.** The function increases on the interval $[-2, 1]$ because the y-values continue to get larger for x-values in that interval. Similarly, the function is constant on the interval $[1, 4]$ because the y-values are always 5 for all x-values there. Finally, the function decreases on the interval $[4, 6]$ because there the y-values continuously get smaller. ***The intervals refer to the x-values where the y-values either increase, decrease, or are constant.***

The formal definitions of these concepts follow.

Very important ✓ Comes at the exam as in the form of choices

Increasing, Decreasing, and Constant Functions

Suppose that a function f is defined over an interval I and x_1 and x_2 are in I.

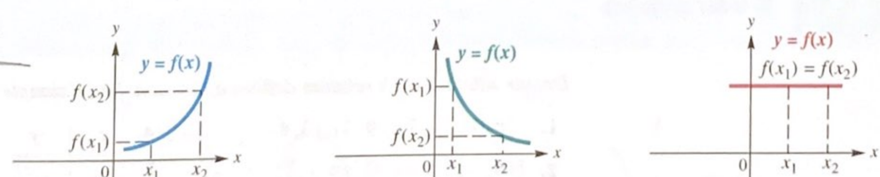
(a) f **increases** on I if, whenever $x_1 < x_2$, $f(x_1) < f(x_2)$.

(b) f **decreases** on I if, whenever $x_1 < x_2$, $f(x_1) > f(x_2)$.

(c) f is **constant** on I if, for every x_1 and x_2, $f(x_1) = f(x_2)$.

Figure 9 illustrates these ideas.

Ex. =



Whenever $x_1 < x_2$, and $f(x_1) < f(x_2)$, f is increasing.
(a)

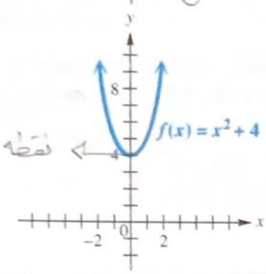
Whenever $x_1 < x_2$, and $f(x_1) > f(x_2)$, f is decreasing.
(b)

For every x_1 and x_2, if $f(x_1) = f(x_2)$, then f is constant.
(c)

Figure 9

NOTE To decide whether a function is increasing, decreasing, or constant on an interval, ask yourself, ***"What does y do as x goes from left to right?"***

لما نشتغل بالتزايد والتناقص
راح نشتغل في محور x



نقطة حرجة

$f(x) = x^2 + 4$ increases on $[0, \infty)$ and decreases on $(-\infty, 0]$.

Figure 10

There can be confusion regarding whether endpoints of an interval should be included when determining intervals over which a function is increasing or decreasing. For example, consider the graph of $y = f(x) = x^2 + 4$, shown in **Figure 10.** Is it increasing on $[0, \infty)$ or just on $(0, \infty)$?

The definition of increasing and decreasing allows us to include 0 as a part of the interval I over which this function is increasing, because if we let $x_1 = 0$, then $f(0) < f(x_2)$ whenever $0 < x_2$. Thus, $f(x) = x^2 + 4$ is increasing on $[0, \infty)$. A similar discussion can be used to show that this function is decreasing on $(-\infty, 0]$. Do not confuse these concepts by saying that f both increases and decreases at the point $(0, 0)$.

The concepts of increasing and decreasing functions apply to intervals of the domain, not to individual points.

It is not incorrect to say that $f(x) = x^2 + 4$ is increasing on $(0, \infty)$—there are infinitely many intervals over which it increases. However, we generally give the largest possible interval when determining where a function increases or decreases. (*Source:* Stewart J., *Calculus*, Fourth Edition, Brooks/Cole Publishing Company, p. 21.)

EXAMPLE 5 **Determining Intervals over Which a Function Is Increasing, Decreasing, or Constant**

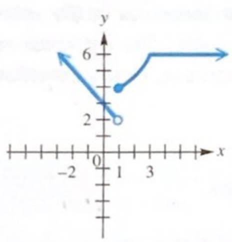


Figure 11

Figure 11 shows the graph of a function. Determine the intervals over which the function is increasing, decreasing, or constant.

SOLUTION We should ask, "What is happening to the y-values as the x-values are getting larger?" Moving from left to right on the graph, we see the following:

- On the interval $(-\infty, 1)$, the y-values are *decreasing*.
- On the interval $[1, 3]$, the y-values are *increasing*.
- On the interval $[3, \infty)$, the y-values are *constant* (and equal to 6).

Therefore, the function is decreasing on $(-\infty, 1)$, increasing on $[1, 3]$, and constant on $[3, \infty)$.

3.1 Exercises

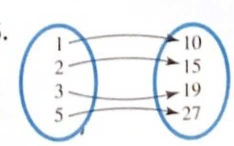
Decide whether each relation defines a function. ***See Example 1.***

1. $\{(8, 0), (5, 7), (9, 3), (3, 8)\}$
2. $\{(9, -2), (-3, 5), (9, 1)\}$
3. $\{(-12, 5), (-10, 3), (8, 3)\}$
4.

x	y
−4	$\sqrt{2}$
0	$\sqrt{2}$
4	$\sqrt{2}$

Decide whether each relation defines a function and give the domain and range. ***See Examples 1–2 and Homework 1–2.***

5. $\{(2, 5), (3, 7), (3, 9), (5, 11)\}$
6.



7.

x	y
0	0
1	-1
2	-2

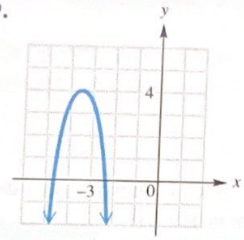
8. **Attendance at NCAA Women's College Basketball Games**

Season* (x)	Attendance (y)
2006	10,878,322
2007	11,120,822
2008	11,160,293
2009	11,134,738

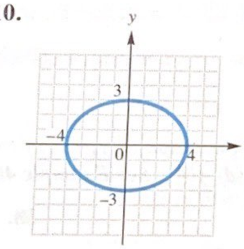
Source: NCAA.

*Each season overlaps the starting year (given) with the following year.

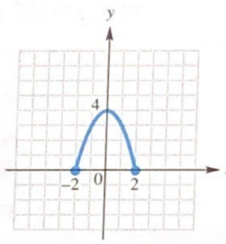
9.



10.



11.



Decide whether each relation defines y as a function of x. Give the domain and range. ***See Example 3.***

12. $y = x^3$ **13.** $x = y^4$ **14.** $y = -6x + 4$ **15.** $x - y < 4$

16. $y = -\sqrt{x}$ **17.** $xy = -6$ **18.** $y = \sqrt{7 - 2x}$ **19.** $y = \dfrac{-7}{x - 5}$

20. **Concept Check** Give an example of a function from everyday life. (*Hint:* Fill in the blanks: _____ depends on _____, so _____ is a function of _____.)

Let $f(x) = -3x + 4$ *and* $g(x) = -x^2 + 4x + 1$. *Find and simplify each of the following.* ***See Homework 3.***

21. $f(-3)$ **22.** $g(10)$ **23.** $f\left(-\dfrac{7}{3}\right)$ **24.** $g\left(-\dfrac{1}{4}\right)$

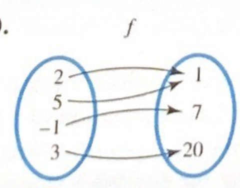
25. $g(k)$ **26.** $g(-x)$ **27.** $f(a + 4)$ **28.** $f(3t - 2)$

turm of

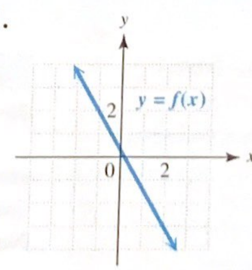
For each function, find **(a)** $f(2)$ *and* **(b)** $f(-1)$. ***See Example 4.***

29. $f = \{(2, 5), (3, 9), (-1, 11), (5, 3)\}$

30.



31.

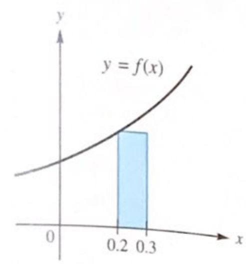


An equation that defines y as a function of x is given. **(a)** *Solve for y in terms of x and replace y with the function notation* $f(x)$. **(b)** *Find* $f(3)$. ***See Homework 4.***

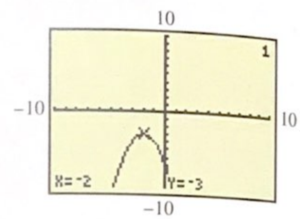
32. $x - 4y = 8$ **33.** $y - 3x^2 = 2 + x$ **34.** $-2x + 5y = 9$

Concept Check *Answer each question.*

35. The figure shows a portion of the graph of $f(x) = x^2 + 3x + 1$ and a rectangle with its base on the x-axis and a vertex on the graph. What is the area of the rectangle? (*Hint:* $f(0.2)$ is the height.)

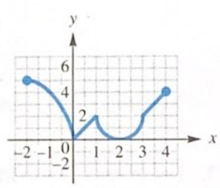


36. The graph of $Y_1 = f(X)$ is shown with a display at the bottom. What is $f(-2)$?



In Exercises 37–38, use the graph of $y = f(x)$ to find each function value: ***(a)*** $f(-2)$, ***(b)*** $f(0)$, ***(c)*** $f(1)$, *and* ***(d)*** $f(4)$. ***See Example 4(d).***

37.



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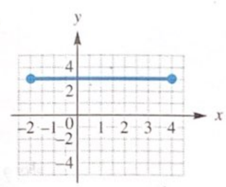
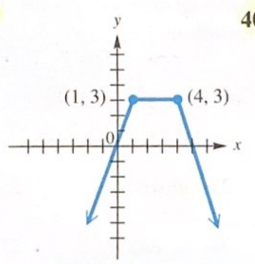


Determine the intervals of the domain for which each function is ***(a)*** *increasing,* ***(b)*** *decreasing, and* ***(c)*** *constant.* ***See Example 5.***

39.



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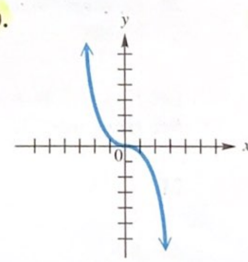

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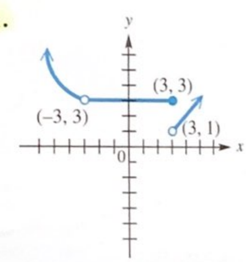


3.2 Equations of Lines and Linear Models

- Point-Slope Form
- Slope-Intercept Form
- Vertical and Horizontal Lines
- Parallel and Perpendicular Lines
- Modeling Data
- Solving Linear Equations in One Variable by Graphing

Point-Slope Form The graph of a linear function is a straight line. We now develop various forms for the equation of a line.

Figure 12 shows the line passing through the fixed point (x_1, y_1) having slope m. (Assuming that the line has a slope guarantees that it is not vertical.) Let (x, y) be any other point on the line. Since the line is not vertical, $x - x_1 \neq 0$. Now use the definition of slope.

$$m = \frac{y - y_1}{x - x_1} \qquad \text{Slope formula}$$

$$m(x - x_1) = y - y_1 \qquad \text{Multiply each side by } x - x_1.$$

Every example and homeworks are Very important.

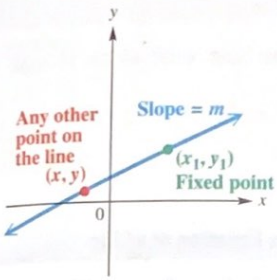


Figure 12

or

$$y - y_1 = m(x - x_1) \quad \text{Interchange sides.}$$

This result is the *point-slope form* of the equation of a line.

Point-Slope Form

The **point-slope form** of the equation of the line with slope m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

LOOKING AHEAD TO CALCULUS

A standard problem in calculus is to find the equation of the line tangent to a curve at a given point. The derivative is used to find the slope of the desired line, and then the slope and the given point are used in the point-slope form to solve the problem.

EXAMPLE 1 Using the Point-Slope Form (Given a Point and the Slope)

Write an equation of the line through $(-4, 1)$ having slope -3.

SOLUTION Here $x_1 = -4$, $y_1 = 1$, and $m = -3$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - 1 = -3[x - (-4)] \quad x_1 = -4,\ y_1 = 1,\ m = -3$$

Be careful with signs.

$$y - 1 = -3(x + 4)$$
$$y - 1 = -3x - 12 \quad \text{Distributive property (Section 1.2)}$$
$$y = -3x - 11 \quad \text{Add 1. (Section 2.1)}$$

* الميل يكون + إذا كان
الخط يصنع زاوية حادة مع الاتجاه
الموجب لمحور X
وإذا
* يصنع زاوية منفرجة يكون
الميل سالب.

HOMEWORK 1 Using the Point-Slope Form (Given Two Points)

Write an equation of the line through $(-3, 2)$ and $(2, -4)$. Write the result in standard form $Ax + By = C$.

NOTE The lines in **Example 1 and Homework 1** both have negative slopes. Keep in mind that a slope of the form $-\frac{A}{B}$ may be interpreted as either $\frac{-A}{B}$ or $\frac{A}{-B}$.

* Slope "m"

* The slope of verticle line is undefined.

* The slope of horizantal line equals 0

2

Slope-Intercept Form As a special case of the point-slope form of the equation of a line, suppose that a line passes through the point $(0, b)$, so the line has y-intercept b. If the line has slope m, then using the point-slope form with $x_1 = 0$ and $y_1 = b$ gives the following.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - b = m(x - 0) \quad x_1 = 0,\ y_1 = b$$
$$y = \underset{\text{Slope}}{m}x + \underset{y\text{-intercept}}{b} \quad \text{Solve for } y.$$

Since this result shows the slope of the line and the y-intercept, it is called the *slope-intercept form* of the equation of the line.

Slope-Intercept Form

The **slope-intercept form** of the equation of the line with slope m and y-intercept b is

$$y = mx + b.$$

EXAMPLE 2 Finding the Slope and y-Intercept from an Equation of a Line

Find the slope and y-intercept of the line with equation $4x + 5y = -10$.

SOLUTION Write the equation in slope-intercept form.

$$4x + 5y = -10$$

$$5y = -4x - 10 \quad \text{Subtract } 4x.$$

$$y = \underset{\substack{\uparrow \\ m}}{-\frac{4}{5}}x \underset{\substack{\uparrow \\ b}}{-\,2} \quad \text{Divide by 5.}$$

The slope is $-\frac{4}{5}$ and the y-intercept is -2.

NOTE Generalizing from **Example 2,** the slope m of the graph of

$$Ax + By = C$$

is $-\frac{A}{B}$, and the y-intercept b is $\frac{C}{B}$. , x-intercept is $\frac{C}{A}$

H.w 2:

$m = \frac{4-1}{2-1} = \frac{3}{1} = 3 \rightarrow m$

$b \rightarrow$ y-int

$y - y_1 = m(x - x_1)$ (1,1)

اختار هذه النقطة

$y - 1 = 3(x - 1)$

$y - 1 = 3x - 3$

$y = 3x - 2 \rightarrow b$

$y = mx + b$

$\boxed{y = 3x - 2}$

HOMEWORK 2 Using the Slope-Intercept Form (Given Two Points)

Write an equation of the line through $(1, 1)$ and $(2, 4)$. Then graph the line using the slope-intercept form.

Very important

EXAMPLE 3 Finding an Equation from a Graph

Use the graph of the linear function f shown in **Figure 13** to complete the following.

(a) Find the slope, y-intercept, and x-intercept.

(b) Write the equation that defines f.

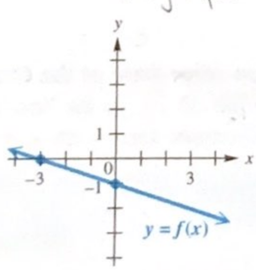


Figure 13

SOLUTION

(a) The line falls 1 unit each time the x-value increases by 3 units. Therefore, the slope is $\frac{-1}{3} = -\frac{1}{3}$. The graph intersects the y-axis at the point $(0, -1)$ and intersects the x-axis at the point $(-3, 0)$. Therefore, the y-intercept is -1 and the x-intercept is -3.

(b) The slope is $m = -\frac{1}{3}$, and the y-intercept is $b = -1$.

$$y = f(x) = mx + b \quad \text{Slope-intercept form}$$

$$f(x) = -\frac{1}{3}x - 1 \quad m = -\tfrac{1}{3},\ b = -1$$

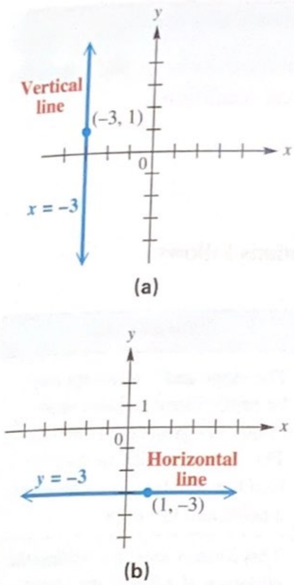


Figure 14

Vertical and Horizontal Lines We have seen graphs of vertical and horizontal lines elsewhere. The vertical line through the point (a, b) passes through all points of the form (a, y), for any value of y. Consequently, the equation of a vertical line through (a, b) is $x = a$. For example, the vertical line through $(-3, 1)$ has equation $x = -3$. See **Figure 14(a).** Since each point on the y-axis has x-coordinate 0, ***the equation of the y-axis is $x = 0$.***

The horizontal line through the point (a, b) passes through all points of the form (x, b), for any value of x. Therefore, the equation of a horizontal line through (a, b) is $y = b$. For example, the horizontal line through $(1, -3)$ has equation $y = -3$. See **Figure 14(b).** Since each point on the x-axis has y-coordinate 0, ***the equation of the x-axis is $y = 0$.***

Equations of Vertical and Horizontal Lines

An equation of the **vertical line** through the point (a, b) is $x = a$.
An equation of the **horizontal line** through the point (a, b) is $y = b$.

Parallel and Perpendicular Lines Since two parallel lines are equally "steep," they should have the same slope. Also, two distinct lines with the same "steepness" are parallel. The following result summarizes this discussion. (The statement "*p* if and only if *q*" means "if *p* then *q and* if *q* then *p*.")

$* \, m_1 = m_2 \rightarrow$ Parallel

$* \, m_1 = \frac{-1}{m_2}$ or $m_1 m_2 = -1 \rightarrow$ Perpendicular

Parallel Lines

Two distinct nonvertical lines are parallel if and only if they have the same slope.

When two lines have slopes with a product of -1, the lines are perpendicular.

Perpendicular Lines

Two lines, neither of which is vertical, are perpendicular if and only if their slopes have a product of -1. Thus, the slopes of perpendicular lines, neither of which is vertical, are *negative reciprocals*.

For example, if the slope of a line is $-\frac{3}{4}$, the slope of any line perpendicular to it is $\frac{4}{3}$, since $-\frac{3}{4}\left(\frac{4}{3}\right) = -1$. (Numbers like $-\frac{3}{4}$ and $\frac{4}{3}$ are **negative reciprocals** of each other.) A proof of this result is outlined in **Exercises 28–34.**

NOTE Because a vertical line has *undefined* slope, it does not follow the *mathematical* rules for parallel and perpendicular lines. We intuitively know that all vertical lines are parallel and that a vertical line and a horizontal line are perpendicular.

Very important ←

HOMEWORK 3 **Finding Equations of Parallel and Perpendicular Lines**

Write the equation in both slope-intercept and standard form of the line that passes through the point $(3, 5)$ and satisfies the given condition.

(a) parallel to the line $2x + 5y = 4$

(b) perpendicular to the line $2x + 5y = 4$

له اعكسه وغيري إشارته (الحل)

a) $m = \frac{-A}{B} = \frac{-2}{5}$

$y - y_1 = m(x - x_1)$

$y - 5 = \frac{-2}{5}(x - 3)$

$y - 5 = -\frac{2}{5}x + \frac{6}{5}$

$* y = \frac{-2}{5}x + \frac{31}{5}$ → Slope-Int-form.

$* \frac{2}{5}x + y = \frac{31}{5}$

$2x + 5y = 31$ → Standard form.

b) $m_1 = \frac{5}{2}$

$y - y_1 = m(x - x_1)$

$y - 5 = \frac{5}{2}(x - 3)$

$y - 5 = \frac{5}{2}x - \frac{15}{2}$

$y = \frac{5}{2}x - \frac{5}{2}$ → Slope-Int form

$y - \frac{5}{2}x = \frac{-5}{2}$

$2y - 5x = -5$

$-5x + 2y = -5$ → Standard form.

A summary of the various forms of linear equations follows.

Equation	Description	When to Use
$y = mx + b$	**Slope-Intercept Form** Slope is m. y-intercept is b.	The slope and y-intercept can be easily identified and used to quickly graph the equation. This form can also be used to find the equation of a line given a point and the slope.
$y - y_1 = m(x - x_1)$	**Point-Slope Form** Slope is m. Line passes through (x_1, y_1).	This form is ideal for finding the equation of a line if the slope and a point on the line or two points on the line are known.
$Ax + By = C$	**Standard Form** (If the coefficients and constant are rational, then A, B, and C are expressed as relatively prime integers, with $A \geq 0$.) Slope is $-\frac{A}{B}$ $(B \neq 0)$. x-intercept is $\frac{C}{A}$ $(A \neq 0)$. y-intercept is $\frac{C}{B}$ $(B \neq 0)$.	The x- and y-intercepts can be found quickly and used to graph the equation. The slope must be calculated.
$y = b$	**Horizontal Line** Slope is 0. y-intercept is b.	If the graph intersects only the y-axis, then y is the only variable in the equation.
$x = a$	**Vertical Line** Slope is undefined. x-intercept is a.	If the graph intersects only the x-axis, then x is the only variable in the equation.

ملخص ←

Modeling Data We can write equations of lines that mathematically describe, or model, real data if the data change at a fairly constant rate. In this case, the data fit a linear pattern, and the rate of change is the slope of the line.

Guidelines for Modeling

Step 1 Make a scatter diagram of the data.

Step 2 Find an equation that models the data. For a line, this involves selecting two data points and finding the equation of the line through them.

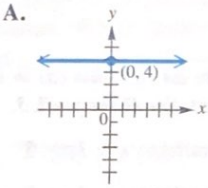
3.2 Exercises

Concept Check *Match each equation in Exercises 1–2 to the correct graph in A–D.*

1. $y = \frac{1}{4}x + 2$

2. $y - (-1) = \frac{3}{2}(x - 1)$

A.



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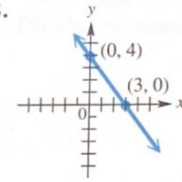

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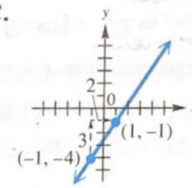
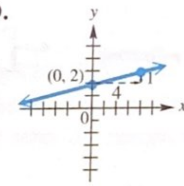


D.



In Exercises 3–13, write an equation for the line described. Give answers in standard form for Exercises 3–7 and in slope-intercept form (if possible) for Exercises 3–13. ***See Examples 1–2 and Homework 1–2.***

3. through $(1, 3)$, $m = -2$

4. through $(-5, 4)$, $m = -\frac{3}{2}$

5. through $(-8, 4)$, undefined slope

6. through $(5, -8)$, $m = 0$

7. through $(-1, 3)$ and $(3, 4)$

8. x-intercept 3, y-intercept -2

9. vertical, through $(-6, 4)$

10. horizontal, through $(-7, 4)$

11. $m = 5$, $b = 15$

12. through $(-2, 5)$ having slope -4

13. slope 0, y-intercept $\frac{3}{2}$

14. *Concept Check* Fill in each blank with the appropriate response: The line $x + 2 = 0$ has x-intercept ________. It ________ (does/does not) have a y-intercept. The slope of this line is ________ (0/undefined). The line $4y = 2$ has y-intercept ________. It ________ (does/does not) have an x-intercept. The slope of this line is ________ (0/undefined).

Give the slope and y-intercept of each line, and graph it. ***See Example 2.***

15. $y = 3x - 1$

16. $4x - y = 7$

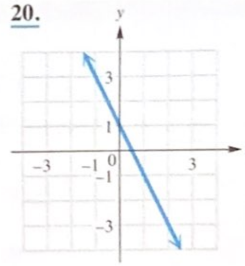
17. $4y = -3x$

18. $x + 2y = -4$

19. $y - \frac{3}{2}x - 1 = 0$

Connecting Graphs with Equations *The graph of a linear function f is shown.* ***(a)*** *Identify the slope, y-intercept, and x-intercept.* ***(b)*** *Write the equation that defines f.* ***See Example 3.***

20.



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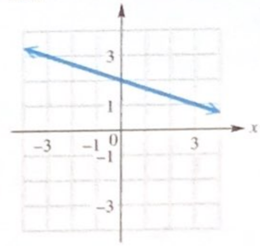

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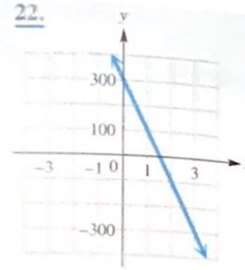

*In Exercises 23–26, write an equation (**a**) in standard form and (**b**) in slope-intercept form for the line described.* ***See Homework 3.***

23. through $(-1, 4)$, parallel to $x + 3y = 5$

24. through $(1, 6)$, perpendicular to $3x + 5y = 1$

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Very important

25. through $(4, 1)$, parallel to $y = -5$ = $y=1$

علشان عامودي

26. through $(-5, 6)$, perpendicular to $x = -2$ = $y=6$

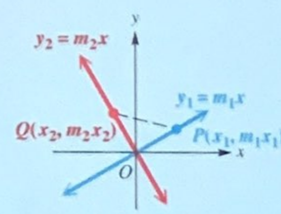
27. Find k so that the line through $(4, -1)$ and $(k, 2)$ is

(**a**) parallel to $3y + 2x = 6$; (**b**) perpendicular to $2y - 5x = 1$.

Relating Concepts

For individual or collaborative investigation *(Exercises 28–34)*

In this section we state that two lines, neither of which is vertical, are perpendicular if and only if their slopes have a product of -1. In Exercises 28–34, we outline a partial proof of this for the case where the two lines intersect at the origin. ***Work these exercises in order,*** *and refer to the figure as needed.*



By the converse of the Pythagorean theorem, if

$$[d(O, P)]^2 + [d(O, Q)]^2 = [d(P, Q)]^2,$$

then triangle POQ is a right triangle with right angle at O.

28. Find an expression for the distance $d(O, P)$.

29. Find an expression for the distance $d(O, Q)$.

30. Find an expression for the distance $d(P, Q)$.

31. Use your results from **Exercises 28–30,** and substitute into the equation from the Pythagorean theorem. Simplify to show that this leads to the equation

$$-2m_1m_2x_1x_2 - 2x_1x_2 = 0.$$

32. Factor $-2x_1x_2$ from the final form of the equation in **Exercise 31.**

33. Use the property that if $ab = 0$ then $a = 0$ or $b = 0$ to solve the equation in **Exercise 32,** showing that $m_1m_2 = -1$.

34. State your conclusion based on **Exercises 28–33.**

3.3 Function Operations and Composition

- Arithmetic Operations on Functions
- The Difference Quotient
- Composition of Functions and Domain

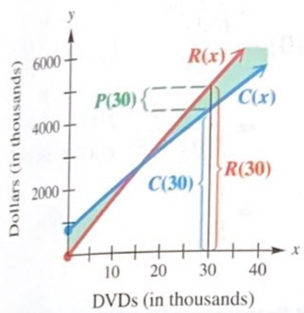


Figure 15

Arithmetic Operations on Functions

Figure 15 shows the situation for a company that manufactures DVDs. The two lines are the graphs of the linear functions for revenue $R(x) = 168x$ and cost $C(x) = 118x + 800$, where x is the number of DVDs produced and sold, and x, $R(x)$, and $C(x)$ are given in thousands. When 30,000 (that is, 30 thousand) DVDs are produced and sold, profit is found as follows.

$$P(x) = R(x) - C(x) \quad \text{Profit function}$$

$$P(30) = R(30) - C(30) \quad \text{Let } x = 30.$$

$$= 5040 - 4340 \quad R(30) = 168(30);\ C(30) = 118(30) + 800$$

$$P(30) = 700 \quad \text{Subtract.}$$

Thus, the profit from the sale of 30,000 DVDs is \$700,000.

The profit function is found by *subtracting* the cost function from the revenue function. New functions can be formed by using other operations as well.

Operations on Functions and Domains

Given two functions f and g, then for all values of x for which both $f(x)$ and $g(x)$ are defined, the functions $f + g$, $f - g$, fg, and $\frac{f}{g}$ are defined as follows.

$$D_f \cap D_g \begin{cases} (f + g)(x) = f(x) + g(x) & \text{Sum} \\ (f - g)(x) = f(x) - g(x) & \text{Difference} \\ (fg)(x) = f(x) \cdot g(x) & \text{Product} \end{cases}$$

$$D_f \cap D_g - \{\text{أصفار/ المقام}\} \leftarrow \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0 \quad \text{Quotient}$$

The **domains of $f + g$, $f - g$,** and **fg** include all real numbers in the intersection of the domains of f and g, while the **domain of $\frac{f}{g}$** includes those real numbers in the intersection of the domains of f and g for which $g(x) \neq 0$.

NOTE The condition $g(x) \neq 0$ in the definition of the quotient means that the domain of $\left(\frac{f}{g}\right)(x)$ is restricted to all values of x for which $g(x)$ is not 0. The condition does not mean that $g(x)$ is a function that is never 0.

EXAMPLE 1 Using Operations on Functions

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find each of the following.

(a) $(f + g)(1)$ **(b)** $(f - g)(-3)$ **(c)** $(fg)(5)$ **(d)** $\left(\frac{f}{g}\right)(0)$

SOLUTION

(a) First determine $f(1) = 2$ and $g(1) = 8$. Then use the definition.

$$\begin{aligned}(f+g)(1) &= f(1) + g(1) && (f+g)(x) = f(x) + g(x)\\ &= 2 + 8 && f(1) = 1^2 + 1;\ g(1) = 3(1) + 5\\ &= 10 && \text{Add.}\end{aligned}$$

(b)
$$\begin{aligned}(f-g)(-3) &= f(-3) - g(-3) && (f-g)(x) = f(x) - g(x)\\ &= 10 - (-4) && f(-3) = (-3)^2 + 1;\ g(-3) = 3(-3) + 5\\ &= 14 && \text{Subtract.}\end{aligned}$$

(c)
$$\begin{aligned}(fg)(5) &= f(5) \cdot g(5)\\ &= (5^2 + 1)(3 \cdot 5 + 5)\\ &= 26 \cdot 20\\ &= 520\end{aligned}$$

(d)
$$\begin{aligned}\left(\frac{f}{g}\right)(0) &= \frac{f(0)}{g(0)} && \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}\\ &= \frac{0^2 + 1}{3(0) + 5} && f(x) = x^2 + 1,\ g(x) = 3x + 5\\ &= \frac{1}{5} && \text{Simplify.}\end{aligned}$$

HOMEWORK 1 **Using Operations on Functions and Determining Domains**

Let $f(x) = 8x - 9$ and $g(x) = \sqrt{2x - 1}$. Find each function in (a)–(d).

(a) $(f+g)(x)$ **(b)** $(f-g)(x)$ **(c)** $(fg)(x)$ very important **(d)** $\left(\frac{f}{g}\right)(x)$

(e) Give the domains of the functions in parts (a)–(d).

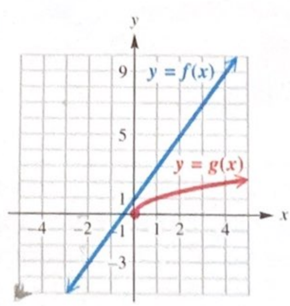
Very importanta

EXAMPLE 2 **Evaluating Combinations of Functions**

If possible, use the given representations of functions f and g to evaluate

$$(f+g)(4), \quad (f-g)(-2), \quad (fg)(1), \quad \text{and} \quad \left(\frac{f}{g}\right)(0).$$ undefined

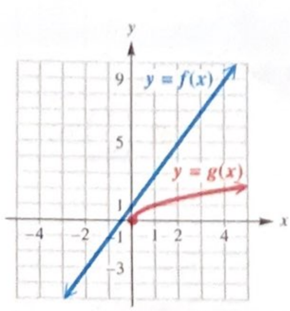
(a)



(b)

x	$f(x)$	$g(x)$
-2	-3	undefined
0	1	0
1	3	1
4	9	2

(c) $f(x) = 2x + 1, \quad g(x) = \sqrt{x}$



SOLUTION

(a) From the figure, repeated in the margin, $f(4) = 9$ and $g(4) = 2$.

$$\begin{aligned}(f+g)(4) &= f(4) + g(4) && (f+g)(x) = f(x) + g(x)\\ &= 9 + 2 && \text{Substitute.}\\ &= 11 && \text{Add.}\end{aligned}$$

For $(f-g)(-2)$, although $f(-2)=-3$, $g(-2)$ is undefined because -2 is not in the domain of g. Thus $(f-g)(-2)$ is undefined.

The domains of f and g include 1.

$$(fg)(1) = f(1)\cdot g(1) \quad (fg)(x)=f(x)\cdot g(x)$$
$$= 3\cdot 1 \quad \text{Substitute.}$$
$$= 3 \quad \text{Multiply.}$$

The graph of g includes the origin, so $g(0)=0$. Thus $\left(\frac{f}{g}\right)(0)$ is undefined.

x	$f(x)$	$g(x)$
-2	-3	undefined
0	1	0
1	3	1
4	9	2

(b) From the table, repeated in the margin, $f(4)=9$ and $g(4)=2$.

$$(f+g)(4) = f(4)+g(4) \quad (f+g)(x)=f(x)+g(x)$$
$$= 9+2 \quad \text{Substitute.}$$
$$= 11 \quad \text{Add.}$$

In the table, $g(-2)$ is undefined, and thus $(f-g)(-2)$ is also undefined.

$$(fg)(1) = f(1)\cdot g(1) \quad (fg)(x)=f(x)\cdot g(x)$$
$$= 3\cdot 1 \quad f(1)=3 \text{ and } g(1)=1$$
$$= 3 \quad \text{Multiply.}$$

The quotient function value $\left(\frac{f}{g}\right)(0)$ is undefined since the denominator, $g(0)$, equals 0.

(c) Using $f(x)=2x+1$ and $g(x)=\sqrt{x}$, we can find $(f+g)(4)$ and $(fg)(1)$. Since -2 is not in the domain of g, $(f-g)(-2)$ is not defined.

$$\begin{aligned}(f+g)(4) &= f(4)+g(4)\\ &= (2\cdot 4+1)+\sqrt{4}\\ &= 9+2\\ &= 11\end{aligned} \qquad \begin{aligned}(fg)(1) &= f(1)\cdot g(1)\\ &= (2\cdot 1+1)\cdot\sqrt{1}\\ &= 3(1)\\ &= 3\end{aligned}$$

$\left(\frac{f}{g}\right)(0)$ is undefined since $g(0)=0$.

The Difference Quotient Suppose the point P lies on the graph of $y=f(x)$ as in **Figure 16,** and suppose h is a positive number. If we let $(x, f(x))$ denote the coordinates of P and let $(x+h, f(x+h))$ denote the coordinates of Q, then the line joining P and Q has slope as follows.

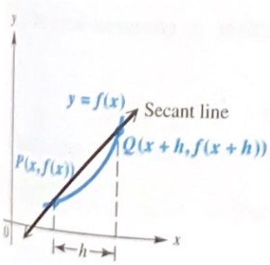


Figure 16

$$m = \frac{f(x+h)-f(x)}{(x+h)-x} \quad \text{Slope formula}$$

$$= \boldsymbol{\frac{f(x+h)-f(x)}{h}}, \quad h\neq 0 \quad \text{Difference quotient}$$

This boldface expression is called the **difference quotient.**

Figure 16 shows the graph of the line PQ (called a **secant line**). As h approaches 0, the slope of this secant line approaches the slope of the line tangent to the curve at P. Important applications of this idea are developed in calculus.

H.W

HOMEWORK 2 Finding the Difference Quotient

Let $f(x) = 2x^2 - 3x$. Find and simplify the expression for the difference quotient,

$$\frac{f(x+h) - f(x)}{h}.$$

CAUTION In **Home Work 2,** notice that the expression $f(x+h)$ is not equivalent to $f(x) + f(h)$.

$$f(x+h) = 2(x+h)^2 - 3(x+h) = 2x^2 + 4xh + 2h^2 - 3x - 3h$$

$$f(x) + f(h) = (2x^2 - 3x) + (2h^2 - 3h) = 2x^2 - 3x + 2h^2 - 3h$$

These expressions differ by $4xh$. In general, for a function f, $f(x+h)$ is *not* equivalent to $f(x) + f(h)$.

LOOKING AHEAD TO CALCULUS

The difference quotient is essential in the definition of the **derivative of a function** in calculus. The derivative provides a formula, in function form, for finding the slope of the tangent line to the graph of the function at a given point.

To illustrate, it is shown in calculus that the derivative of $f(x) = x^2 + 3$ is given by the function $f'(x) = 2x$. Now, $f'(0) = 2(0) = 0$, meaning that the slope of the tangent line to $f(x) = x^2 + 3$ at $x = 0$ is 0, which implies that the tangent line is horizontal. If you draw this tangent line, you will see that it is the line $y = 3$, which is indeed a horizontal line.

Composition of Functions and Domain The diagram in **Figure 17** shows a function f that assigns to each x in its domain a value $f(x)$. Then another function g assigns to each $f(x)$ in its domain a value $g(f(x))$. This two-step process takes an element x and produces a corresponding element $g(f(x))$.

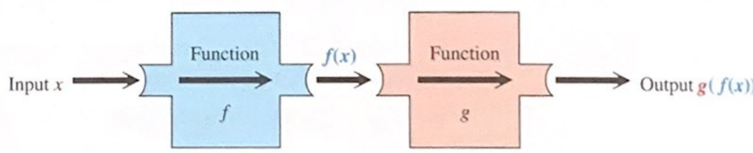


Figure 17

The function with y-values $g(f(x))$ is called the *composition* of functions g and f, which is written $g \circ f$ and read "**g of f**."

Composition of Functions and Domain

If f and g are functions, then the **composite function,** or **composition,** of g and f is defined by

$$(g \circ f)(x) = g(f(x)).$$ $D_{g\circ f} = D_g \cap D_f$

The **domain of $g \circ f$** is the set of all numbers x in the domain of f such that $f(x)$ is in the domain of g. $(f\circ g)(x) = f(g(x))$ $D_{f\circ g} = D_f \cap D_g$

As a real-life example of how composite functions occur, consider the following retail situation:

A \$40 pair of blue jeans is on sale for 25% off. If you purchase the jeans before noon, the retailer offers an additional 10% off. What is the final sale price of the blue jeans?

You might be tempted to say that the jeans are 35% off and calculate $\$40(0.35) = \$14$, giving a final sale price of

$$\$40 - \$14 = \$26$$

for the jeans. ***This is not correct.*** To find the final sale price, we must first find the price after taking 25% off and then take an additional 10% off *that* price. See **Figure 18.**

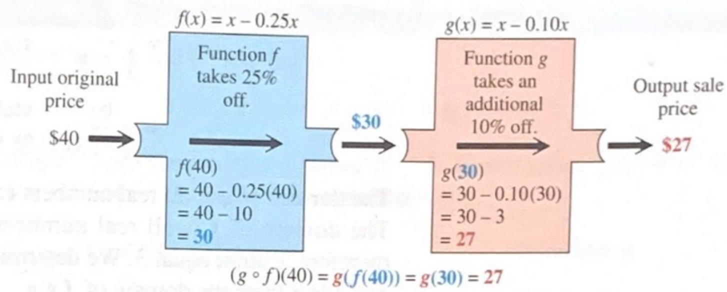


Figure 18

Important ←

EXAMPLE 3 Evaluating Composite Functions

Let $f(x) = 2x - 1$ and $g(x) = \frac{4}{x-1}$.

(a) Find $(f \circ g)(2)$. **(b)** Find $(g \circ f)(-3)$.

SOLUTION

(a) First find $g(2)$: $g(2) = \dfrac{4}{2-1} = \dfrac{4}{1} = 4.$

Now find $(f \circ g)(2)$.

$$\begin{aligned}(f \circ g)(2) &= f(g(2)) && \text{Definition of composition}\\ &= f(4) && \text{See above.}\\ &= 2(4) - 1 && \text{Definition of } f\\ &= 7 && \text{Simplify.}\end{aligned}$$

(b)
$$\begin{aligned}(g \circ f)(-3) &= g(f(-3)) && \text{Definition of composition}\\ &= g[2(-3) - 1] && f(-3) = 2(-3) - 1\\ &= g(-7) && 2(-3) - 1 = -7\\ &= \frac{4}{-7-1} && g(x) = \tfrac{4}{x+1}\\ &= \frac{4}{-8}, \text{ or } -\frac{1}{2} && \text{Simplify.}\end{aligned}$$

Very important —

HOMEWORK 3 Determining Composite Functions and Their Domains

Given that $f(x) = \sqrt{x}$ and $g(x) = 4x + 2$, find each of the following.

(a) $(f \circ g)(x)$ and its domain **(b)** $(g \circ f)(x)$ and its domain

EXAMPLE 4 **Determining Composite Functions and Their Domains**

Given that $f(x) = \frac{6}{x-3}$ and $g(x) = \frac{1}{x}$, find each of the following.

(a) $(f \circ g)(x)$ and its domain **(b)** $(g \circ f)(x)$ and its domain

SOLUTION

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) \qquad g(x) = \tfrac{1}{x}$$

$$= \frac{6}{\frac{1}{x} - 3} \qquad f(x) = \tfrac{6}{x-3}$$

$$= \frac{6x}{1-3x} \qquad \text{Multiply the numerator and denominator by } x. \text{ (Section 1.5)}$$

The domain of g is all real numbers *except* 0, which makes $g(x)$ undefined. The domain of f is all real numbers *except* 3. The expression for $g(x)$, therefore, cannot equal 3. We determine the value that makes $g(x) = 3$ and *exclude* it from the domain of $f \circ g$.

$$\frac{1}{x} = 3 \qquad \text{The solution must be excluded.}$$

$$1 = 3x \qquad \text{Multiply by } x.$$

$$x = \frac{1}{3} \qquad \text{Divide by 3.}$$

Therefore, the domain of $f \circ g$ is the set of all real numbers *except* 0 and $\frac{1}{3}$, written in interval notation as

$$(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right).$$

(b) $$(g \circ f)(x) = g(f(x)) = g\left(\frac{6}{x-3}\right)$$

$$= \frac{1}{\frac{6}{x-3}} \qquad \text{Note that this is meaningless if } x = 3.$$

$$= \frac{x-3}{6} \qquad \frac{1}{\frac{a}{b}} = 1 \div \frac{a}{b} = 1 \cdot \frac{b}{a} = \frac{b}{a}$$

The domain of f is all real numbers *except* 3, and the domain of g is all real numbers *except* 0. The expression for $f(x)$, which is $\frac{6}{x-3}$, is never zero, since the numerator is the nonzero number 6. Therefore, the domain of $g \circ f$ is the set of all real numbers *except* 3, written

$$(-\infty, 3) \cup (3, \infty).$$

LOOKING AHEAD TO CALCULUS

Finding the derivative of a function in calculus is called **differentiation.** To differentiate a composite function such as $h(x) = (3x+2)^4$, we interpret $h(x)$ as $(f \circ g)(x)$, where $g(x) = 3x + 2$ and $f(x) = x^4$. The **chain rule** allows us to differentiate composite functions. Notice the use of the composition symbol and function notation in the following, which comes from the chain rule.

If $h(x) = (f \circ g)(x)$, then
$h'(x) = f'(g(x)) \cdot g'(x)$.

NOTE In a situation like **Example 4(b),** it often helps to consider the *unsimplified* form of the composition expression when determining the domain.

HOMEWORK 4 **Showing That $(g \circ f)(x)$ Is Not Equivalent to $(f \circ g)(x)$**

Let $f(x) = 4x + 1$ and $g(x) = 2x^2 + 5x$. Show that $(g \circ f)(x) \neq (f \circ g)(x)$. (This is sufficient to prove that this inequality is true in general.)

As **Home Work 4** shows, ***it is not always true that*** $f \circ g = g \circ f$. In fact, the composite functions $f \circ g$ and $g \circ f$ are equal only for a special class of functions.

In calculus it is sometimes necessary to treat a function as a composition of two functions. The next example shows how this can be done.

EXAMPLE 5 Finding Functions That Form a Given Composite

Find functions f and g such that

$$(f \circ g)(x) = (x^2 - 5)^3 - 4(x^2 - 5) + 3.$$

SOLUTION Note the repeated quantity $x^2 - 5$. If we choose $g(x) = x^2 - 5$ and $f(x) = x^3 - 4x + 3$, then we have the following.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{By definition} \\ &= f(x^2 - 5) && g(x) = x^2 - 5 \\ &= (x^2 - 5)^3 - 4(x^2 - 5) + 3 && \text{Use the rule for } f. \end{aligned}$$

There are other pairs of functions f and g that also satisfy these conditions. Here is another such pair.

$$f(x) = (x - 5)^3 - 4(x - 5) + 3 \quad \text{and} \quad g(x) = x^2$$

3.3 Exercises

Let $f(x) = x^2 + 3$ *and* $g(x) = -2x + 6$. *Find each of the following.* ***See Example 1.***

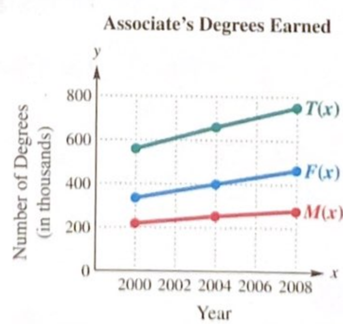
1. $(f + g)(3)$
2. $(f - g)(-1)$
3. $(fg)(4)$
4. $\left(\frac{f}{g}\right)(-1)$

For the pair of functions defined, find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, *and* $\left(\frac{f}{g}\right)(x)$. *Give the domain of each.* ***See Homework 1.***

5. $f(x) = 3x + 4$, $g(x) = 2x - 5$
6. $f(x) = 2x^2 - 3x$, $g(x) = x^2 - x + 3$
7. $f(x) = \sqrt{4x - 1}$, $g(x) = \frac{1}{x}$

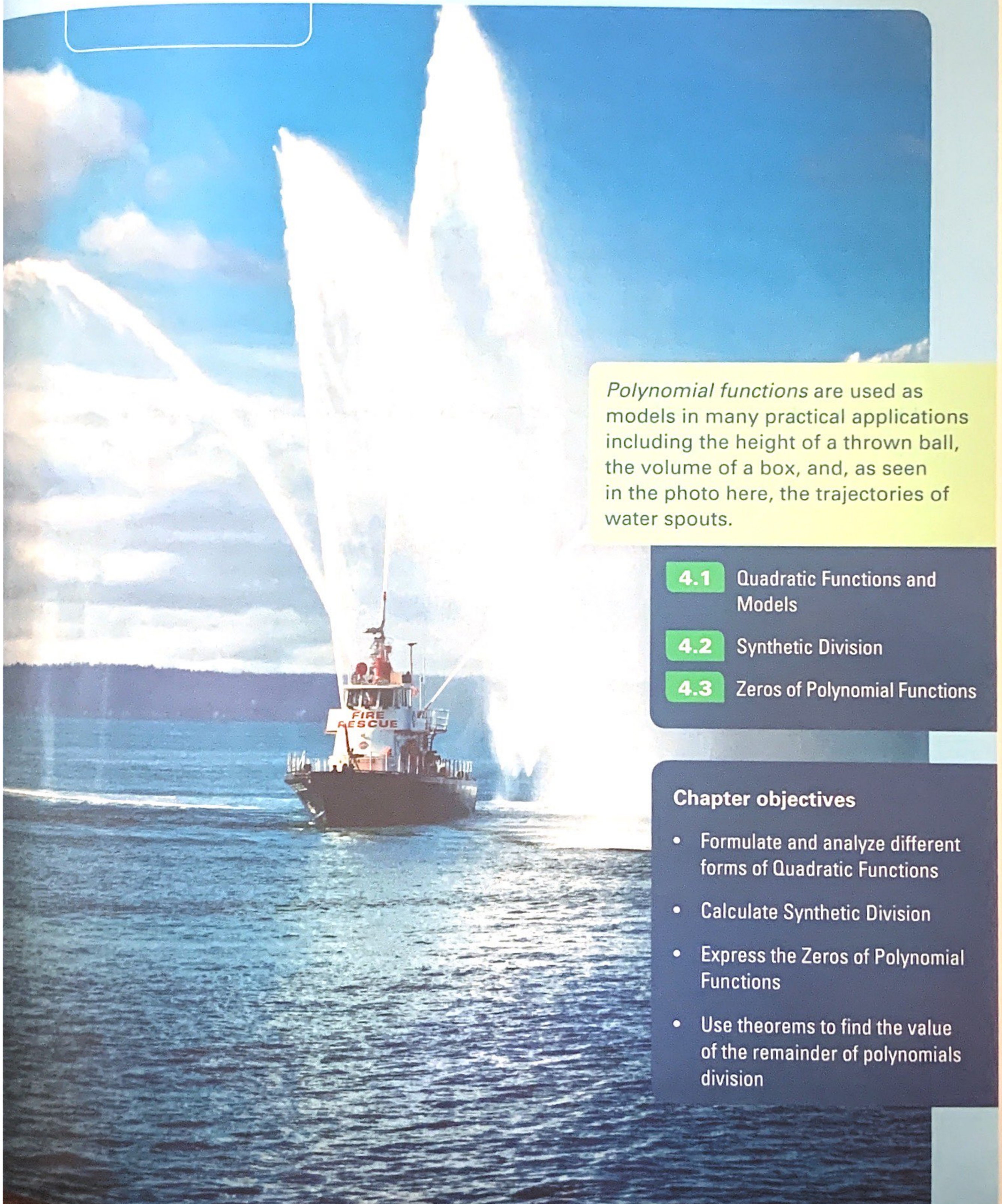
Associate's Degrees Earned *The graph shows the number of associate's degrees earned (in thousands) in the United States from 2000 through 2008.* $M(x)$ *gives the number of degrees earned by males,* $F(x)$ *gives the number earned by females, and* $T(x)$ *gives the total number for both groups. Use the graph in Exercises 8–9.*

8. Estimate $M(2004)$ and $F(2004)$, and use your results to estimate $T(2004)$.
9. Use the slopes of the line segments to decide in which period (2000–2004 or 2004–2008) the total number of associate's degrees earned increased more rapidly.



Source: U.S. National Center for Education Statistics.

4 Polynomial and Rational Functions



Polynomial functions are used as models in many practical applications including the height of a thrown ball, the volume of a box, and, as seen in the photo here, the trajectories of water spouts.

Chapter objectives

- Formulate and analyze different forms of Quadratic Functions
- Calculate Synthetic Division
- Express the Zeros of Polynomial Functions
- Use theorems to find the value of the remainder of polynomials division

4.1 Quadratic Functions and Models

- Quadratic Functions
- Graphing Techniques
- Completing the Square
- The Vertex Formula
- Quadratic Models

LOOKING AHEAD TO CALCULUS

In calculus, polynomial functions are used to approximate more complicated functions. For example, the trigonometric function $\sin x$ is approximated by the polynomial

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}.$$ Important

A *polynomial function* is defined as follows.

Polynomial Function

A **polynomial function** f of degree n, where n is a nonnegative integer, is given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1$, and a_0 are real numbers, with $a_n \neq 0$.

When we are analyzing a polynomial function, the degree n and the **leading coefficient** a_n play an important role. These are both given in the **leading term** $a_n x^n$. The table provides examples.

Polynomial Function	Function Name	Degree n	Leading Coefficient a_n
$f(x) = 2$	Constant	0	2
$f(x) = 5x - 1$	Linear	1	5
$f(x) = 4x^2 - x + 1$	Quadratic	2	4
$f(x) = 2x^3 - \frac{1}{2}x + 5$	Cubic	3	2
$f(x) = x^4 + \sqrt{2}x^3 - 3x^2$	Quartic	4	1

The function $f(x) = 0$ is the **zero polynomial** and has no degree.

Quadratic Functions Earlier we discussed constant and linear polynomial functions. Polynomial functions of degree 2 are *quadratic functions*.

Quadratic Function

A function f is a **quadratic function** if

$$f(x) = ax^2 + bx + c,$$ → General form.

where a, b, and c are real numbers, with $a \neq 0$.

The simplest quadratic function is

$$f(x) = x^2, \quad \text{Squaring function}$$

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

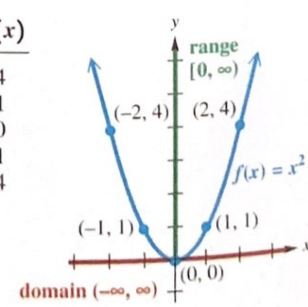


Figure 1

as shown in **Figure 1.** This graph is a **parabola.** Every quadratic function defined over the real numbers has a graph that is a parabola.

The domain of $f(x) = x^2$ is $(-\infty, \infty)$, and the range is $[0, \infty)$. The lowest point on the graph occurs at the origin $(0, 0)$. Thus, the function decreases on the interval $(-\infty, 0]$ and increases on the interval $[0, \infty)$. (Remember that these intervals indicate x-values.)

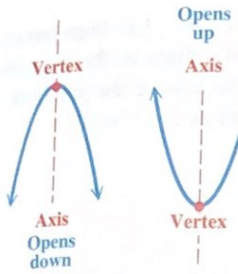


Figure 2

Parabolas are symmetric with respect to a line (the y-axis in **Figure 1**). This line is the **axis of symmetry,** or **axis,** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola. As **Figure 2** shows, the vertex of a parabola that opens down is the highest point of the graph, and the vertex of a parabola that opens up is the lowest point of the graph.

Graphing Techniques The graphing techniques may be applied to the graph of $f(x) = x^2$ to give the graph of *any* quadratic function. Compared to the basic graph of $f(x) = x^2$, the graph of $F(x) = a(x-h)^2 + k$ has the following characteristics.

$$F(x) = a(x-h)^2 + k$$ → Standard form.

	Horizontal shift:	Vertical shift:
• Opens up if $a > 0$		
• Opens down if $a < 0$	• h units right if $h > 0$	• k units up if $k > 0$
• Vertically stretched (narrower) if $\lvert a \rvert > 1$	• $\lvert h \rvert$ units left if $h < 0$	• $\lvert k \rvert$ units down if $k < 0$
• Vertically shrunk (wider) if $0 < \lvert a \rvert < 1$		

EXAMPLE 1 **Graphing Quadratic Functions**

$4 - 4(2) - 2$
$4 - 8 - 2$
$4 - 10$
$= -6$

Graph each function. Give the domain and range.

(a) $f(x) = x^2 - 4x - 2$

(b) $g(x) = -\frac{1}{2}x^2$ (and compare to $y = x^2$ and $y = \frac{1}{2}x^2$)

(c) $F(x) = -\frac{1}{2}(x-4)^2 + 3$ (and compare to the graph in part (b))

فقدري تجيبيها بدون رسم
لأنها في الصورة القياسية

SOLUTION

(a) See the table with **Figure 3**. The domain of $f(x) = x^2 - 4x - 2$ is $(-\infty, \infty)$, the range is $[-6, \infty)$, the vertex is $(2, -6)$, and the axis has equation $x = 2$.

يجي بالاختبار

* كل ما كانت قيمة
الـ x قريبة
من الصفر كل
ما كانت الرسمة
واضحة وكل ما
نبعد عنها راح تضيع.

x	$f(x)$
−1	3
0	−2
1	−5
2	−6
3	−5
4	−2
5	3

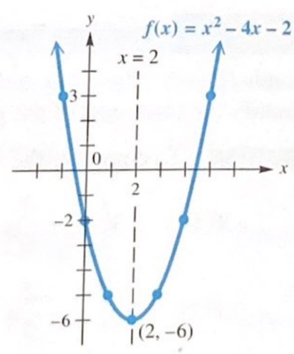


Figure 3

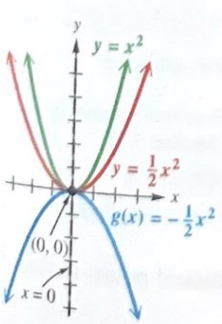


Figure 4

(b) Think of $g(x) = -\frac{1}{2}x^2$ as $g(x) = -\left(\frac{1}{2}x^2\right)$. The graph of $y = \frac{1}{2}x^2$ is a wider version of the graph of $y = x^2$, and the graph of $g(x) = -\left(\frac{1}{2}x^2\right)$ is a reflection of the graph of $y = \frac{1}{2}x^2$ across the x-axis. See **Figure 4**. The vertex is $(0,0)$, and the axis of the parabola is the line $x = 0$ (the y-axis). The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0]$.

(c) Notice that $F(x) = -\frac{1}{2}(x-4)^2 + 3$ is related to $g(x) = -\frac{1}{2}x^2$ from part (b). The graph of $F(x)$ is the graph of $g(x)$ translated 4 units to the right and 3 units up. See **Figure 5.** The vertex is (4, 3) and the axis of the parabola is the line $x = 4$. The domain is $(-\infty, \infty)$, and the range is $(-\infty, 3]$.

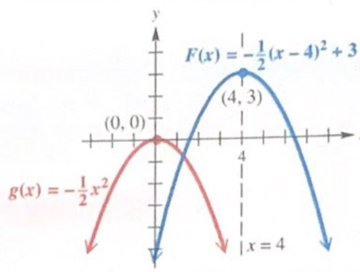


Figure 5

Completing the Square In general, the graph of the quadratic function

$$f(x) = a(x-h)^2 + k$$

is a parabola with *vertex* (h, k) and *axis* $x = h$. The parabola opens up if a is positive and down if a is negative. With these facts in mind, we *complete the square* to graph the general quadratic function

$$f(x) = ax^2 + bx + c.$$

إذا جاء أي سؤال بهذه الطريقة جاوبه بطريقة الـ vertex → أحصل من طريقة إكمال المربع

نجيب نصها وبعدين نربعها و نضيفها لجوا القوس وبرا القوس

$(x^2 - 6x) + 7$

$= (x^2 - 6x + 9) - 9 + 7$

$f(x) = (x-3)^2 - 2$

Vertex: $(3, -2)$

axis of sym = $x = 3$

$D = \mathbb{R}$

$R =$

HOMEWORK 1 Graphing a Parabola by Completing the Square ($a = 1$)

Graph $f(x) = x^2 - 6x + 7$ by completing the square and locating the vertex. Find the intervals over which the function is increasing or decreasing. $(3, -2)$

$h = \frac{-b}{2a} = \frac{6}{2(1)} = 3$ $\quad k = 3^2 - 6(3) + 7 = 9 - 18 + 7 = 9 - 11 = -2$ $\quad h = 3$

$D: \mathbb{R}$

$R = [-2, \infty)$

NOTE In **Homework 1** we added and subtracted 9 *on the same side* of the equation to complete the square. This differs from adding the same number to *each side of the equation,* as when we completed the square in **Section 1.3.** Since we want $f(x)$ (or y) alone on one side of the equation, we adjusted that step in the process of completing the square.

EXAMPLE 2 Graphing a Parabola by Completing the Square ($a \neq 1$)

Graph $f(x) = -3x^2 - 2x + 1$ by completing the square and locating the vertex. Identify the intercepts of the graph.

SOLUTION To complete the square, the coefficient of x^2 must be 1.

$$f(x) = -3\left(x^2 + \frac{2}{3}x \qquad \right) + 1$$ Factor -3 from the first two terms.

$$f(x) = -3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right) + 1$$ $\left[\frac{1}{2}\left(\frac{2}{3}\right)\right]^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$, so add and subtract $\frac{1}{9}$.

$$f(x) = -3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - 3\left(-\frac{1}{9}\right) + 1$$ Distributive property **(Section 1.2)**

Be careful here.

$$f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}$$ Factor and simplify.

The vertex is $\left(-\frac{1}{3}, \frac{4}{3}\right)$. The intercepts are good additional points to find. The y-intercept is found by evaluating $f(0)$.

$$f(0) = -3(0)^2 - 2(0) + 1 = 1 \longleftarrow \text{The } y\text{-intercept is 1.}$$

The x-intercepts are found by setting $f(x)$ equal to 0 and solving for x.

$$0 = -3x^2 - 2x + 1 \qquad \text{Set } f(x) = 0.$$

$$0 = 3x^2 + 2x - 1 \qquad \text{Multiply by } -1.$$

$$0 = (3x - 1)(x + 1) \qquad \text{Factor.}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -1 \qquad \text{Zero-factor property (Section 2.3)}$$

Therefore, the x-intercepts are $\frac{1}{3}$ and -1. The graph is shown in **Figure 6.**

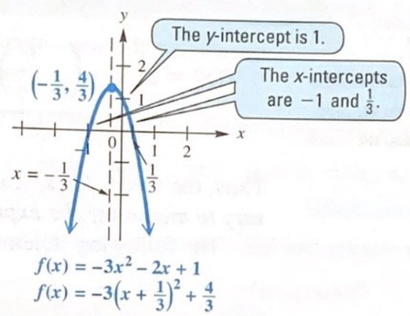


Figure 6

NOTE It is possible to reverse the process of **Example 2** and write the quadratic function from its graph if the vertex and any other point on the graph are known. Since quadratic functions take the form

$$f(x) = a(x - h)^2 + k,$$

substitute the x- and y-values of the vertex, $\left(-\frac{1}{3}, \frac{4}{3}\right)$, for h and k, respectively.

$$f(x) = a\left[x - \left(-\frac{1}{3}\right)\right]^2 + \frac{4}{3} \qquad \text{Let } h = -\tfrac{1}{3} \text{ and } k = \tfrac{4}{3}.$$

$$f(x) = a\left(x + \frac{1}{3}\right)^2 + \frac{4}{3} \qquad \text{Simplify.}$$

Now find the value of a by substituting the x- and y-coordinates of any other point on the graph, say $(0, 1)$, into this equation and solving for a.

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أي قيمة في الرسمة

$$1 = a\left(0 + \frac{1}{3}\right)^2 + \frac{4}{3} \qquad \text{Let } x = 0 \text{ and } y = 1.$$

$$1 = a\left(\frac{1}{9}\right) + \frac{4}{3} \qquad \text{Square.}$$

$$-\frac{1}{3} = \frac{1}{9}a \qquad \text{Subtract } \tfrac{4}{3}.$$

$$a = -3 \qquad \text{Multiply by 9. Interchange sides.}$$

Verify in **Example 2** that the vertex form of the function is

$$f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}.$$

In the Exercise set, problems of this type are labeled ***Connecting Graphs with Equations***.

LOOKING AHEAD TO CALCULUS

An important concept in calculus is the **definite integral.** If the graph of f lies above the x-axis, the symbol

$$\int_a^b f(x)\,dx$$

represents the area of the region above the x-axis and below the graph of f from $x = a$ to $x = b$. For example, in **Figure 6** with

$$f(x) = -3x^2 - 2x + 1,$$

$a = -1$, and $b = \frac{1}{3}$, calculus provides the tools for determining that the area enclosed by the parabola and the x-axis is $\frac{32}{27}$ (square units).

The Vertex Formula We can generalize the earlier work to obtain a formula for the vertex of a parabola.

$$f(x) = ax^2 + bx + c \qquad \text{General quadratic form}$$

$$= a\left(x^2 + \frac{b}{a}x \qquad \right) + c \qquad \text{Factor } a \text{ from the first two terms.}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \qquad \text{Add } \left[\tfrac{1}{2}\left(\tfrac{b}{a}\right)\right]^2 = \tfrac{b^2}{4a^2} \text{ inside the parentheses. Subtract } a\left(\tfrac{b^2}{4a^2}\right) \text{ outside the parentheses.}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \qquad \text{Factor and simplify.}$$

$$f(x) = a\Big[x - \underbrace{\left(-\frac{b}{2a}\right)}_{h}\Big]^2 + \underbrace{\frac{4ac - b^2}{4a}}_{k} \qquad \text{Vertex form of } f(x) = a(x-h)^2 + k$$

Thus, the vertex (h, k) can be expressed in terms of a, b, and c. ***It is not necessary to memorize the expression for k, since it is equal to $f(h) = f\left(-\frac{b}{2a}\right)$.***

The following statements summarize this discussion.

Graph of a Quadratic Function

The quadratic function defined by $f(x) = ax^2 + bx + c$ can be written as

$$y = f(x) = a(x - h)^2 + k, \quad a \neq 0,$$

where $h = -\frac{b}{2a}$ and $k = f(h)$. Vertex formula

The graph of f has the following characteristics.

1. It is a parabola with vertex (h, k) and the vertical line $x = h$ as axis.
2. It opens up if $a > 0$ and down if $a < 0$.
3. It is wider than the graph of $y = x^2$ if $|a| < 1$ and narrower if $|a| > 1$.
4. The y-intercept is $f(0) = c$.
5. The x-intercepts are found by solving the equation $ax^2 + bx + c = 0$.
   - If $b^2 - 4ac > 0$, the x-intercepts are $\frac{\overset{-2}{-b} \pm \sqrt{b^2 - 4ac}}{2a}$.
   - If $b^2 - 4ac = 0$, the x-intercept is $-\frac{b}{2a}$. $= 1$
   - If $b^2 - 4ac < 0$, there are no x-intercepts. $= 0$

$h = \frac{-b}{2a} = \frac{-4}{2(2)} = 1$

$K = 2(1)^2 + 4(1) + 5$
$= 2 + 4 + 5$
$= 11$

Vertex $= (1, 11)$

axis $= x = h =$ خط

$\underline{x = 1}$

HOMEWORK 2 **Using the Vertex Formula**

Find the axis and vertex of the parabola having equation $f(x) = 2x^2 + 4x + 5$.

Quadratic Models Since the vertex of a vertical parabola is the highest or lowest point on the graph, equations of the form

$$y = ax^2 + bx + c$$

are important in certain problems where we must find the maximum or minimum value of some quantity.

- When $a < 0$, the y-coordinate of the vertex gives the maximum value of y.
- When $a > 0$, the y-coordinate of the vertex gives the minimum value of y.

The x-coordinate of the vertex tells *where* the maximum or minimum value occurs.

If air resistance is neglected, the height s (in feet) of an object projected directly upward from an initial height s_0 feet with initial velocity v_0 feet per second is

$$s(t) = -16t^2 + v_0t + s_0,$$

where t is the number of seconds after the object is projected. The coefficient of t^2 (that is, -16) is a constant based on the gravitational force of Earth. This constant is different on other surfaces, such as the moon and the other planets.

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EXAMPLE 3 Solving a Problem Involving Projectile Motion

A ball is projected directly upward from an initial height of 100 ft with an initial velocity of 80 ft per sec.

(a) Give the function that describes the height of the ball in terms of time t.

(b) After how many seconds does the ball reach its maximum height? What is this maximum height?

(c) For what interval of time is the height of the ball greater than 160 ft?

(d) After how many seconds will the ball hit the ground?

SOLUTION

(a) Use the projectile height function with $v_0 = 80$ and $s_0 = 100$.

$$s(t) = -16t^2 + v_0t + s_0$$

$$s(t) = -16t^2 + 80t + 100$$

(b) Since the coefficient of t^2 is -16, the graph of the projectile function is a parabola that opens downward. Find the coordinates of the vertex to determine the maximum height and when it occurs. Let $a = -16$ and $b = 80$ in the vertex formula.

$$t = -\frac{b}{2a} = -\frac{80}{2(-16)} = 2.5$$

$$s(t) = -16t^2 + 80t + 100$$

$$s(2.5) = -16(2.5)^2 + 80(2.5) + 100$$

$$s(2.5) = 200$$

Therefore, after 2.5 sec the ball reaches its maximum height of 200 ft.

(c) We must solve the quadratic *inequality*

$$-16t^2 + 80t + 100 > 160.$$

$$-16t^2 + 80t - 60 > 0 \quad \text{Subtract 160.}$$

$$4t^2 - 20t + 15 < 0 \quad \text{(Section 2.4)}$$

Divide by -4; reverse the inequality symbol.

Use the quadratic formula to find the solutions of $4t^2 - 20t + 15 = 0$.

$$t = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(15)}}{2(4)} \quad \text{(Section 2.3)}$$

$$t = \frac{5 - \sqrt{10}}{2} \approx 0.92 \quad \text{or} \quad t = \frac{5 + \sqrt{10}}{2} \approx 4.08$$

These numbers divide the number line into three intervals: $(-\infty, 0.92)$, $(0.92, 4.08)$, and $(4.08, \infty)$. Using a test value from each interval shows that $(0.92, 4.08)$ satisfies the *inequality*. The ball is more than 160 ft above the ground between 0.92 sec and 4.08 sec.

(d) The height is 0 when the ball hits the ground. We use the quadratic formula to find the *positive* solution of

$$-16t^2 + 80t + 100 = 0.$$

Here, $a = -16$, $b = 80$, and $c = 100$.

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(100)}}{2(-16)}$$

$$t \approx \cancel{-1.04} \quad \text{or} \quad t \approx 6.04$$

Reject

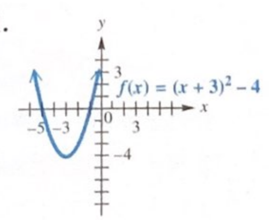
The ball hits the ground after about 6.04 sec.

4.1 Exercises

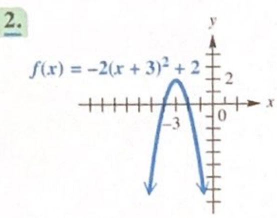
In Exercises 1–4, you are given an equation and the graph of a quadratic function. Do each of the following. ***See Examples 1–2 and Homework 1–2.***

(a) Give the domain and range.
(b) Give the coordinates of the vertex.
(c) Give the equation of the axis.
(d) Find the y-intercept.
(e) Find the x-intercepts.

1.



2.



3. Graph the following on the same coordinate system.

(a) $y = x^2$ **(b)** $y = 3x^2$ **(c)** $y = \frac{1}{3}x^2$

(d) How does the coefficient of x^2 affect the shape of the graph?

4. Graph the following on the same coordinate system.

(a) $y = (x - 2)^2$ **(b)** $y = (x + 1)^2$ **(c)** $y = (x + 3)^2$

(d) How do these graphs differ from the graph of $y = x^2$?

Graph each quadratic function. Give the ***(a)*** *vertex,* ***(b)*** *axis,* ***(c)*** *domain, and* ***(d)*** *range. Then determine* ***(e)*** *the interval of the domain for which the function is increasing and* ***(f)*** *the interval for which the function is decreasing.* ***See Examples 1–2 and Homework 1–2.***

5. $f(x) = (x - 2)^2$

6. $f(x) = (x + 3)^2 - 4$

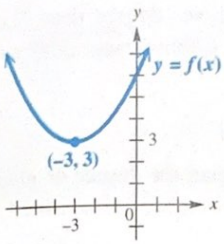
7. $f(x) = -\frac{1}{2}(x + 1)^2 - 3$

8. $f(x) = x^2 - 2x + 3$

9. $f(x) = x^2 - 10x + 21$

10. $f(x) = -2x^2 - 12x - 16$

11. $f(x) = -\frac{1}{2}x^2 - 3x - \frac{1}{2}$



Concept Check *The figure shows the graph of a quadratic function $y = f(x)$. Use it to work Exercises 12–13.*

12. What is the minimum value of $f(x)$?

13. How many real solutions are there to the equation $f(x) = 1$?

Concept Check *Several possible graphs of the quadratic function*

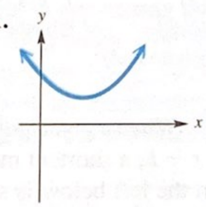
$$f(x) = ax^2 + bx + c$$

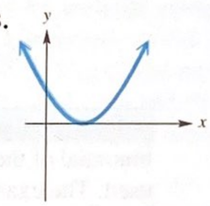
are shown below. For the restrictions on a, b, and c given in Exercises 14–16, select the corresponding graph from choices A–F. (Hint: Use the discriminant. See **Section 2.3** *if necessary.)*

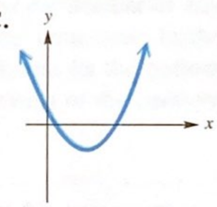
14. $a < 0$; $b^2 - 4ac = 0$

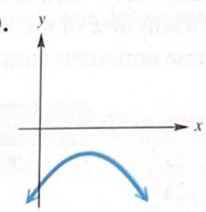
15. $a < 0$; $b^2 - 4ac < 0$

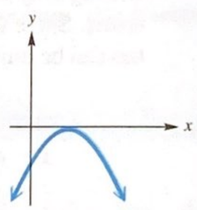
16. $a > 0$; $b^2 - 4ac > 0$

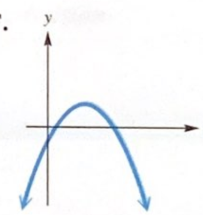
A.


B.


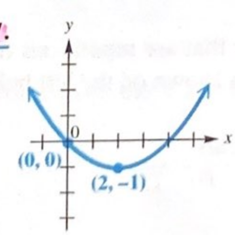
C.


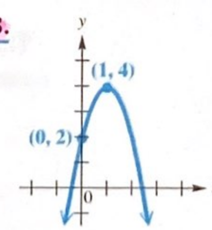
D.


E.


F.


Connecting Graphs with Equations *In Exercises 17–18, find a quadratic function f whose graph matches the one in the figure. (Hint: See the Note following* **Example 2.**)

17.


18.


4.2 Synthetic Division القسمة التركيبية

- Synthetic Division
- Evaluating Polynomial Functions Using the Remainder Theorem
- Testing Potential Zeros

* تستخدم القسمة التركيبية إذا كان المقسوم عليه من الدرجة الأولى

* درجة ال quotient أقل من درجة المقسوم بـ 1

A division problem can be written using multiplication, even when the division involves polynomials. The **division algorithm** illustrates this.

Division Algorithm

Let $f(x)$ and $g(x)$ be polynomials with $g(x)$ of lesser degree than $f(x)$ and $g(x)$ of degree 1 or more. There exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x) \cdot q(x) + r(x),$$

where either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$.

For instance, we saw in **Example 6** of **Section 1.3** that

$$\frac{3x^3 - 2x^2 - 150}{x^2 - 4} = 3x - 2 + \frac{12x - 158}{x^2 - 4}. \qquad \text{(Section 1.3)}$$

We can express this result using the division algorithm.

$$\underbrace{3x^3 - 2x^2 - 150}_{\substack{f(x) \\ \text{Dividend} \\ \text{(original polynomial)}}} = \underbrace{(x^2 - 4)}_{\substack{g(x) \\ \text{Divisor}}} \cdot \underbrace{(3x - 2)}_{\substack{q(x) \\ \text{Quotient}}} + \underbrace{12x - 158}_{\substack{r(x) \\ \text{Remainder}}}$$

Synthetic Division When a given polynomial in x is divided by a first-degree binomial of the form $x - k$, a shortcut method called **synthetic division** may be used. The example on the left below is simplified by omitting all variables and writing only coefficients, with 0 used to represent the coefficient of any missing terms. Since the coefficient of x in the divisor is always 1 in these divisions, it too can be omitted. These omissions simplify the problem, as shown on the right.

$$\begin{array}{r|l}
 & 3x^2 + 10x + 40 \\ \hline
x - 4 & 3x^3 - 2x^2 + 0x - 150 \\
 & \underline{3x^3 - 12x^2} \\
 & \quad 10x^2 + 0x \\
 & \quad \underline{10x^2 - 40x} \\
 & \qquad 40x - 150 \\
 & \qquad \underline{40x - 160} \\
 & \qquad\qquad 10
\end{array}
\qquad\qquad
\begin{array}{r|rrrr}
 & 3 & 10 & 40 & \\ \hline
-4 & 3 & -2 & 0 & -150 \\
 & \underline{3} & \underline{-12} & & \\
 & & 10 & 0 & \\
 & & \underline{10} & \underline{-40} & \\
 & & & 40 & -150 \\
 & & & \underline{40} & \underline{-160} \\
 & & & & 10
\end{array}$$

The numbers in color that are repetitions of the numbers directly above them can also be omitted, as shown on the left below.

$$\begin{array}{r|rrrr}
 & 3 & 10 & 40 & \\ \hline
-4 & 3 & -2 & 0 & -150 \\
 & & \underline{-12} & & \\
 & & 10 & 0 & \\
 & & & \underline{-40} & \\
 & & & 40 & -150 \\
 & & & & \underline{-160} \\
 & & & & 10
\end{array}
\qquad\qquad
\begin{array}{r|rrrr}
 & 3 & 10 & 40 & \\ \hline
-4 & 3 & -2 & 0 & -150 \\
 & & \underline{-12} & & \\
 & & 10 & & \\
 & & & \underline{-40} & \\
 & & & 40 & \\
 & & & & \underline{-160} \\
 & & & & 10
\end{array}$$

The numbers in color are again repetitions of those directly above them. They may be omitted, as shown on the right above.

The entire process can now be condensed vertically. The top row of numbers can be omitted since it duplicates the bottom row if the 3 is brought down.

$$
\begin{array}{r|rrrr}
-4 & 3 & -2 & 0 & -150 \\
 & & -12 & -40 & -160 \\
\hline
 & 3 & 10 & 40 & 10
\end{array}
$$

The rest of the bottom row is obtained by subtracting -12, -40, and -160 from the corresponding terms above them.

To simplify the arithmetic, we replace subtraction in the second row by addition and compensate by changing the -4 at the upper left to its additive inverse, 4.

Additive inverse $\longrightarrow$

$$
\begin{array}{r|rrrrl}
4 & 3 & -2 & 0 & -150 & \\
 & & 12 & 40 & 160 & \longleftarrow \text{Signs changed} \\
\hline
 & 3 & 10 & 40 & 10 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \\
\end{array}
$$

$$\text{Quotient} \longrightarrow 3x^2 + 10x + 40 + \frac{10}{x-4} \quad \longleftarrow \text{Remainder}$$

Synthetic division provides an efficient process for dividing a polynomial by a binomial of the form $x - k$. Begin by writing the coefficients of the polynomial in decreasing powers of the variable, using 0 as the coefficient of any missing powers. The number k is written to the left in the same row. In the example above, $x - k$ is $x - 4$, so k is 4. The answer is found on the bottom row with the remainder farthest to the right and the coefficients of the quotient on the left when written in order of decreasing degree.

CAUTION ***To avoid errors, use 0 as the coefficient for any missing terms, including a missing constant, when setting up the division.***

EXAMPLE 1 Using Synthetic Division

Use synthetic division to divide.

$$\frac{5x^3 - 6x^2 - 28x - 2}{x + 2}$$

SOLUTION Express $x + 2$ in the form $x - k$ by writing it as $x - (-2)$.

$x + 2$ leads to -2.

$$
\begin{array}{r|rrrrl}
-2 & 5 & -6 & -28 & -2 & \longleftarrow \text{Coefficients of the polynomial}
\end{array}
$$

Bring down the 5, and multiply: $-2(5) = -10$.

$$
\begin{array}{r|rrrr}
-2 & 5 & -6 & -28 & -2 \\
 & \downarrow & -10 & & \\
\hline
 & 5 & & &
\end{array}
$$

Add -6 and -10 to obtain -16. Multiply: $-2(-16) = 32$.

$$
\begin{array}{r|rrrr}
-2 & 5 & -6 & -28 & -2 \\
 & & -10 & 32 & \\
\hline
 & 5 & -16 & &
\end{array}
$$

Add -28 and 32, obtaining 4. Finally, $-2(4) = -8$.

$$\begin{array}{r|rrrr} -2 & 5 & -6 & -28 & -2 \\ & & -10 & 32 & -8 \\ \hline & 5 & -16 & 4 & \end{array}$$

Add columns.
Be careful with signs.

Add -2 and -8 to obtain -10.

$$\begin{array}{r|rrrr} -2 & 5 & -6 & -28 & -2 \\ & & -10 & 32 & -8 \\ \hline & \underbrace{5 \quad -16 \quad 4}_{\text{Quotient}} & & & -10 \leftarrow \text{Remainder} \end{array}$$

Since the divisor $x - k$ has degree 1, the degree of the quotient will always be one less than the degree of the polynomial to be divided.

$$\frac{5x^3 - 6x^2 - 28x - 2}{x + 2} = 5x^2 - 16x + 4 + \frac{-10}{x + 2}$$

Remember to add $\frac{\text{remainder}}{\text{divisor}}$.

The result of the division in **Example 1** can be written as

$$5x^3 - 6x^2 - 28x - 2 = (x + 2)(5x^2 - 16x + 4) + (-10)$$

by multiplying each side by the denominator $x + 2$. The theorem that follows is a generalization of this product form.

Special Case of the Division Algorithm

For any polynomial $f(x)$ and any complex number k, there exists a unique polynomial $q(x)$ and number r such that the following holds.

$$f(x) = (x - k)q(x) + r$$

The mathematical statement

$$\underbrace{5x^3 - 6x^2 - 28x - 2}_{f(x)} = \underbrace{(x + 2)}_{(x-k)}\cdot\underbrace{(5x^2 - 16x + 4)}_{q(x)} + \underbrace{(-10)}_{r}$$

illustrates this connection. This form of the division algorithm is useful in developing the *remainder theorem*.

Evaluating Polynomial Functions Using the Remainder Theorem Suppose that $f(x)$ is written as $f(x) = (x - k)q(x) + r$. This equality is true for all complex values of x, so it is true for $x = k$. Replace x with k.

$$f(k) = (k - k)q(k) + r, \quad \text{or} \quad f(k) = r$$

This proves the following **remainder theorem,** which gives a new method of evaluating polynomial functions.

Remainder Theorem

If the polynomial $f(x)$ is divided by $x - k$, then the remainder is equal to $f(k)$.

In **Example 1,** when $f(x) = 5x^3 - 6x^2 - 28x - 2$ was divided by $x + 2$, or $x - (-2)$, the remainder was -10. Substitute -2 for x in $f(x)$.

$$\begin{aligned} f(-2) &= 5(-2)^3 - 6(-2)^2 - 28(-2) - 2 \\ &= -40 - 24 + 56 - 2 \\ &= -10 \end{aligned}$$

Use parentheses around substituted values to avoid errors.

As shown below, an alternative way to find the value of a polynomial is to use synthetic division. By the remainder theorem, instead of replacing x by -2 to find $f(-2)$, divide $f(x)$ by $x + 2$ using synthetic division as in **Example 1.** Then $f(-2)$ is the remainder, -10.

$$\begin{array}{r|rrrr} -2 & 5 & -6 & -28 & -2 \\ & & -10 & 32 & -8 \\ \hline & 5 & -16 & 4 & -10 \end{array} \leftarrow f(-2)$$

تقدري تجيبها بالآلة بالتعويض

HOMEWORK 1 **Applying the Remainder Theorem**

Let $f(x) = -x^4 + 3x^2 - 4x - 5$. Use the remainder theorem to find $f(-3)$.

Testing Potential Zeros A **zero** of a polynomial function $f(x)$ is a number k such that $f(k) = 0$. *The real number zeros are the x-intercepts of the graph of the function.*

The remainder theorem gives a quick way to decide whether a number k is a zero of a polynomial function defined by $f(x)$, as follows.

1. Use synthetic division to find $f(k)$.
2. If the remainder is 0, then $f(k) = 0$ and k is a zero of $f(x)$. If the remainder is not 0, then k is not a zero of $f(x)$.

A zero of $f(x)$ is a **root,** or **solution,** of the equation $f(x) = 0$.

إذا كان العدد المركب 1+2i صفر للدالة فإن الـ 1-2i هو أيضا يكون صفر للدالة (مرافقه)

تقدري تحلها بالآلة (التعويض)

EXAMPLE 2 **Deciding Whether a Number Is a Zero**

Decide whether the given number k is a zero of $f(x)$.

(a) $f(x) = x^3 - 4x^2 + 9x - 6;\quad k = 1$

(b) $f(x) = x^4 + x^2 - 3x + 1;\quad k = -1$

(c) $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5;\quad k = 1 + 2i$

إذا كان الـ k عدد مركب يفضل استخدام القسمة التركيبية

$f(x) = x^3 - 4x^2 + 9x - 6$

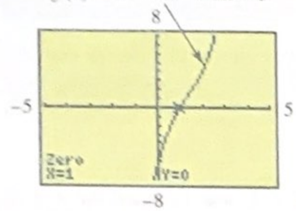


Figure 7

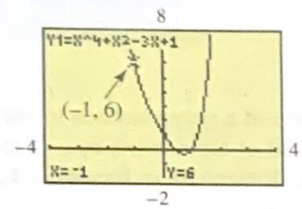


Figure 8

SOLUTION

(a) Use synthetic division to decide whether 1 is a zero of $f(x) = x^3 - 4x^2 + 9x - 6$.

$$\text{Proposed zero} \longrightarrow \begin{array}{r|rrrr} 1 & 1 & -4 & 9 & -6 \\ & & 1 & -3 & 6 \\ \hline & 1 & -3 & 6 & 0 \end{array} \begin{array}{l} \longleftarrow f(x) = x^3 - 4x^2 + 9x - 6 \\ \\ \longleftarrow \text{Remainder} \end{array}$$

Since the remainder is 0, $f(1) = 0$, and 1 is a zero of the given polynomial function. An x-intercept of the graph of $f(x) = x^3 - 4x^2 + 9x - 6$ is 1, so the graph includes the point $(1, 0)$. The graph in **Figure 7** supports this.

(b) For $f(x) = x^4 + x^2 - 3x + 1$, remember to use 0 as coefficient for the missing x^3-term in the synthetic division.

$$\text{Proposed zero} \longrightarrow \begin{array}{r|rrrrr} -1 & 1 & 0 & 1 & -3 & 1 \\ & & -1 & 1 & -2 & 5 \\ \hline & 1 & -1 & 2 & -5 & 6 \end{array} \longleftarrow \text{Remainder}$$

The remainder is not 0, so -1 is not a zero of $f(x) = x^4 + x^2 - 3x + 1$. In fact, $f(-1) = 6$, indicating that $(-1, 6)$ is on the graph of $f(x)$. The graph in **Figure 8** supports this.

(c) Use synthetic division and operations with complex numbers to determine whether $1 + 2i$ is a zero of $f(x) = x^4 - 2x^3 + 4x^2 + 2x - 5$.

$$\begin{array}{r|rrrrr} 1+2i & 1 & -2 & 4 & 2 & -5 \\ & & 1+2i & -5 & -1-2i & 5 \\ \hline & 1 & -1+2i & -1 & 1-2i & 0 \end{array} \begin{array}{l} \\ i^2 = -1 \text{ (Section 1.2)} \\ \longleftarrow \text{Remainder} \end{array}$$

$(1 + 2i)(-1 + 2i)$
$= -1 + 4i^2$
$= -5$

Since the remainder is 0, $1 + 2i$ is a zero of the given polynomial function. Notice that $1 + 2i$ is *not* a real number zero. Therefore, it cannot appear as an x-intercept on the graph of $f(x)$.

4.2 Exercises

Use synthetic division to perform each division. ***See Example 1.***

1. $\dfrac{x^3 + 7x^2 + 13x + 6}{x + 2}$

2. $\dfrac{2x^4 - x^3 - 7x^2 + 7x - 10}{x - 2}$

3. $\dfrac{x^4 + 5x^3 + 4x^2 - 3x + 9}{x + 3}$

4. $\dfrac{x^6 - 3x^4 + 2x^3 - 6x^2 - 5x + 3}{x + 2}$

5. $\dfrac{-11x^4 + 2x^3 - 8x^2 - 4}{x + 1}$

6. $\dfrac{x^3 + x^2 + \frac{1}{2}x + \frac{1}{8}}{x + \frac{1}{2}}$

7. $\dfrac{x^4 - x^3 - 5x^2 - 3x}{x + 1}$

8. $\dfrac{x^4 - 1}{x - 1}$

9. $\dfrac{x^7 + 1}{x + 1}$

Express $f(x)$ in the form $f(x) = (x - k)q(x) + r$ for the given value of k.

10. $f(x) = 2x^3 + 3x^2 - 16x + 10;\ k = -4$

11. $f(x) = -x^3 + x^2 + 3x - 2;\ k = 2$

12. $f(x) = 2x^4 + x^3 - 15x^2 + 3x;\ k = -3$

13. $f(x) = -5x^4 + x^3 + 2x^2 + 3x + 1;\ k = 1$

For each polynomial function, use the remainder theorem and synthetic division to find $f(k)$. ***See Homework 1.***

14. $f(x) = x^2 - 4x - 5;\ k = 5$

15. $f(x) = -x^3 + 8x^2 + 63;\ k = 4$

16. $f(x) = 2x^3 - 3x^2 - 5x + 4;\ k = 2$

17. $f(x) = x^4 + 6x^3 + 9x^2 + 3x - 3;\ k = 4$

18. $f(x) = 6x^3 - 31x^2 - 15x;\ k = -\frac{1}{2}$

19. $f(x) = x^2 - x + 3;\ k = 3 - 2i$

20. $f(x) = 2x^2 + 10;\ k = i\sqrt{5}$

Use synthetic division to decide whether the given number k is a zero of the given polynomial function. If it is not, give the value of $f(k)$. ***See Example 2 and Homework 1.***

21. $f(x) = x^2 + 4x - 5;\ k = -5$

22. $f(x) = x^3 + 2x^2 - x + 6;\ k = -3$

23. $f(x) = 2x^3 + 9x^2 - 16x + 12;\ k = 1$

24. $f(x) = 2x^3 - 3x^2 - 5x;\ k = 0$

25. $f(x) = 3x^4 + 13x^3 - 10x + 8;\ k = -\frac{4}{3}$

26. $f(x) = 16x^4 + 3x^2 - 2;\ k = \frac{1}{2}$

27. $f(x) = x^2 - 4x + 5;\ k = 2 - i$

28. $f(x) = x^2 - 3x + 5;\ k = 1 - 2i$

29. $f(x) = 2x^3 - x^2 + 3x - 5;\ k = 2 - i$

Relating Concepts

For individual or collaborative investigation *(Exercises 30–34)*

The remainder theorem indicates that when a polynomial $f(x)$ *is divided by* $x - k$, *the remainder is equal to* $f(k)$. *For*

$$f(x) = x^3 - 2x^2 - x + 2,$$

use the remainder theorem to find each of the following. Then determine the coordinates of the corresponding point on the graph of $f(x)$.

30. $f(-1)$

31. $f(0)$

32. $f\left(\frac{3}{2}\right)$

33. $f(3)$

34. Use the results from **Exercises 30–33** to plot eight points on the graph of $f(x)$. Connect these points with a smooth curve. Describe a method for graphing polynomial functions using the remainder theorem.

4.3 Zeros of Polynomial Functions

- Factor Theorem
- Rational Zeros Theorem
- Number of Zeros
- Conjugate Zeros Theorem
- Finding Zeros of a Polynomial Function
- Descartes' Rule of Signs

Factor Theorem Consider the polynomial function

$$f(x) = x^2 + x - 2,$$

which is written in factored form as

factors for the function f(x)

$$f(x) = (x - 1)(x + 2). \quad \text{(Section 1.4)}$$

For this function, $f(1) = 0$ and $f(-2) = 0$, and thus 1 and -2 are zeros of $f(x)$. Notice the special relationship between each linear factor and its corresponding zero. The **factor theorem** summarizes this relationship.

Factor Theorem

For any polynomial function $f(x)$, $x - k$ is a factor of the polynomial if and only if $f(k) = 0$.

EXAMPLE 1 **Deciding Whether $x - k$ Is a Factor**

Determine whether $x - 1$ is a factor of each polynomial.

تقدري تحليها بالتعويض راح تساوي
تعوضيها بـ $x = 1$ ← $x - 1 = 0$
بالمعادلة.
إذا المعادلة تساوي صفر
راح يكون factor بس
العكس لا
لازم الـ Remaidor يساوي
صفر.

(a) $f(x) = 2x^4 + 3x^2 - 5x + 7$

(b) $f(x) = 3x^5 - 2x^4 + x^3 - 8x^2 + 5x + 1$

SOLUTION

(a) By the factor theorem, $x - 1$ will be a factor if $f(1) = 0$. Use synthetic division and the remainder theorem to decide.

$$\begin{array}{r|rrrrr} 1 & 2 & 0 & 3 & -5 & 7 \\ & & 2 & 2 & 5 & 0 \\ \hline & 2 & 2 & 5 & 0 & 7 \leftarrow f(1) = 7 \end{array}$$

Use a zero coefficient for the missing term.

(Section 4.2)

The remainder is 7 and not 0, so $x - 1$ is not a factor of $2x^4 + 3x^2 - 5x + 7$.

(b)

$$\begin{array}{r|rrrrrr} 1 & 3 & -2 & 1 & -8 & 5 & 1 \\ & & 3 & 1 & 2 & -6 & -1 \\ \hline & 3 & 1 & 2 & -6 & -1 & 0 \leftarrow f(1) = 0 \end{array}$$

Because the remainder is 0, $x - 1$ is a factor. Additionally, we can determine from the coefficients in the bottom row that the other factor is

$$3x^4 + x^3 + 2x^2 - 6x - 1.$$

Thus, we can express the polynomial in factored form.

$$f(x) = (x - 1)(3x^4 + x^3 + 2x^2 - 6x - 1)$$

We can use the factor theorem to factor a polynomial of greater degree into linear factors of the form $ax - b$.

HOMEWORK 1 **Factoring a Polynomial Given a Zero**

Factor $f(x) = 6x^3 + 19x^2 + 2x - 3$ into linear factors if -3 is a zero of f.

Rational Zeros Theorem The **rational zeros theorem** gives a method to determine all possible candidates for rational zeros of a polynomial function with integer coefficients.

Rational Zeros Theorem

If $\frac{p}{q}$ is a rational number written in lowest terms, and if $\frac{p}{q}$ is a zero of f, a polynomial function with integer coefficients, then p is a factor of the constant term, and q is a factor of the leading coefficient.

Proof $f\left(\frac{p}{q}\right) = 0$ since $\frac{p}{q}$ is a zero of $f(x)$.

LOOKING AHEAD TO CALCULUS

Finding the derivative of a polynomial function is one of the basic skills required in a first calculus course. For the functions

$$f(x) = x^4 - x^2 + 5x - 4,$$
$$g(x) = -x^6 + x^2 - 3x + 4,$$
and $$h(x) = 3x^3 - x^2 + 2x - 4,$$

the derivatives are

$$f'(x) = 4x^3 - 2x + 5,$$
$$g'(x) = -6x^5 + 2x - 3,$$
and $$h'(x) = 9x^2 - 2x + 2.$$

Notice the use of the "prime" notation. For example, the derivative of $f(x)$ is denoted $f'(x)$.

Look for the pattern among the exponents and the coefficients. Using this pattern, what is the derivative of

$$F(x) = 4x^4 - 3x^3 + 6x - 4?$$

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0 \qquad \text{Definition of zero of } f$$

$$a_n\left(\frac{p^n}{q^n}\right) + a_{n-1}\left(\frac{p^{n-1}}{q^{n-1}}\right) + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0 \qquad \text{Power rule for exponents (Section 1.3)}$$

$$a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} = -a_0q^n \qquad \text{Multiply by } q^n \text{ and subtract } a_0q^n.$$

$$p(a_np^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1q^{n-1}) = -a_0q^n \qquad \text{Factor out } p.$$

This result shows that $-a_0q^n$ equals the product of the two factors p and $(a_np^{n-1} + \cdots + a_1q^{n-1})$. For this reason, p must be a factor of $-a_0q^n$. Since it was assumed that $\frac{p}{q}$ is written in lowest terms, p and q have no common factor other than 1, so p is not a factor of q^n. Thus, p must be a factor of a_0. In a similar way, it can be shown that q is a factor of a_n.

EXAMPLE 2 Using the Rational Zeros Theorem

Consider the polynomial function.

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

(a) List all possible rational zeros.

(b) Find all rational zeros and factor $f(x)$ into linear factors.

SOLUTION

(a) For a rational number $\frac{p}{q}$ to be a zero, p must be a factor of $a_0 = 2$, and q must be a factor of $a_4 = 6$. Thus, p can be ± 1 or ± 2, and q can be ± 1, ± 2, ± 3, or ± 6. The possible rational zeros, $\frac{p}{q}$, are ± 1, ± 2, $\pm \frac{1}{2}$, $\pm \frac{1}{3}$, $\pm \frac{1}{6}$, and $\pm \frac{2}{3}$.

(b) Use the remainder theorem to show that 1 is a zero.

Use "trial and error" to find zeros.

$$\begin{array}{r|rrrrr} 1 & 6 & 7 & -12 & -3 & 2 \\ & & 6 & 13 & 1 & -2 \\ \hline & 6 & 13 & 1 & -2 & 0 \end{array} \longleftarrow f(1) = 0$$

The 0 remainder shows that 1 is a zero. The quotient is $6x^3 + 13x^2 + x - 2$.

$$f(x) = (x - 1)(6x^3 + 13x^2 + x - 2) \qquad \text{Begin factoring } f(x).$$

Now, use the quotient polynomial and synthetic division to find that -2 is a zero.

$$\begin{array}{r|rrrr} -2 & 6 & 13 & 1 & -2 \\ & & -12 & -2 & 2 \\ \hline & 6 & 1 & -1 & 0 \end{array} \longleftarrow f(-2) = 0$$

The new quotient polynomial is $6x^2 + x - 1$. Therefore, $f(x)$ can now be completely factored as follows.

$$f(x) = (x - 1)(x + 2)(6x^2 + x - 1)$$
$$f(x) = (x - 1)(x + 2)(3x - 1)(2x + 1)$$

Setting $3x - 1 = 0$ and $2x + 1 = 0$ yields the zeros $\frac{1}{3}$ and $-\frac{1}{2}$. In summary, the rational zeros are 1, -2, $\frac{1}{3}$, and $-\frac{1}{2}$, and they can be seen as x-intercepts on the graph of $f(x)$ in **Figure 9.** The linear factorization of $f(x)$ is

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$
$$f(x) = (x - 1)(x + 2)(3x - 1)(2x + 1).$$

Check by multiplying these factors.

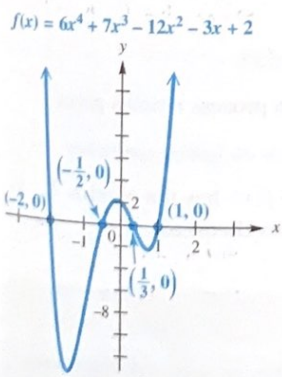


Figure 9

NOTE In **Example 2,** once we obtained the quadratic factor

$$6x^2 + x - 1,$$

we were able to complete the work by factoring it directly. Had it not been easily factorable, we could have used the quadratic formula to find the other two zeros (and factors).

CAUTION ***The rational zeros theorem gives only possible rational zeros. It does not tell us whether these rational numbers are actual zeros.*** We must rely on other methods to determine whether or not they are indeed zeros. Furthermore, the polynomial must have integer coefficients.

To apply the rational zeros theorem to a polynomial with fractional coefficients, multiply through by the least common denominator of all the fractions. For example, any rational zeros of $p(x)$ defined below will also be rational zeros of $q(x)$.

$$p(x) = x^4 - \frac{1}{6}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{3}$$

$$q(x) = 6x^4 - x^3 + 4x^2 - x - 2 \qquad \text{Multiply the terms of } p(x) \text{ by 6.}$$

Number of Zeros The **fundamental theorem of algebra** says that every function defined by a polynomial of degree 1 or more has a zero, which means that every such polynomial can be factored.

Fundamental Theorem of Algebra

Every function defined by a polynomial of degree 1 or more has at least one complex zero.

From the fundamental theorem, if $f(x)$ is of degree 1 or more, then there is some number k_1 such that $f(k_1) = 0$. By the factor theorem,

$$f(x) = (x - k_1)q_1(x), \quad \text{for some polynomial } q_1(x).$$

If $q_1(x)$ is of degree 1 or more, the fundamental theorem and the factor theorem can be used to factor $q_1(x)$ in the same way. There is some number k_2 such that $q_1(k_2) = 0$, so

$$q_1(x) = (x - k_2)q_2(x)$$

and

$$f(x) = (x - k_1)(x - k_2)q_2(x).$$

Assuming that $f(x)$ has degree n and repeating this process n times gives

$$f(x) = a(x - k_1)(x - k_2)\cdots(x - k_n). \qquad a \text{ is the leading coefficient.}$$

Each of these factors leads to a zero of $f(x)$, so $f(x)$ has the n zeros k_1, k_2, $k_3, \ldots, k_n$. This result suggests the **number of zeros theorem.**

Number of Zeros Theorem

A function defined by a polynomial of degree n has *at most* n distinct zeros.

For example, a polynomial function of degree 3 has *at most* three distinct zeros but can have as few as one zero. Consider the following polynomial.

$$f(x) = x^3 + 3x^2 + 3x + 1$$
$$f(x) = (x + 1)^3$$

The function f is of degree 3 but has only one zero, -1. Actually, the zero -1 occurs *three* times, since there are three factors of $x + 1$. The number of times a zero occurs is referred to as the **multiplicity of the zero.**

HOMEWORK 2 Finding a Polynomial Function That Satisfies Given Conditions (Real Zeros)

Find a function f defined by a polynomial of degree 3 that satisfies the given conditions.

(a) Zeros of -1, 2, and 4; $f(1) = 3$

(b) -2 is a zero of multiplicity 3; $f(-1) = 4$

NOTE In **Home Work 2(a),** we cannot clear the denominators in $f(x)$ by multiplying each side by 2 because the result would equal $2 \cdot f(x)$, not $f(x)$.

Conjugate Zeros Theorem The following properties of complex conjugates are needed to prove the **conjugate zeros theorem.** We use a simplified notation for conjugates here. If $z = a + bi$, then the conjugate of z is written $\overline{z}$, where $\overline{z} = a - bi$. For example, if $z = -5 + 2i$, then $\overline{z} = -5 - 2i$. The proofs of the first of these properties is left for **Exercise 57.**

Properties of Conjugates

For any complex numbers c and d, the following properties hold.

$$\overline{c + d} = \overline{c} + \overline{d}, \quad \overline{c \cdot d} = \overline{c} \cdot \overline{d}, \quad \text{and} \quad \overline{c^n} = (\overline{c})^n$$

The remainder theorem can be used to show that both $2 + i$ and $2 - i$ are zeros of $f(x) = x^3 - x^2 - 7x + 15$. In general, if z is a zero of a polynomial function with *real* coefficients, then so is $\overline{z}$.

Conjugate Zeros Theorem

If $f(x)$ defines a polynomial function ***having only real coefficients*** and if $z = a + bi$ is a zero of $f(x)$, where a and b are real numbers, then

$$\overline{z} = a - bi \text{ is also a zero of } f(x).$$

Proof Start with the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where all coefficients are real numbers. If the complex number z is a zero of $f(x)$, then we have the following.

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0 \quad \text{From the preceding discussion}$$

Take the conjugate of both sides of this equation.

$$\overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \overline{0}$$

$$\overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \cdots + \overline{a_1 z} + \overline{a_0} = \overline{0} \quad \text{Use generalizations of the properties } \overline{c+d} = \overline{c} + \overline{d} \text{ and } \overline{c \cdot d} = \overline{c} \cdot \overline{d}.$$

$$\overline{a_n}\, \overline{z^n} + \overline{a_{n-1}}\, \overline{z^{n-1}} + \cdots + \overline{a_1}\, \overline{z} + \overline{a_0} = \overline{0}$$

$$a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots + a_1(\overline{z}) + a_0 = 0 \quad \text{Use the property } \overline{c^n} = (\overline{c})^n \text{ and the fact that for any real number } a, \overline{a} = a.$$

$$f(\overline{z}) = 0$$

Hence $\overline{z}$ is also a zero of $f(x)$, which completes the proof.

CAUTION ***When the conjugate zeros theorem is applied, it is essential that the polynomial have only real coefficients.*** For example,

$$f(x) = x - (1 + i)$$

has $1 + i$ as a zero, but the conjugate $1 - i$ is not a zero.

EXAMPLE 3 Finding a Polynomial Function That Satisfies Given Conditions (Complex Zeros)

Find a polynomial function of least degree having only real coefficients and zeros 3 and $2 + i$.

SOLUTION The complex number $2 - i$ must also be a zero, so the polynomial has at least three zeros: 3, $2 + i$, and $2 - i$. For the polynomial to be of least degree, these must be the only zeros. By the factor theorem there must be three factors: $x - 3$, $x - (2 + i)$, and $x - (2 - i)$.

$$f(x) = (x - 3)[x - (2 + i)][x - (2 - i)] \quad \text{Factor theorem}$$

$$f(x) = (x - 3)(x - 2 - i)(x - 2 + i) \quad \text{Distribute negative signs.}$$

$$f(x) = (x - 3)(x^2 - 4x + 5) \quad \text{Multiply and combine like terms; } i^2 = -1. \text{ (Section 2.2)}$$

$$f(x) = x^3 - 7x^2 + 17x - 15 \quad \text{Multiply again.}$$

Any nonzero multiple of $x^3 - 7x^2 + 17x - 15$ also satisfies the given conditions on zeros. The information on zeros given in the problem is not sufficient to give a specific value for the leading coefficient.

Finding Zeros of a Polynomial Function The theorem on conjugate zeros helps predict the number of real zeros of polynomial functions with real coefficients.

- A polynomial function with real coefficients of odd degree n, where $n \geq 1$, must have at least one real zero (since zeros of the form $a + bi$, where $b \neq 0$, occur in conjugate pairs).
- A polynomial function with real coefficients of even degree n may have no real zeros.

HOMEWORK 3 Finding All Zeros Given One Zero

Find all zeros of $f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12$, given that $1 - i$ is a zero.

NOTE In **Home Work 3,** if we had been unable to factor $x^2 - 5x + 6$ into linear factors, we would have used the quadratic formula to solve the equation $x^2 - 5x + 6 = 0$ to find the remaining two zeros of the function.

Descartes' Rule of Signs The following rule helps to determine the number of positive and negative real zeros of a polynomial function. A **variation in sign** is a change from positive to negative or from negative to positive in successive terms of the polynomial when they are written in order of descending powers of the variable. ***Missing terms (those with 0 coefficients) are counted as no change in sign and can be ignored.***

Descartes' Rule of Signs

Let $f(x)$ define a polynomial function with real coefficients and a nonzero constant term, with terms in descending powers of x.

(a) The number of positive real zeros of f either equals the number of variations in sign occurring in the coefficients of $f(x)$, or is less than the number of variations by a positive even integer.

(b) The number of negative real zeros of f either equals the number of variations in sign occurring in the coefficients of $f(-x)$, or is less than the number of variations by a positive even integer.

EXAMPLE 4 Applying Descartes' Rule of Signs

Determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros of

$$f(x) = x^4 - 6x^3 + 8x^2 + 2x - 1.$$

SOLUTION We first consider the possible number of positive zeros by observing that $f(x)$ has three variations in signs.

$$f(x) = +x^4 - 6x^3 + 8x^2 + 2x - 1$$

1 2 3

Thus, by Descartes' rule of signs, $f(x)$ has either three or one (since $3 - 2 = 1$) positive real zeros.

For negative zeros, consider the variations in signs for $f(-x)$.

$$\begin{aligned} f(-x) &= (-x)^4 - 6(-x)^3 + 8(-x)^2 + 2(-x) - 1 \\ &= x^4 + 6x^3 + 8x^2 - 2x - 1 \end{aligned}$$

1

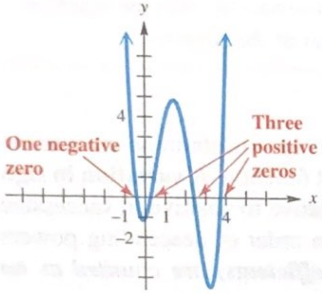
Since there is only one variation in sign, $f(x)$ has exactly one negative real zero.

Because $f(x)$ is a fourth-degree polynomial function, it must have four complex zeros, some of which may be repeated. Descartes' rule of signs has indicated that exactly one of these zeros is a negative real number.

- One possible combination of the zeros is one negative real zero, three positive real zeros, and no nonreal complex zeros.
- Another possible combination of the zeros is one negative real zero, one positive real zero, and two nonreal complex zeros.

By the conjugate zeros theorem, any possible nonreal complex zeros must occur in conjugate pairs since $f(x)$ has real coefficients. The table below summarizes these possibilities.

Possible Number of Zeros		
Positive	Negative	Nonreal Complex
3	1	0
1	1	2



$f(x) = x^4 - 6x^3 + 8x^2 + 2x - 1$

Figure 10

The graph of $f(x)$ in **Figure 10** verifies the correct combination of three positive real zeros with one negative real zero, as seen in the first row of the table.*

NOTE ***Descartes' rule of signs does not identify the multiplicity of the zeros of a function.*** For example, if it indicates that a function $f(x)$ has exactly two positive real zeros, then $f(x)$ may have two distinct positive real zeros or one positive real zero of multiplicity 2.

4.3 Exercises

Concept Check Decide whether each statement is true *or* false. *If false, tell why.*

1. Since $x - 1$ is a factor of $f(x) = x^6 - x^4 + 2x^2 - 2$, we can conclude that $f(1) = 0$.

2. For $f(x) = (x + 2)^4(x - 3)$, 2 is a zero of multiplicity 4.

Use the factor theorem and synthetic division to decide whether the second polynomial is a factor of the first. **See Example 1.**

3. $x^3 - 5x^2 + 3x + 1$; $x - 1$

4. $2x^4 + 5x^3 - 8x^2 + 3x + 13$; $x + 1$

5. $-x^3 + 3x - 2$; $x + 2$

6. $4x^2 + 2x + 54$; $x - 4$

7. $x^3 + 2x^2 + 3$; $x - 1$

8. $2x^4 + 5x^3 - 2x^2 + 5x + 6$; $x + 3$

Factor $f(x)$ into linear factors given that k is a zero of $f(x)$. **See Homework 1.**

9. $f(x) = 2x^3 - 3x^2 - 17x + 30$; $k = 2$

10. $f(x) = 6x^3 + 13x^2 - 14x + 3$; $k = -3$

11. $f(x) = 6x^3 + 25x^2 + 3x - 4$; $k = -4$

12. $f(x) = x^3 + (7 - 3i)x^2 + (12 - 21i)x - 36i$; $k = 3i$

13. $f(x) = 2x^3 + (3 - 2i)x^2 + (-8 - 5i)x + (3 + 3i)$; $k = 1 + i$

14. $f(x) = x^4 + 2x^3 - 7x^2 - 20x - 12$; $k = -2$ (multiplicity 2)

For each polynomial function, one zero is given. Find all others. **See Homework 1 and 3.**

15. $f(x) = x^3 - x^2 - 4x - 6$; 3

16. $f(x) = x^3 - 7x^2 + 17x - 15$; $2 - i$

17. $f(x) = x^4 + 5x^2 + 4$; $-i$

For each polynomial function, **(a)** *list all possible rational zeros,* **(b)** *find all rational zeros, and* **(c)** *factor $f(x)$.* **See Example 2.**

* The authors would like to thank Mary Hill of College of DuPage for her input into **Example 4.**

18. $f(x) = x^3 - 2x^2 - 13x - 10$

19. $f(x) = x^3 + 6x^2 - x - 30$

20. $f(x) = 6x^3 + 17x^2 - 31x - 12$

21. $f(x) = 24x^3 + 40x^2 - 2x - 12$

For each polynomial function, find all zeros and their multiplicities.

22. $f(x) = (x - 2)^3(x^2 - 7)$

23. $f(x) = 3x(x - 2)(x + 3)(x^2 - 1)$

24. $f(x) = (x^2 + x - 2)^5\left(x - 1 + \sqrt{3}\right)^2$

Find a polynomial function $f(x)$ of degree 3 with real coefficients that satisfies the given conditions. **See Home Work 2.**

25. Zeros of -3, 1, and 4; $f(2) = 30$

26. Zeros of -2, 1, and 0; $f(-1) = -1$

27. Zero of -3 having multiplicity 3; $f(3) = 36$

28. Zero of 0 and zero of 1 having multiplicity 2; $f(2) = 10$

Find a polynomial function $f(x)$ of least degree having only real coefficients with zeros as given. **See Homework 2–3 and Example 3.**

29. $5 + i$ and $5 - i$

30. $0, i,$ and $1 + i$

31. $1 + \sqrt{2}$, $1 - \sqrt{2}$, and 1

32. $2 - i$, 3, and -1

33. 2 and $3 + i$

34. $1 - \sqrt{2}$, $1 + \sqrt{2}$, and $1 - i$

35. $2 - i$ and $6 - 3i$

36. 4, $1 - 2i$, and $3 + 4i$

37. $1 + 2i$ and 2 (multiplicity 2)

Use Descartes' rule of signs to determine the different possibilities for the numbers of positive, negative, and nonreal complex zeros for each function. **See Example 4.**

38. $f(x) = 2x^3 - 4x^2 + 2x + 7$

39. $f(x) = 4x^3 - x^2 + 2x - 7$

40. $f(x) = 5x^4 + 3x^2 + 2x - 9$

41. $f(x) = -8x^4 + 3x^3 - 6x^2 + 5x - 7$

42. $f(x) = x^5 + 3x^4 - x^3 + 2x + 3$

43. $f(x) = 7x^5 + 6x^4 + 2x^3 + 9x^2 + x + 5$

44. $f(x) = 2x^5 - 7x^3 + 6x + 8$

45. $f(x) = 5x^6 - 6x^5 + 7x^3 - 4x^2 + x + 2$

*Find all complex zeros of each polynomial function. Give exact values. List multiple zeros as necessary.**

46. $f(x) = x^4 + 2x^3 - 3x^2 + 24x - 180$

47. $f(x) = x^4 + x^3 - 9x^2 + 11x - 4$

48. $f(x) = 2x^5 + 11x^4 + 16x^3 + 15x^2 + 36x$

49. $f(x) = x^5 - 6x^4 + 14x^3 - 20x^2 + 24x - 16$

50. $f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$

51. $f(x) = 5x^3 - 9x^2 + 28x + 6$

52. $f(x) = x^4 + 29x^2 + 100$

53. $f(x) = x^4 + 2x^2 + 1$

54. $f(x) = x^4 - 6x^3 + 7x^2$

55. $f(x) = x^4 - 8x^3 + 29x^2 - 66x + 72$

56. $f(x) = x^6 - 9x^4 - 16x^2 + 144$

If c and d are complex numbers, prove each statement. (Hint: Let $c = a + bi$ and $d = m + ni$ and form all the conjugates, the sums, and the products.)

57. $\overline{c + d} = \overline{c} + \overline{d}$

58. $\overline{a} = a$ for any real number a

* The authors would like to thank Aileen Solomon of Trident Technical College for preparing and suggesting the inclusion of **Exercises 46–52.**

In 1545, a method of solving a cubic equation of the form

$$x^3 + mx = n,$$

developed by Niccolo Tartaglia, was published in the Ars Magna, *a work by Girolamo Cardano. The formula for finding the one real solution of the equation is*

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{\frac{-n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

(Source: Gullberg, J., *Mathematics from the Birth of Numbers,* W.W. Norton & Company.)
Use the formula to solve each equation for the one real solution.

59. $x^3 + 9x = 26$

Glossary

axis (of symmetry) of a parabola The line of symmetry for a parabola is the axis of the parabola.

محور (التماثل) لقطع مكافئ خط التماثل لأي قطع مكافئ هو محور القطع المكافئ.

leading coefficient In a polynomial function of degree n, the leading coefficient is a_n. It is the coefficient of the term of greatest degree.

المعامل القيادي في أي دالة متعددة الحدود للدرجة n، يكون المعامل القيادي هو a_n. هو معامل عنصر الدرجة العليا.

parabola (function definition) A parabola is a curve that is the graph of a quadratic function defined over the set of all real numbers.

القطع المكافئ (تعريف الدالة) القطع المكافئ هو منحنى رسم بياني لدالة تربيعية معرفة فوق مجموعة من الأعداد الحقيقية.

polynomial function of degree n A polynomial function of degree n, where n is a nonnegative integer, is a function defined by an expression of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_1$, and a_0 are real numbers, with $a_n \neq 0$.

الدالة متعددة الحدود من الدرجة n أي دالة متعددة الحدود من الدرجة n، حيث تكون n عدد صحيح غير سلبي، هي دالة تُعرَّف من خلال تعبير بصيغة $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$، حيث إن $a_n, a_{n-1}, \ldots, a_1$ و a_0 تعد أعدادًا حقيقية مع $a_n \neq 0$.

quadratic function A function f of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, with $a \neq 0$, is a quadratic function.

المعادلة التربيعية أي معادلة f بصيغة $f(x) = ax^2 + bx + c$، حيث تكون a و b و c أعدادًا حقيقية مع $a \neq 0$ هي معادلة تربيعية.

root (solution) of an equation A zero k of $f(x)$ is a number such that $f(k) = 0$ is true.

جذر (حل) لمعادلة أي صفر k لمعادلة $f(x)$ هي عدد مثل $f(k) = 0$ بحيث تكون صحيحة.

synthetic division Synthetic division is a shortcut method of dividing a polynomial by a binomial of the form $x - k$.

القسمة المختزلة هي طريقة مختصرة لقسمة كثير الحدود على ثنائي الحد بالصيغة $x - k$.

vertex of a parabola The vertex of a parabola is the point where the axis of symmetry intersects the parabola. For the graph of a quadratic function, this is the turning point of the parabola.

رأس القطع المكافئ هي النقطة التي يقطع عندها محور التماثل القطع المكافئ. ويطلق عليه نقطة تحول القطع المكافئ في مخطط دوال الدرجة الثانية

zero of multiplicity n A polynomial function has a zero k of multiplicity n if the zero k occurs exactly n times. The polynomial has exactly n factors of $x - k$.

الصفر الذي عدد تكراره n دالة متعددة الحدود تتضمن صفر k متكررًا n من المرات إذا كان الصفر k متكررًا n مرة بالضبط. متعددة الحدود لها n كعامل $x - k$.

zero polynomial The function f defined by $f(x) = 0$ is the zero polynomial.

متعددة الحدود الصفرية هي دالة f محددة بـ $f(x) = 0$.

5.1 Inverse Functions

- One-to-One Functions
- Inverse Functions
- Equations of Inverses
- An Application of Inverse Functions to Cryptography

One-to-One Functions

Suppose we define the function

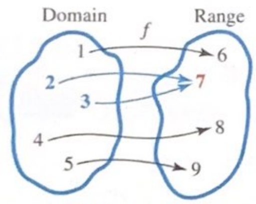
$$F = \{(-2, 2), (-1, 1), (0, 0), (1, 3), (2, 5)\}.$$

(Notice that we have defined F so that each *second* component is used only once.) We can form another set of ordered pairs from F by interchanging the x- and y-values of each pair in F. We call this set G, so

$$G = \{(2, -2), (1, -1), (0, 0), (3, 1), (5, 2)\}.$$

To show that these two sets are related, G is called the *inverse* of F. For a function f to have an inverse, f must be a *one-to-one function*.

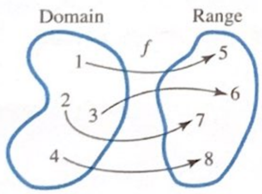
In a one-to-one function, each x-value corresponds to only one y-value, and each y-value corresponds to only one x-value.



Not One-to-One

Figure 1

The function f shown in **Figure 1** is not one-to-one because the y-value 7 corresponds to *two* x-values, 2 and 3. That is, the ordered pairs $(2, 7)$ and $(3, 7)$ both belong to the function. The function f in **Figure 2** is one-to-one.



One-to-One

Figure 2

One-to-One Function

A function f is a **one-to-one function** if, for elements a and b in the domain of f,

تؤدي إلى

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

Using the concept of the *contrapositive* from the study of logic, the last line in the preceding box is equivalent to

one-to-one من خلاله نعرفه هل الداله هي

$$f(a) = f(b) \quad \text{implies} \quad a = b.$$

We use this statement to decide whether a function f is one-to-one in the next example.

أي داله odd حتكون one-to-one function.

أي داله even حتكون not one-to-one function حتى لو كانوا تحت الجذر .

EXAMPLE 1 Deciding Whether Functions Are One-to-One

Decide whether each function is one-to-one.

(a) $f(x) = -4x + 12$ **(b)** $f(x) = \sqrt{25 - x^2}$

SOLUTION

(a) We must show that $f(a) = f(b)$ leads to the result $a = b$.

$$f(a) = f(b)$$

$$-4a + 12 = -4b + 12 \qquad f(x) = -4x + 12$$

$$-4a = -4b \qquad \text{Subtract 12.}$$

$$a = b \qquad \text{Divide by } -4.$$

By the definition, $f(x) = -4x + 12$ is one-to-one.

(b) For the function $f(x) = \sqrt{25 - x^2}$, if we choose $a = 3$ and $b = -3$, then $3 \neq -3$, but

$$f(3) = \sqrt{25 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

and $$f(-3) = \sqrt{25 - (-3)^2} = \sqrt{25 - 9} = 4.$$

Here, even though $3 \neq -3$, $f(3) = f(-3) = 4$. By the definition, f is *not* a one-to-one function.

As illustrated in **Example 1(b),** a way to show that a function is *not* one-to-one is to produce a pair of different domain elements that lead to the same function value. There is also a useful graphical test, the **horizontal line test,** that tells whether or not a function is one-to-one.

Horizontal Line Test

A function is one-to-one if every horizontal line intersects the graph of the function at most once.

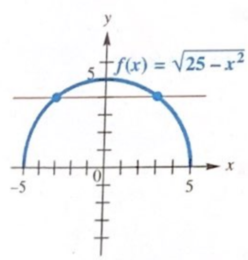


Figure 3

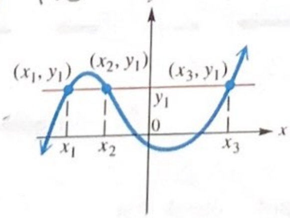
NOTE In **Example 1(b),** the graph of the function is a semicircle, as shown in **Figure 3.** Because there is at least one horizontal line that intersects the graph in more than one point, this function is not one-to-one.

HOMEWORK 1 **Using the Horizontal Line Test**

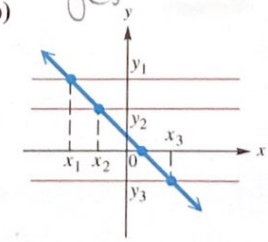
Determine whether each graph is the graph of a one-to-one function.

باختبار الخط الأفقي

(a) No



(b) Yes



Notice that the function graphed in **Homework 1(b)** decreases on its entire domain. ***In general, a function that is either increasing or decreasing on its entire domain, such as $f(x) = -x$, $g(x) = x^3$, and $h(x) = \sqrt{x}$, must be one-to-one.***

In summary, there are four ways to decide whether a function is one-to-one.

Tests to Determine Whether a Function Is One-to-One

1. Show that $f(a) = f(b)$ implies $a = b$. This means that f is one-to-one. **(Example 1(a))**
2. In a one-to-one function, every y-value corresponds to no more than one x-value. To show that a function is not one-to-one, find at least two x-values that produce the same y-value. **(Example 1(b))**
3. Sketch the graph and use the horizontal line test. **(Homework 1)**
4. If the function either increases or decreases on its entire domain, then it is one-to-one. A sketch is helpful here, too. **(Homework 1(b))**

Inverse Functions Consider the functions

$$f(x) = 8x + 5 \quad \text{and} \quad g(x) = \frac{1}{8}x - \frac{5}{8}.$$

Let us choose an arbitrary element from the domain of f, say 10. Evaluate $f(10)$.

$$f(10) = 8 \cdot 10 + 5 \qquad \text{Let } x = 10.$$

$$f(10) = 85 \qquad \text{Multiply and then add.}$$

Now, we evaluate $g(85)$.

$$g(85) = \frac{1}{8}(85) - \frac{5}{8} \qquad \text{Let } x = 85.$$

$$= \frac{85}{8} - \frac{5}{8} \qquad \text{Multiply.}$$

$$= \frac{80}{8} \qquad \text{Subtract.}$$

$$g(85) = 10 \qquad \text{Divide.}$$

Starting with 10, we "applied" function f and then "applied" function g to the result, which returned the number 10. See **Figure 4.**

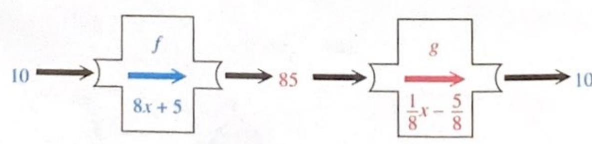


Figure 4

As further examples, check that

$$f(3) = 29 \quad \text{and} \quad g(29) = 3,$$

$$f(-5) = -35 \quad \text{and} \quad g(-35) = -5,$$

$$g(2) = -\frac{3}{8} \quad \text{and} \quad f\left(-\frac{3}{8}\right) = 2.$$

In particular, for this pair of functions,

$$f(g(2)) = 2 \quad \text{and} \quad g(f(2)) = 2.$$

In fact, for *any* value of x,

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

Using the notation for composition introduced in **Section 3.3,** these two equations can be written as follows.

$$(f \circ g)(x) = x \quad \text{and} \quad (g \circ f)(x) = x \qquad \text{The result is the identity function.}$$

Because the compositions of f and g yield the *identity* function, they are *inverses* of each other.

Important

Inverse Function

Let f be a one-to-one function. Then g is the **inverse function** of f if

$$(f \circ g)(x) = x \quad \text{for every } x \text{ in the domain of } g,$$

and

$$(g \circ f)(x) = x \quad \text{for every } x \text{ in the domain of } f.$$

The condition that f is one-to-one in the definition of inverse function is essential. Otherwise, g will not define a function.

EXAMPLE 2 Deciding Whether Two Functions Are Inverses

Let functions f and g be defined by

$$f(x) = x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{x + 1},$$

respectively. Is g the inverse function of f?

SOLUTION As shown in **Figure 5,** the horizontal line test applied to the graph indicates that f is one-to-one, so the function does have an inverse. Since it is one-to-one, we now find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= \left(\sqrt[3]{x + 1}\right)^3 - 1 \quad \text{(Section 3.3)}\\ &= x + 1 - 1\\ &= x\end{aligned}$$

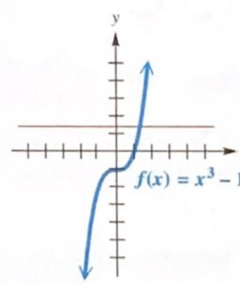


Figure 5

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= \sqrt[3]{(x^3 - 1) + 1}\\ &= \sqrt[3]{x^3}\\ &= x\end{aligned}$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, function g is the inverse of function f.

A special notation is used for inverse functions: If g is the inverse of a function f, then g is written as f^{-1} (read **"f-inverse"**). In **Example 2,**

$$f(x) = x^3 - 1 \quad \text{has inverse} \quad f^{-1}(x) = \sqrt[3]{x + 1}.$$

CAUTION ***Do not confuse the −1 in f^{-1} with a negative exponent.*** The symbol $f^{-1}(x)$ does not represent $\frac{1}{f(x)}$. It represents the inverse function of f.

By the definition of inverse function, the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}. See **Figure 6.**

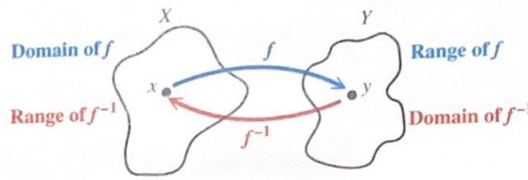


Figure 6

HOMEWORK 2 Finding Inverses of One-to-One Functions

Find the inverse of each function that is one-to-one.

(a) $F = \{(-2, 1), (-1, 0), (0, 1), (1, 2), (2, 2)\}$

(b) $G = \{(3, 1), (0, 2), (2, 3), (4, 0)\}$

(c) The table in the margin shows the number of days in Illinois that were unhealthy for sensitive groups for selected years using the Air Quality Index (AQI). Let f be the function defined in the table, with the years forming the domain and the numbers of unhealthy days forming the range.

Year	Number of Unhealthy Days
2004	7
2005	32
2006	8
2007	24
2008	14
2009	13

Source: Illinois Environmental Protection Agency.

* لما يكون الدالة one-to-one
لازم يكون لها Inverses.

Equations of Inverses The inverse of a one-to-one function is found by interchanging the x- and y-values of each of its ordered pairs. The equation of the inverse of a function defined by $y = f(x)$ is found in the same way, as given in the box on the next page.

Finding the Equation of the Inverse of $y = f(x)$

For a one-to-one function f defined by an equation $y = f(x)$, find the defining equation of the inverse as follows. (If necessary, replace $f(x)$ with y first. Any restrictions on x and y should be considered.)

1 احتيل الـ f(x) واحط بدالها y.

2 ***Step 1*** Interchange x and y.

3 ***Step 2*** Solve for y.

4 ***Step 3*** Replace y with $f^{-1}(x)$.

EXAMPLE 3 Finding Equations of Inverses

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(a) $f(x) = 2x + 5$ **(b)** $y = x^2 + 2$ **(c)** $f(x) = (x - 2)^3$

SOLUTION

(a) The graph of $y = 2x + 5$ is a nonhorizontal line, so by the horizontal line test, f is a one-to-one function. To find the equation of the inverse, follow the steps in the preceding box, first replacing $f(x)$ with y.

$$
\begin{aligned}
f(x) &= 2x + 5 \\
y &= 2x + 5 && \text{Let } y = f(x). \\
x &= 2y + 5 && \text{Interchange } x \text{ and } y. \text{ (Step 1)} \\
x - 5 &= 2y && \left.\begin{array}{l}\text{Subtract 5.} \\ \text{Divide by 2.} \\ \text{Rewrite.}\end{array}\right\} \text{ Solve for } y. \text{ (Step 2)} \\
y &= \frac{x-5}{2} && \\
f^{-1}(x) &= \frac{1}{2}x - \frac{5}{2} && \text{Replace } y \text{ with } f^{-1}(x). \text{ (Step 3)} \quad \frac{a-b}{c} = \left(\frac{1}{c}\right)a - \frac{b}{c}
\end{aligned}
$$

Thus, $f^{-1}(x) = \frac{x-5}{2} = \frac{1}{2}x - \frac{5}{2}$ is a linear function. In the function defined by $y = 2x + 5$, the value of y is found by starting with a value of x, multiplying by 2, and adding 5.

The form $f^{-1}(x) = \frac{x-5}{2}$ for the equation of the inverse has us *subtract* 5 and then *divide* by 2. This shows how an inverse is used to "undo" what a function does to the variable x.

(b) The equation $y = x^2 + 2$ has a parabola opening up as its graph, so some horizontal lines will intersect the graph at two points. For example, both $x = 3$ and $x = -3$ correspond to $y = 11$. Because of the presence of the x^2-term, there are many pairs of x-values that correspond to the same y-value. This means that the function defined by $y = x^2 + 2$ is not one-to-one and does not have an inverse.

The steps for finding the equation of an inverse lead to the following.

$$
\begin{aligned}
y &= x^2 + 2 \\
x &= y^2 + 2 && \text{Interchange } x \text{ and } y. \\
x - 2 &= y^2 && \text{Solve for } y. \\
\pm\sqrt{x-2} &= y && \text{Square root property (Section 2.3)}
\end{aligned}
$$

Remember *both* roots.

The last step shows that there are two y-values for each choice of x greater than 2, so the given function is not one-to-one and cannot have an inverse.

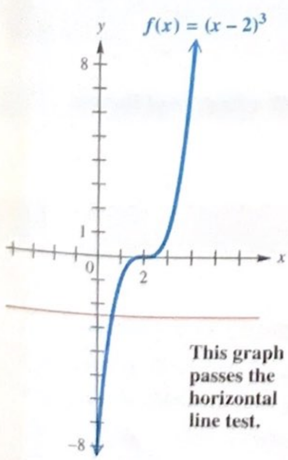


Figure 7

(c) **Figure 7** shows that the horizontal line test assures us that this horizontal translation of the graph of the cubing function is one-to-one.

$$
\begin{aligned}
f(x) &= (x-2)^3 && \text{Given function} \\
y &= (x-2)^3 && \text{Replace } f(x) \text{ with } y. \\
x &= (y-2)^3 && \text{Interchange } x \text{ and } y. \\
\sqrt[3]{x} &= \sqrt[3]{(y-2)^3} && \text{Take the cube root on each side.} \\
\sqrt[3]{x} &= y - 2 && \sqrt[3]{a^3} = a \text{ (Section 1.6)} \\
\sqrt[3]{x} + 2 &= y && \text{Solve for } y \text{ by adding 2.} \\
f^{-1}(x) &= \sqrt[3]{x} + 2 && \text{Replace } y \text{ with } f^{-1}(x). \text{ Rewrite.}
\end{aligned}
$$

Very important

HOMEWORK 3 Finding the Equation of the Inverse of a Rational Function

$y = \frac{2x+3}{x-4}$

$x = \frac{2y+3}{y-4}$

$x(y-4) = 2y+3$

$xy - 4x = 2y + 3$

$xy - 2y = 4x + 3$

$y(x-2) = 4x + 3$

$y = \frac{4x+3}{x-2}$

The rational function

$$f(x) = \frac{2x+3}{x-4}, \quad x \neq 4$$

is a one-to-one function. Find its inverse.

One way to graph the inverse of a function f whose equation is known follows.

Step 1 Find some ordered pairs that are on the graph of f.

Step 2 Interchange x and y to get ordered pairs that are on the graph of f^{-1}.

Step 3 Plot those points, and sketch the graph of f^{-1} through them.

Another way is to select points on the graph of f and use symmetry to find corresponding points on the graph of f^{-1}.

For example, suppose the point (a, b) shown in **Figure 8** is on the graph of a one-to-one function f. Then the point (b, a) is on the graph of f^{-1}. The line segment connecting (a, b) and (b, a) is perpendicular to, and cut in half by, the line $y = x$. The points (a, b) and (b, a) are "mirror images" of each other with respect to $y = x$. ***Thus, we can find the graph of f^{-1} from the graph of f by locating the mirror image of each point in f with respect to the line $y = x$.***

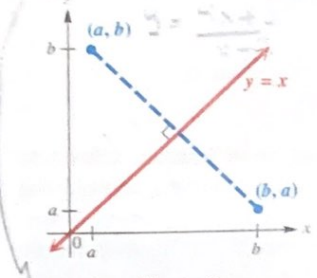


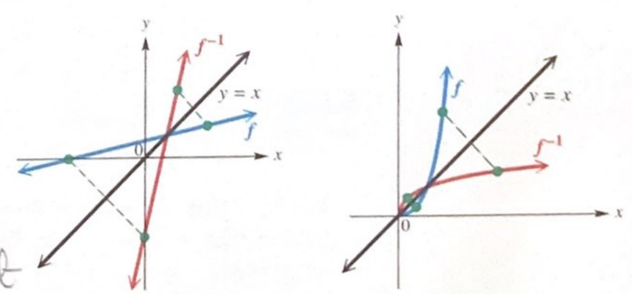
Figure 8

$f^{-1}(x) = \frac{4x+3}{x-2}, \quad x \neq 2$

EXAMPLE 4 Graphing f^{-1} Given the Graph of f

In each set of axes in **Figure 9**, the graph of a one-to-one function f is shown in blue. Graph f^{-1} in red.

SOLUTION In **Figure 9**, the graphs of two functions f shown in blue are given with their inverses shown in red. In each case, the graph of f^{-1} is a reflection of the graph of f with respect to the line $y = x$.



Important

Figure 9

HOMEWORK 4 Finding the Inverse of a Function with a Restricted Domain

Let $f(x) = \sqrt{x+5}, \quad x \geq -5$. Find $f^{-1}(x)$.

Important Facts about Inverses

1. If f is one-to-one, then f^{-1} exists.
2. The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.
3. If the point (a, b) lies on the graph of f, then (b, a) lies on the graph of f^{-1}. The graphs of f and f^{-1} are reflections of each other across the line $y = x$.
4. To find the equation for f^{-1}, replace $f(x)$ with y, interchange x and y, and solve for y. This gives $f^{-1}(x)$.

An Application of Inverse Functions to Cryptography A one-to-one function and its inverse can be used to make information secure. The function is used to encode a message, and its inverse is used to decode the coded message. In practice, complicated functions are used. We illustrate the process with a simple function in **Example 5.**

EXAMPLE 5 Using Functions to Encode and Decode a Message

Use the one-to-one function $f(x) = 3x + 1$ and the following numerical values assigned to each letter of the alphabet to encode and decode the message BE MY FACEBOOK FRIEND.

A	1	**H**	8	**O**	15	**V**	22
B	2	**I**	9	**P**	16	**W**	23
C	3	**J**	10	**Q**	17	**X**	24
D	4	**K**	11	**R**	18	**Y**	25
E	5	**L**	12	**S**	19	**Z**	26
F	6	**M**	13	**T**	20		
G	7	**N**	14	**U**	21		

SOLUTION The message BE MY FACEBOOK FRIEND would be encoded as

7 16 40 76 19 4 10 16 7
46 46 34 19 55 28 16 43 13

because

B corresponds to 2 and $f(2) = 3(2) + 1 = 7$,

E corresponds to 5 and $f(5) = 3(5) + 1 = 16$, and so on.

Using the inverse $f^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$ to decode yields

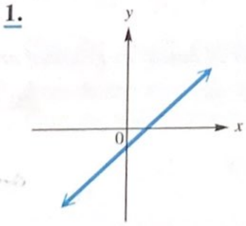
$$f^{-1}(7) = \frac{1}{3}(7) - \frac{1}{3} = 2, \quad \text{which corresponds to B,}$$

$$f^{-1}(16) = \frac{1}{3}(16) - \frac{1}{3} = 5, \quad \text{which corresponds to E,} \quad \text{and so on.}$$

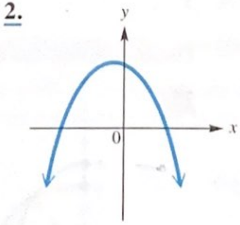
5.1 Exercises

Decide whether each function as graphed or defined is one-to-one. **See Example 1 and Homework 1.**

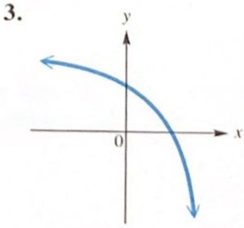
1.



2.



3.



4. $y = 2x - 8$

5. $y = \sqrt{36 - x^2}$

6. $y = 2x^3 - 1$

7. $y = \dfrac{-1}{x + 2}$

8. $y = 2(x + 1)^2 - 6$

9. $y = \sqrt[3]{x + 1} - 3$

10. Explain why a constant function, such as $f(x) = 3$, defined over the set of real numbers, cannot be one-to-one.

Concept Check *Answer each of the following.*

11. For a function to have an inverse, it must be ____________.

12. The domain of f is equal to the ____________ of f^{-1}, and the range of f is equal to the ____________ of f^{-1}.

13. *True* or *false*: If $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$.

14. If a function f has an inverse and $f(-3) = 6$, then $f^{-1}(6) =$ ____________.

Concept Check *In Exercises 15–17, an everyday activity is described. Keeping in mind that an inverse operation "undoes" what an operation does, describe each inverse activity.*

15. tying your shoelaces

16. entering a room

17. screwing in a light bulb

Decide whether the given functions are inverses. **See Homework 2.**

18.

x	$f(x)$
3	−4
2	−6
5	8
1	9
4	3

x	$g(x)$
−4	3
−6	2
8	5
9	1
3	4

19. $f = \{(2, 5), (3, 5), (4, 5)\}$; $g = \{(5, 2)\}$

Use the definition of inverses to determine whether f and g are inverses. **See Example 2.**

20. $f(x) = 2x + 4$, $g(x) = \frac{1}{2}x - 2$

21. $f(x) = -3x + 12$, $g(x) = -\frac{1}{3}x - 12$

22. $f(x) = \frac{x + 1}{x - 2}$, $g(x) = \frac{2x + 1}{x - 1}$

23. $f(x) = \frac{2}{x + 6}$, $g(x) = \frac{6x + 2}{x}$

24. $f(x) = x^2 + 3$, $x \geq 0$; $g(x) = \sqrt{x - 3}$, $x \geq 3$

If the function is one-to-one, find its inverse. **See Homework 2.**

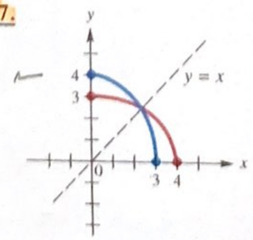
25. $\{(-3, 6), (2, 1), (5, 8)\}$

26. $\{(1, -3), (2, -7), (4, -3), (5, -5)\}$

Decide whether each pair of functions graphed are inverses. **See Example 3.**

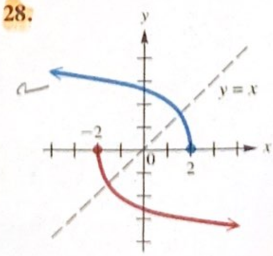
27.

yes



28.

no



For each function as defined that is one-to-one, **(a)** *write an equation for the inverse function in the form* $y = f^{-1}(x)$, **(b)** *graph f and* f^{-1} *on the same axes, and* **(c)** *give the domain and the range of f and* f^{-1}. *If the function is not one-to-one, say so.* ***See Examples 3–4.***

29. $y = 3x - 4$ **30.** $f(x) = -4x + 3$ **31.** $f(x) = x^3 + 1$

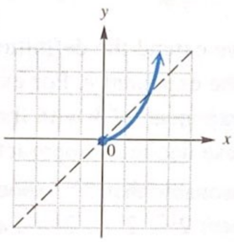
32. $y = x^2 + 8$ **33.** $y = \frac{1}{x}, \quad x \neq 0$ **34.** $f(x) = \frac{1}{x-3}, \quad x \neq 3$

35. $f(x) = \frac{x+1}{x-3}, \quad x \neq 3$ **36.** $f(x) = \frac{2x+6}{x-3}, \quad x \neq 3$

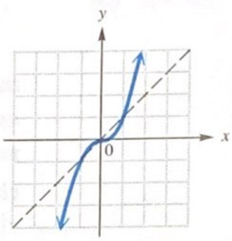
37. $f(x) = \sqrt{6 + x}, \quad x \geq -6$

Graph the inverse of each one-to-one function. ***See Example 4.***

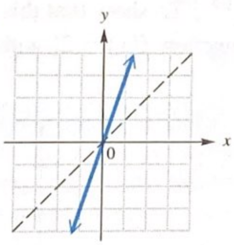
38.



39.



40.



Concept Check *The graph of a function f is shown in the figure. Use the graph to find each value.*

41. $f^{-1}(4)$ **42.** $f^{-1}(0)$ **43.** $f^{-1}(-3)$

Concept Check *Answer each of the following.*

44. Suppose $f(x)$ is the number of cars that can be built for x dollars. What does $f^{-1}(1000)$ represent?

45. If a line has slope a, what is the slope of its reflection across the line $y = x$?

Use the alphabet coding assignment given in ***Example 5*** *for Exercises 46–47.*

46. The function $f(x) = 3x - 2$ was used to encode a message as

37 25 19 61 13 34 22 1 55 1 52 52 25 64 13 10.

Find the inverse function and determine the message.

47. Encode the message SEND HELP, using the one-to-one function $f(x) = x^3 - 1$. Give the inverse function that the decoder will need when the message is received.

الدالة الأسية

5.2 Exponential Functions

- Exponents and Properties
- Exponential Functions
- Exponential Equations
- Compound Interest
- The Number e and Continuous Compounding
- Exponential Models

Exponents and Properties Recall the definition of $a^{m/n}$: If a is a real number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number, then

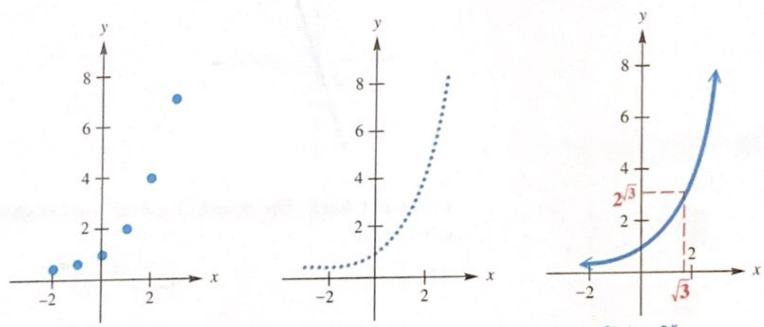
$$a^{m/n} = \left(\sqrt[n]{a}\right)^m. \quad \text{(Section 1.6)}$$

For example,

$$16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8,$$

$$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}, \quad \text{and} \quad 64^{-1/2} = \frac{1}{64^{1/2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

In this section we extend the definition of a^r to include all real (not just rational) values of the exponent r. For example, $2^{\sqrt{3}}$ might be evaluated by *approximating* the exponent $\sqrt{3}$ with the rational numbers 1.7, 1.73, 1.732, and so on. Since these decimals approach the value of $\sqrt{3}$ more and more closely, it seems reasonable that $2^{\sqrt{3}}$ should be approximated more and more closely by the numbers $2^{1.7}$, $2^{1.73}$, $2^{1.732}$, and so on. (Recall, for example, that $2^{1.7} = 2^{17/10} = \left(\sqrt[10]{2}\right)^{17}$.) To show that this assumption is reasonable, **Figure 10** gives graphs of the function $f(x) = 2^x$ with three different domains.



$f(x) = 2^x$; integers as domain

$f(x) = 2^x$; selected rational numbers as domain

$f(x) = 2^x$; real numbers as domain

Figure 10

Using this interpretation of real exponents, all rules and theorems for exponents are valid for all real number exponents, not just rational ones. In addition to the rules for exponents presented earlier, we use several new properties in this chapter. These properties are generalized below. Proofs of the properties are not given here, because they require more advanced mathematics.

Additional Properties of Exponents

For any real number $a > 0$, $a \neq 1$, the following statements are true.

(a) a^x **is a unique real number for all real numbers** x.

Very important **(b)** $a^b = a^c$ **if and only if** $b = c$.

(c) If $a > 1$ **and** $m < n$, **then** $a^m < a^n$.

(d) If $0 < a < 1$ **and** $m < n$, **then** $a^m > a^n$.

Properties (a) and (b) require $a > 0$ so that a^x is always defined. For example, $(-6)^x$ is not a real number if $x = \frac{1}{2}$. This means that a^x will always be positive, since a must be positive. In property (a), a cannot equal 1 because $1^x = 1$ for every real number value of x, so each value of x leads to the same real number, 1. For property (b) to hold, a must not equal 1 since, for example, $1^4 = 1^5$, even though $4 \neq 5$.

Properties (c) and (d) say that ***when $a > 1$, increasing the exponent on "a" leads to a greater number, but when $0 < a < 1$, increasing the exponent on "a" leads to a lesser number.***

EXAMPLE 1 Evaluating an Exponential Expression

If $f(x) = 2^x$, find each of the following.

(a) $f(-1)$ (b) $f(3)$ (c) $f\left(\frac{5}{2}\right)$ (d) $f(4.92)$

SOLUTION

(a) $f(-1) = 2^{-1} = \frac{1}{2}$ Replace x with -1.

(b) $f(3) = 2^3 = 8$

(c) $f\left(\frac{5}{2}\right) = 2^{5/2} = (2^5)^{1/2} = 32^{1/2} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$ (Section 1.6)

(d) $f(4.92) = 2^{4.92} \approx 30.2738447$

Exponential Functions We now define a function $f(x) = a^x$ whose domain is the set of all real numbers. Notice how the independent variable x appears in the exponent in this function. In earlier chapters the independent variable did not appear in exponents.

Exponential Function

If $a > 0$ and $a \neq 1$, then

$$f(x) = a^x$$

defines the **exponential function with base a.**

NOTE ***We do not allow 1 as the base for an exponential function.*** If $a = 1$, the function becomes the constant function defined by $f(x) = 1$, which is not an exponential function.

Figure 10 showed the graph of $f(x) = 2^x$ with three different domains. We repeat the final graph (with real numbers as domain) here.

- The y-intercept is $y = 2^0 = 1$.
- Since $2^x > 0$ for all x and $2^x \to 0$ as $x \to -\infty$, the x-axis is a horizontal asymptote.
- As the graph suggests, the domain of the function is $(-\infty, \infty)$ and the range is $(0, \infty)$.
- The function is increasing on its entire domain, and it therefore is one-to-one.

These observations from **Figure 10** lead to the following generalizations about the graphs of exponential functions.

Exponential Function $f(x) = a^x$

Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

For $f(x) = 2^x$:

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

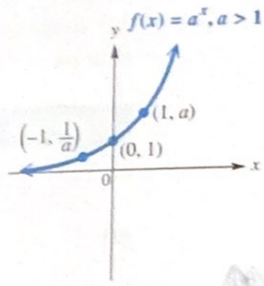


Figure 11

- $f(x) = a^x$**, for** $a > 1$**,** is increasing and continuous on its entire domain, $(-\infty, \infty)$.

important ←
- The x-axis is a horizontal asymptote as $x \to -\infty$.
- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.

For $f(x) = \left(\frac{1}{2}\right)^x$:

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

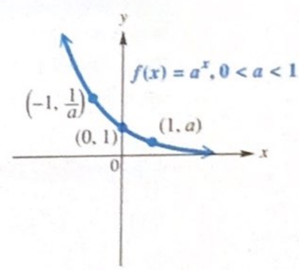


Figure 12

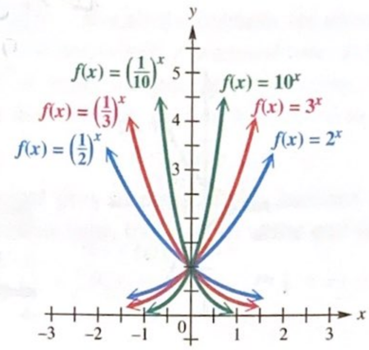
- $f(x) = a^x$**, for** $0 < a < 1$**,** is decreasing and continuous on its entire domain, $(-\infty, \infty)$.
- The x-axis is a horizontal asymptote as $x \to \infty$.
- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.

The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis. Thus, we have the following.

$$\text{If } f(x) = 2^x, \quad \text{then} \quad f(-x) = 2^{-x} = 2^{-1 \cdot x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x.$$

This is supported by the graphs in **Figures 11 and 12.**

The graph of $f(x) = 2^x$ is typical of graphs of $f(x) = a^x$ where $a > 1$. For larger values of a, the graphs rise more steeply, but the general shape is similar to the graph in **Figure 11.** When $0 < a < 1$, the graph decreases in a manner similar to the graph of $f(x) = \left(\frac{1}{2}\right)^x$. In **Figure 13** on the next page, the graphs of several typical exponential functions illustrate these facts.



$f(x) = a^x$

Domain: $(-\infty, \infty)$; Range: $(0, \infty)$

- When $a > 1$, the function is increasing.
- When $0 < a < 1$, the function is decreasing.
- In every case, the x-axis is a horizontal asymptote.

Figure 13

In summary, the graph of a function of the form $f(x) = a^x$ has the following features.

Characteristics of the Graph of $f(x) = a^x$

1. The points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$ are on the graph.
2. If $a > 1$, then f is an increasing function. If $0 < a < 1$, then f is a decreasing function. important
3. The x-axis is a horizontal asymptote. or $y=0$ important
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

HOMEWORK 1 **Graphing an Exponential Function**

Graph $f(x) = 5^x$. Give the domain and range. D: ℝ R= $(0,\infty)$

increasing

EXAMPLE 2 **Graphing Reflections and Translations**

Graph each function. Show the graph of $y = 2^x$ for comparison. Give the domain and range.

(a) $f(x) = -2^x$ **(b)** $f(x) = 2^{x+3}$ **(c)** $f(x) = 2^{x-2} - 1$

SOLUTION In each graph, we show in particular how the point $(0, 1)$ on the graph of $y = 2^x$ has been translated.

(a) The graph of $f(x) = -2^x$ is that of $f(x) = 2^x$ reflected across the x-axis. The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0)$. See **Figure 14.**

(b) The graph of $f(x) = 2^{x+3}$ is the graph of $f(x) = 2^x$ translated 3 units to the left, as shown in **Figure 15.** The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

(c) The graph of $f(x) = 2^{x-2} - 1$ is that of $f(x) = 2^x$ translated 2 units to the right and 1 unit down. See **Figure 16.** The domain is $(-\infty, \infty)$, and the range is $(-1, \infty)$.

وجود ال K بغير ال asymptote و Range ال line

وجود ال ما لا يغير حتى ولكنه بعمل ازاحة

$2^{x+3} \to 2^{x+c} + K$

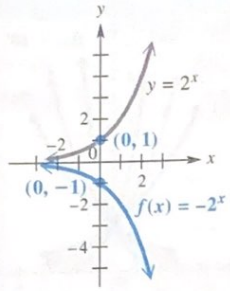


Figure 14

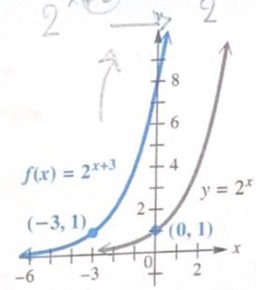


Figure 15

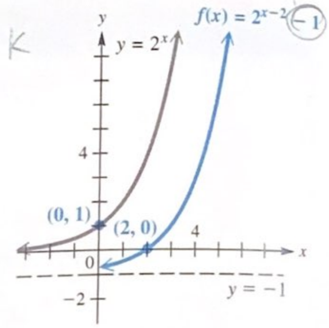


Figure 16

Exponential Equations Because the graph of $y = a^x$ is that of a one-to-one function, to solve $a^{x_1} = a^{x_2}$, we need only show that $x_1 = x_2$. Property (b) given earlier in this section is used to solve **exponential equations,** which are equations with variables as exponents.

بنحل المسألة

HOMEWORK 2 **Solving an Exponential Equation**

Solve $\left(\frac{1}{3}\right)^x = 81.$ → $(3^{-1})^x = 81$

$3^{-x} = 3^4$

$-x = 4 \to x = -4 \to$ Solution Set $= \{-4\}$

EXAMPLE 3 **Solving an Exponential Equation**

Solve $2^{x+4} = 8^{x-6}$.

SOLUTION Write each side of the equation using a common base.

$$2^{x+4} = 8^{x-6}$$

$$2^{x+4} = (2^3)^{x-6} \quad \text{Write 8 as a power of 2.}$$

$$2^{x+4} = 2^{3x-18} \quad (a^m)^n = a^{mn}$$

$$x + 4 = 3x - 18 \quad \text{Set exponents equal (Property (b)).}$$

$$-2x = -22 \quad \text{Subtract } 3x \text{ and 4. (Section 2.1)}$$

$$x = 11 \quad \text{Divide by } -2.$$

Check by substituting 11 for x in the original equation. The solution set is $\{11\}$.

Later in this chapter, we describe a general method for solving exponential equations where the approach used in **Example 3 and Home Work 2** is not possible. For instance, the above method could not be used to solve an equation like

$$7^x = 12,$$

since it is not easy to express both sides as exponential expressions with the same base.

Very important

HOMEWORK 3 **Solving an Equation with a Fractional Exponent**

Solve $x^{4/3} = 81.$ $= -27$

سـ حلي

Compound Interest Recall the formula for simple interest, $I = Prt$, where P is principal (amount deposited), r is annual rate of interest expressed as a decimal, and t is time in years that the principal earns interest. Suppose $t = 1$ yr. Then at the end of the year, the amount has grown to

$$P + Pr = P(1 + r),$$

the original principal plus interest. If this balance earns interest at the same interest rate for another year, the balance at the end of *that* year will be

$$\begin{aligned}[P(1 + r)] + [P(1 + r)]r &= [P(1 + r)](1 + r) && \text{Factor.}\\ &= P(1 + r)^2. && a \cdot a = a^2\end{aligned}$$

After the third year, this will grow to

$$\begin{aligned}[P(1 + r)^2] + [P(1 + r)^2]r &= [P(1 + r)^2](1 + r) && \text{Factor.}\\ &= P(1 + r)^3. && a^2 \cdot a = a^3\end{aligned}$$

Continuing in this way produces a formula for interest compounded annually.

$$A = P(1 + r)^t$$

The general formula for compound interest can be derived in the same way.

Compound Interest

If P dollars are deposited in an account paying an annual rate of interest r compounded (paid) n times per year, then after t years the account will contain A dollars, according to the following formula.

$$A = P\left(1 + \frac{r}{n}\right)^{tn}$$

5.2 Exercises

For $f(x) = 3^x$ and $g(x) = \left(\frac{1}{4}\right)^x$, find each of the following. In Exercises 7 and 8, round the answer to the nearest thousandth. ***See Example 1.***

1. $f(2)$
2. $f(-2)$
3. $g(2)$
4. $g(-2)$
5. $f\left(\frac{3}{2}\right)$
6. $g\left(\frac{3}{2}\right)$
7. $f(2.34)$
8. $g(-1.68)$

Graph each function. ***See Homework 1.***

9. $f(x) = 3^x$
10. $f(x) = \left(\frac{1}{3}\right)^x$
11. $f(x) = \left(\frac{3}{2}\right)^x$
12. $f(x) = \left(\frac{1}{10}\right)^{-x}$
13. $f(x) = 4^{-x}$
14. $f(x) = 2^{|x|}$

Sketch the graph of $f(x) = 2^x$. Then refer to it and use the techniques of ***Chapter 3*** *to graph each function as defined.* ***See Example 2.***

15. $f(x) = 2^x + 1$
16. $f(x) = 2^{x+1}$
17. $f(x) = -2^{x+2}$
18. $f(x) = 2^{-x}$
19. $f(x) = 2^{x-1} + 2$
20. $f(x) = 2^{x+2} - 4$

Sketch the graph of $f(x) = \left(\frac{1}{3}\right)^x$. Then refer to it and use the techniques of ***Chapter 3*** *to graph each function as defined.* ***See Example 2.***

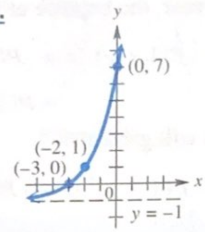
21. $f(x) = \left(\frac{1}{3}\right)^x - 2$
22. $f(x) = \left(\frac{1}{3}\right)^{x+2}$
23. $f(x) = \left(\frac{1}{3}\right)^{-x+1}$

24. $f(x) = \left(\frac{1}{3}\right)^{-x}$ 25. $f(x) = \left(\frac{1}{3}\right)^{x-2} + 2$ **26.** $f(x) = \left(\frac{1}{3}\right)^{x+2} - 1$

27. ***Concept Check*** Fill in the blank: The graph of $f(x) = a^{-x}$ is the same as that of $g(x) = (\_\_\_\_\_)^x$.

Connecting Graphs with Equations *Write an equation for the graph given. Each represents an exponential function f, with base 2 or 3, translated and/or reflected.*

28.



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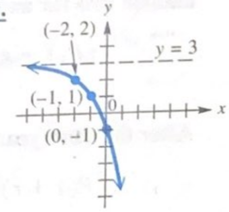

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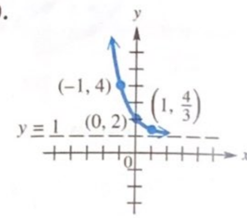


Solve each equation. See Example 3 and Homework 2–3.

31. $4^x = 2$ 32. $\left(\frac{5}{2}\right)^x = \frac{4}{25}$ **33.** $2^{3-2x} = 8$

34. $e^{4x-1} = (e^2)^x$ **35.** $27^{4x} = 9^{x+1}$ **36.** $4^{x-2} = 2^{3x+3}$

37. $\left(\frac{1}{e}\right)^{-x} = \left(\frac{1}{e^2}\right)^{x+1}$ 38. $(\sqrt{2})^{x+4} = 4^x$ 39. $\frac{1}{27} = x^{-3}$

40. $x^{2/3} = 4$ 41. $x^{5/2} = 32$ 42. $x^{-6} = \frac{1}{64}$

43. $x^{5/3} = -243$

5.3 Logarithmic Functions

- Logarithms
- Logarithmic Equations
- Logarithmic Functions
- Properties of Logarithms

Logarithms The previous section dealt with exponential functions of the form $y = a^x$ for all positive values of a, where $a \neq 1$. The horizontal line test shows that exponential functions are one-to-one and thus have inverse functions. The equation defining the inverse of a function is found by interchanging x and y in the equation that defines the function. Starting with $y = a^x$ and interchanging x and y yields

$$x = a^y.$$

Here y is the exponent to which a must be raised in order to obtain x. We call this exponent a **logarithm,** symbolized by the abbreviation **"log."** The expression $\log_a x$ represents the logarithm in this discussion. The number a is called the **base** of the logarithm, and x is called the **argument** of the expression. It is read **"logarithm with base a of x,"** or **"logarithm of x with base a,"** or **"base a logarithm of x."**

Logarithm

For all real numbers y and all positive numbers a and x, where $a \neq 1$,

important $y = \log_a x$ is equivalent to $x = a^y$.

The expression $\log_a x$ represents the exponent to which the base a must be raised in order to obtain x.

EXAMPLE 1 Writing Equivalent Logarithmic and Exponential Forms

The table shows several pairs of equivalent statements, written in both logarithmic and exponential forms.

SOLUTION

Logarithmic Form	Exponential Form
$\log_2 8 = 3$	$2^3 = 8$
$\log_{1/2} 16 = -4$	$\left(\frac{1}{2}\right)^{-4} = 16$
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_3 \frac{1}{81} = -4$	$3^{-4} = \frac{1}{81}$
$\log_5 5 = 1$	$5^1 = 5$
$\log_{3/4} 1 = 0$	$\left(\frac{3}{4}\right)^0 = 1$

$\log_a a = 1$ *

$\log 1 = 0$ *

Bases لأي

To remember the relationships among a, x, and y in the two equivalent forms $y = \log_a x$ and $x = a^y$, refer to these diagrams.

A logarithm is an exponent.

$$\text{Logarithmic form: } y = \log_{\underset{\text{Base}}{\uparrow}{a}} x \quad (\overset{\text{Exponent}}{\downarrow} y)$$

$$\text{Exponential form: } \underset{\text{Base}}{a}^{\overset{\text{Exponent}}{y}} = x$$

Logarithmic Equations The definition of logarithm can be used to solve a **logarithmic equation,** which is an equation with a logarithm in at least one term. Many logarithmic equations can be solved by first writing the equation in exponential form.

HOMEWORK 1 Solving Logarithmic Equations

لحل المعادلات اللوغاريتمية
التي تحتوي على log في
طرف والطرف الآخر
عدد نحولها لمعادلة
أسية

Solve each equation.

(a) $\log_x \frac{8}{27} = 3$ (b) $\log_4 x = \frac{5}{2}$ (c) $\log_{49} \sqrt[3]{7} = x$

Logarithmic Functions We define the logarithmic function with base a as follows.

Logarithmic Function

important

If $a > 0$, $a \neq 1$, and $x > 0$, then

$$f(x) = \log_a x$$

defines the **logarithmic function with base a.**

Exponential and logarithmic functions are inverses of each other. The graph of $y = 2^x$ is shown in red in **Figure 17.** The graph of its inverse is found by reflecting the graph of $y = 2^x$ across the line $y = x$. The graph of the inverse function, defined by $y = \log_2 x$, shown in blue, has the y-axis as a vertical asymptote.

x	2^x	x	$\log_2 x$
-2	0.25	0.25	-2
-1	0.5	0.5	-1
0	1	1	0
1	2	2	1
2	4	4	2

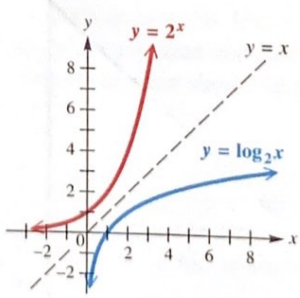


Figure 17

Since the domain of an exponential function is the set of all real numbers, the range of a logarithmic function also will be the set of all real numbers. In the same way, both the range of an exponential function and the domain of a logarithmic function are the set of all positive real numbers.

Thus, logarithms can be found for positive numbers only.

Logarithmic Function $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

x	$f(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

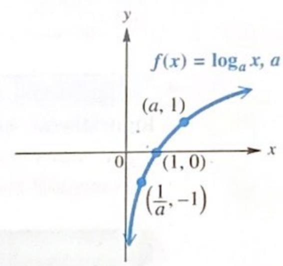


Figure 18

- $f(x) = \log_a x$**, for** $a > 1$**,** is increasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

For $f(x) = \log_{1/2} x$:

x	$f(x)$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2
8	-3

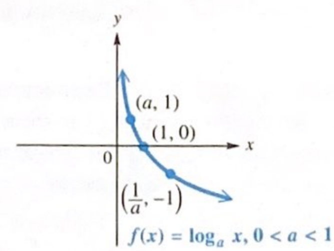


Figure 19

- $f(x) = \log_a x$**, for** $0 < a < 1$**,** is decreasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

صفحة ٥٦ مهمة

The graphs in **Figures 18 and 19** and the information with them suggest the following generalizations about the graphs of logarithmic functions of the form $f(x) = \log_a x$.

Characteristics of the Graph of $f(x) = \log_a x$

1. The points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$ are on the graph.
2. If $a > 1$, then f is an increasing function. If $0 < a < 1$, then f is a decreasing function. Important
3. The y-axis is a vertical asymptote. $x = 0$
4. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

* **EXAMPLE 2 Graphing Logarithmic Functions**

Graph each function. Domain – Range – Increasing or decreasing – Asymptote line

(a) $f(x) = \log_{1/2} x$ * D = (0, ∞) * R = ℝ * Decreasing * asymptote line = Vertical

(b) $f(x) = \log_3 x$ * D = (0, ∞) R = ℝ * Increasing * y-axis

SOLUTION

(a) One approach is to first graph $y = \left(\frac{1}{2}\right)^x$, which defines the inverse function of f, by plotting points. Some ordered pairs are given in the table with the graph shown in red in **Figure 20.** The graph of $f(x) = \log_{1/2} x$ is the reflection of the graph of $y = \left(\frac{1}{2}\right)^x$ across the line $y = x$. The ordered pairs for $y = \log_{1/2} x$ are found by interchanging the x- and y-values in the ordered pairs for $y = \left(\frac{1}{2}\right)^x$. See the graph in blue in **Figure 20.**

x	$y = \left(\frac{1}{2}\right)^x$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
4	$\frac{1}{16}$

x	$f(x) = \log_{1/2} x$
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2
$\frac{1}{16}$	4

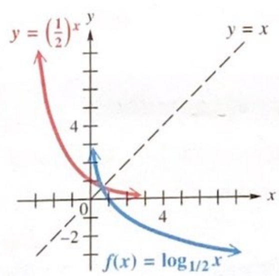


Figure 20

x	$f(x) = \log_3 x$
$\frac{1}{3}$	-1
1	0
3	1
9	2

Think: $x = 3^y$

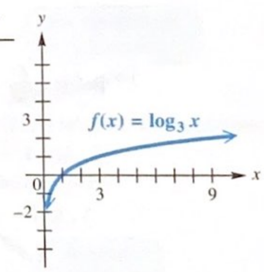


Figure 21

(b) Another way to graph a logarithmic function is to write $f(x) = y = \log_3 x$ in exponential form as $x = 3^y$, and then select y-values and calculate corresponding x-values. Several selected ordered pairs are shown in the table for the graph in **Figure 21.**

CAUTION If you write a logarithmic function in exponential form to graph, as in **Example 2(b),** start *first* with y-values to calculate corresponding x-values. ***Be careful to write the values in the ordered pairs in the correct order.***

More general logarithmic functions can be obtained by forming the composition of $f(x) = \log_a x$ with a function $g(x)$. For example, if $f(x) = \log_2 x$ and $g(x) = x - 1$, then

$$(f \circ g)(x) = f(g(x)) = \log_2(x - 1). \quad \text{(Section 3.3)}$$

The next example shows how to graph such functions.

Important

HOMEWORK 2 **Graphing Translated Logarithmic Functions**

Graph each function. Give the domain and range.

(a) $f(x) = \log_2(x - 1)$ D=(1,∞) / Increasing R=ℝ / x=1 → Vertical asymptote

(b) $f(x) = (\log_3 x) - 1$ D= (0,∞) R= ℝ Increasing. X=0 → Vertical asymptote

(c) $f(x) = \log_4(x + 2) + 1$
* D= (-2,∞) * R= ℝ
* Increasing * X= -2 → Vertical asymptote

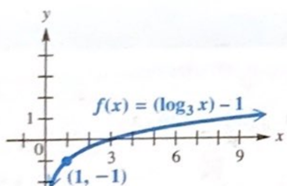


Figure 22

NOTE If we are given a graph such as the one in **Figure 22** and are asked to find its equation, we could reason as follows: The point $(1, 0)$ on the basic logarithmic graph has been shifted *down* 1 unit, and the point $(3, 0)$ on the given graph is 1 unit lower than $(3, 1)$, which is on the graph of $y = \log_3 x$. Thus, the equation will be

$$y = (\log_3 x) - 1.$$

Properties of Logarithms The properties of logarithms enable us to change the form of logarithmic statements so that products can be converted to sums, quotients can be converted to differences, and powers can be converted to products.

Important

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r, the following properties hold.

Property	Description
Product Property $\log_a xy = \log_a x + \log_a y$	The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.
Quotient Property $\log_a \frac{x}{y} = \log_a x - \log_a y$	The logarithm of the quotient of two numbers is equal to the difference between the logarithms of the numbers.
Power Property $\log_a x^r = r \log_a x$	The logarithm of a number raised to a power is equal to the exponent multiplied by the logarithm of the number.
Logarithm of 1 $\log_a 1 = 0$	The base a logarithm of 1 is 0.
Base *a* Logarithm of *a* $\log_a a = 1$	The base a logarithm of a is 1.

* $\log_{\frac{1}{a}} \frac{1}{x} = \log_a x$

LOOKING AHEAD TO CALCULUS

A technique called **logarithmic differentiation,** which uses the properties of logarithms, can often be used to differentiate complicated functions.

Proof To prove the product property, let $m = \log_a x$ and $n = \log_a y$.

$\log_a x = m$ means $a^m = x$

$\log_a y = n$ means $a^n = y$

Now consider the product xy.

$xy = a^m \cdot a^n$ Substitute.

$xy = a^{m+n}$ Product rule for exponents (Section 1.3)

$\log_a xy = m + n$ Write in logarithmic form.

$\log_a xy = \log_a x + \log_a y$ Substitute.

The last statement is the result we wished to prove. The quotient and power properties are proved similarly.

EXAMPLE 3 **Using the Properties of Logarithms**

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_6(7 \cdot 9)$ **(b)** $\log_9 \frac{15}{7}$ **(c)** $\log_5 \sqrt{8}$

(d) $\log_a \frac{mnq}{p^2t^4}$ **(e)** $\log_a \sqrt[3]{m^2}$ **(f)** $\log_b \sqrt[n]{\frac{x^3y^5}{z^m}}$

SOLUTION

(a) $\log_6(7 \cdot 9) = \log_6 7 + \log_6 9$ Product property

(b) $\log_9 \frac{15}{7} = \log_9 15 - \log_9 7$ Quotient property

(c) $\log_5 \sqrt{8} = \log_5(8^{1/2}) = \frac{1}{2}\log_5 8$ Power property

(d) $\log_a \frac{mnq}{p^2t^4} = \log_a m + \log_a n + \log_a q - (\log_a p^2 + \log_a t^4)$ Product and quotient properties

Use parentheses to avoid errors.

$= \log_a m + \log_a n + \log_a q - (2\log_a p + 4\log_a t)$ Power property

$= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a t$

Be careful with signs.

(e) $\log_a \sqrt[3]{m^2} = \log_a m^{2/3} = \frac{2}{3}\log_a m$

(f) $\log_b \sqrt[n]{\frac{x^3y^5}{z^m}} = \log_b\left(\frac{x^3y^5}{z^m}\right)^{1/n}$ $\sqrt[n]{a} = a^{1/n}$ (Section 1.6)

$= \frac{1}{n}\log_b \frac{x^3y^5}{z^m}$ Power property

$= \frac{1}{n}(\log_b x^3 + \log_b y^5 - \log_b z^m)$ Product and quotient properties

$= \frac{1}{n}(3\log_b x + 5\log_b y - m\log_b z)$ Power property

$= \frac{3}{n}\log_b x + \frac{5}{n}\log_b y - \frac{m}{n}\log_b z$ Distributive property (Section 1.2)

* لتجميع اللوغاريتمات في لوغاريتم
واحد لابد أن يكون معامل
الـ log واحد أو ١ -
إذا كان عدد بخلاف الواحد
نرفعه أس.

* HOMEWORK 3 Using the Properties of Logarithms

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_3(x+2) + \log_3 x - \log_3 2 \rightarrow \log_3 \frac{(x+2)x}{2}$ **(b)** $2 \log_a m - 3 \log_a n$

$= \log_a \frac{m^2}{n^3}$

(c) $\frac{1}{2} \log_b m + \frac{3}{2} \log_b 2n - \log_b m^2 n$

$\log_b m^{\frac{1}{2}} + \log_b (2n)^{\frac{3}{2}} - \log_b m^2 n \rightarrow \log_b \frac{m^{\frac{1}{2}}(2n)^{\frac{3}{2}}}{m^2 n} \rightarrow \log_b \frac{2^{\frac{3}{2}} n^{\frac{1}{2}}}{m^{\frac{3}{2}}}$

Napier's Rods

The search for ways to make calculations easier has been a long, ongoing process. Machines built by Charles Babbage and Blaise Pascal, a system of "rods" used by John Napier, and slide rules were the forerunners of today's calculators and computers. The invention of logarithms by John Napier in the 16th century was a great breakthrough in the search for easier calculation methods.

Source: IBM Corporate Archives.

CAUTION ***There is no property of logarithms to rewrite a logarithm of a sum or difference.*** That is why, in **Homework 3(a),** $\log_3(x+2)$ was not written as $\log_3 x + \log_3 2$. The distributive property does not apply in a situation like this because $\log_3(x+y)$ is one term. The abbreviation "log" is a function name, *not* a factor.

The next example uses = symbols for values of logarithms. These are actually approximations.

* EXAMPLE 4 Using the Properties of Logarithms with Numerical Values

Assume that $\log_{10} 2 = 0.3010$. Find each logarithm.

(a) $\log_{10} 4$ **(b)** $\log_{10} 5$

SOLUTION

(a)
$$\begin{aligned} \log_{10} 4 &= \log_{10} 2^2 \\ &= 2 \log_{10} 2 \\ &= 2(0.3010) \\ &= 0.6020 \end{aligned}$$

(b)
$$\begin{aligned} \log_{10} 5 &= \log_{10} \frac{10}{2} \\ &= \log_{10} 10 - \log_{10} 2 \\ &= 1 - 0.3010 \\ &= 0.6990 \end{aligned}$$

Recall that for inverse functions f and g, $(f \circ g)(x) = (g \circ f)(x) = x$. We can use this property with exponential and logarithmic functions to state two more properties. If $f(x) = a^x$ and $g(x) = \log_a x$, then

$$(f \circ g)(x) = a^{\log_a x} \quad \text{and} \quad (g \circ f)(x) = \log_a(a^x).$$

Theorem on Inverses

For $a > 0$, $a \neq 1$, the following properties hold.

$$a^{\log_a x} = x \ (\text{for } x > 0) \quad \text{and} \quad \log_a a^x = x$$

The following are examples of applications of this theorem.

$$7^{\log_7 10} = 10, \quad \log_5 5^3 = 3, \quad \text{and} \quad \log_r r^{k+1} = k + 1$$

The second statement in the theorem will be useful when we solve other logarithmic and exponential equations.

5.3 Exercises

Concept Check *In Exercise 1, match the logarithm in Column I with its value in Column II. Remember that $\log_a x$ is the exponent to which a must be raised in order to obtain x.*

	I	II
1.	(a) $\log_2 16$	**A.** 0
	(b) $\log_3 1$	**B.** $\frac{1}{2}$
	(c) $\log_{10} 0.1$	**C.** 4
	(d) $\log_2 \sqrt{2}$	**D.** -3
	(e) $\log_e \frac{1}{e^2}$	**E.** -1
	(f) $\log_{1/2} 8$	**F.** -2

If the statement is in exponential form, write it in an equivalent logarithmic form. If the statement is in logarithmic form, write it in exponential form. **See Example 1.**

2. $3^4 = 81$ **3.** $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$ **4.** $\log_6 36 = 2$ **5.** $\log_{\sqrt{3}} 81 = 8$

6. Explain why logarithms of negative numbers are not defined.

Solve each logarithmic equation. **See Homework 1.**

7. $x = \log_5 \frac{1}{625}$ **8.** $\log_x \frac{1}{32} = 5$ **9.** $x = \log_8 \sqrt[4]{8}$

10. $x = 3^{\log_3 8}$ **11.** $x = 2^{\log_2 9}$ **12.** $\log_x 25 = -2$

13. $\log_4 x = 3$ **14.** $x = \log_4 \sqrt[3]{16}$ **15.** $\log_9 x = \frac{5}{2}$

16. $\log_{1/2}(x + 3) = -4$ **17.** $\log_{(x+3)} 6 = 1$

18. $3x - 15 = \log_x 1 \quad (x > 0, x \neq 1)$

19. Compare the summary of characteristics of the graph of $f(x) = \log_a x$ with the similar summary about the graph of $f(x) = a^x$ in **Section 5.2.** Make a list of characteristics that reinforce the idea that these are inverse functions.

Sketch the graph of $f(x) = \log_2 x$. Then refer to it and use the techniques of **Chapter 3** *to graph each function. Give the domain and range.* **See Homework 2.**

20. $f(x) = (\log_2 x) + 3$ **21.** $f(x) = |\log_2(x + 3)|$

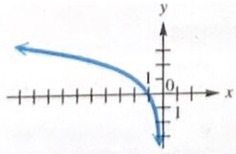
Sketch the graph of $f(x) = \log_{1/2} x$. Then refer to it and use the techniques of **Chapter 3** *to graph each function. Give the domain and range.* **See Homework 2.**

22. $f(x) = \log_{1/2}(x - 2)$

Concept Check *In Exercises 23–25 match the function with its graph from choices A–F.*

23. $f(x) = \log_2 x$ **24.** $f(x) = \log_2 \frac{1}{x}$ **25.** $f(x) = \log_2(x - 1)$

A.



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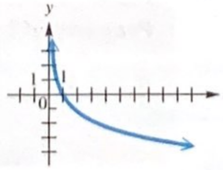

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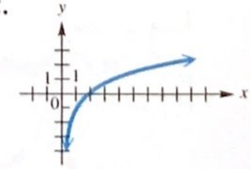

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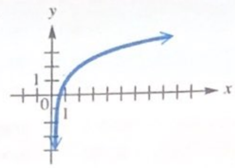

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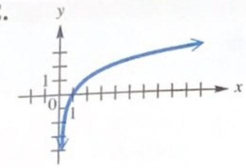

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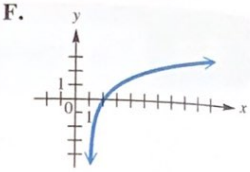
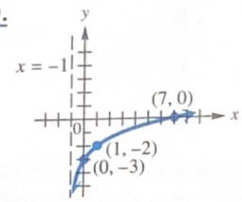


Graph each function. ***See Example 2 and Homework 2.***

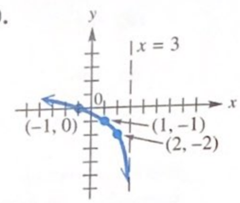
26. $f(x) = \log_5 x$ 27. $f(x) = \log_{1/2}(1 - x)$ 28. $f(x) = \log_3(x - 1) + 2$

Connecting Graphs with Equations *In Exercises 29–31, write an equation for the graph given. Each is a logarithmic function f with base 2 or 3, translated and/or reflected.* ***See the Note following Homework 2.***

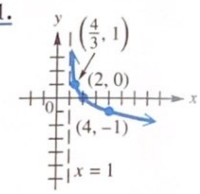
29.



30.



31.



Use the properties of logarithms to rewrite each expression. Simplify the result if possible. Assume all variables represent positive real numbers. ***See Example 3.***

32. $\log_2 \dfrac{6x}{y}$ 33. $\log_5 \dfrac{5\sqrt{7}}{3}$ 34. $\log_4(2x + 5y)$

35. $\log_m \sqrt{\dfrac{5r^3}{z^5}}$ 36. $\log_2 \dfrac{ab}{cd}$ 37. $\log_3 \dfrac{\sqrt{x} \cdot \sqrt[3]{y}}{w^2\sqrt{z}}$

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers. ***See Homework 3.***

38. $\log_a x + \log_a y - \log_a m$ 39. $\log_a m - \log_a n - \log_a t$

40. $\dfrac{1}{3}\log_b x^4y^5 - \dfrac{3}{4}\log_b x^2y$ 41. $2\log_a(z + 1) + \log_a(3z + 2)$

42. $-\dfrac{2}{3}\log_5 5m^2 + \dfrac{1}{2}\log_5 25m^2$

Given the approximations $\log_{10} 2 = 0.3010$ *and* $\log_{10} 3 = 0.4771$, *find each logarithm without using a calculator.* ***See Example 4.***

43. $\log_{10} 6$ 44. $\log_{10} \dfrac{3}{2}$ 45. $\log_{10} \dfrac{9}{4}$ 46. $\log_{10} \sqrt{30}$

5.4 Exponential and Logarithmic Equations

- Exponential Equations
- Logarithmic Equations
- Applications and Models

Exponential Equations We solved exponential equations in earlier sections. General methods for solving these equations depend on the property below, which follows from the fact that logarithmic functions are one-to-one.

Property of Logarithms

If $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$, then the following holds.

$$x = y \quad \text{is equivalent to} \quad \log_a x = \log_a y.$$

log = log$_e$

ln = log$_e$ → natural number

* **EXAMPLE 1 Solving an Exponential Equation**

Solve $7^x = 12$. Give the solution to the nearest thousandth.

SOLUTION The properties of exponents given in **Section 5.2** cannot be used to solve this equation, so we apply the preceding property of logarithms. While any appropriate base b can be used, the best practical base is base 10 or base e. We choose base e (natural) logarithms here.

$$7^x = 12$$ ـه ينفع نحلها بالـ log

$$\ln 7^x = \ln 12 \qquad \text{Property of logarithms}$$

$$x \ln 7 = \ln 12 \qquad \text{Power property (Section 5.3)}$$

This is *exact.*

$$x = \frac{\ln 12}{\ln 7} \qquad \text{Divide by ln 7.}$$

$$x \approx 1.277$$

This is *approximate.*

The solution set is $\{1.277\}$.

* حل المعادلة الأسية يتوقف على شكلها مثلاً :

١- إذا كان الأساس يساوي الأساس فإن الأس يساوي الأس ونكمل حل المعادلة.

٢- إذا كان الأساس لا يساوي الأساس فنأخذ log الطرفين أو ln الطرفين

CAUTION When evaluating a quotient like $\frac{\ln 12}{\ln 7}$ in **Example 1,** do not confuse this quotient with $\ln \frac{12}{7}$, which can be written as $\ln 12 - \ln 7$. ***We cannot change the quotient of two logarithms to a difference of logarithms.***

$$\frac{\ln 12}{\ln 7} \neq \ln \frac{12}{7}$$

---

* وإذا كان الأساس العدد الطبيعي e فيفضل أخذ الـ ln

$$\ln_e = \log_e e = 1$$

* **HOMEWORK 1 Solving an Exponential Equation**

Solve $3^{2x-1} = 0.4^{x+2}$. Give the solution to the nearest thousandth.

* **EXAMPLE 2 Solving Base *e* Exponential Equations**

Solve each equation. Give solutions to the nearest thousandth.

(a) $e^{x^2} = 200$ **(b)** $e^{2x+1} \cdot e^{-4x} = 3e$

SOLUTION

(a)

$$e^{x^2} = 200$$

$$\ln e^{x^2} = \ln 200 \qquad \text{Take the natural logarithm on each side.}$$

$$x^2 = \ln 200 \qquad \ln e^{x^2} = x^2 \text{ (Section 5.3)}$$

Remember both roots.

$$x = \pm\sqrt{\ln 200} \qquad \text{Square root property (Section 1.3)}$$

$$x \approx \pm 2.302 \qquad \text{Use a calculator.}$$

The solution set is $\{\pm 2.302\}$.

(b)

$$e^{2x+1} \cdot e^{-4x} = 3e$$

$$e^{-2x+1} = 3e \qquad a^m \cdot a^n = a^{m+n} \text{ (Section 1.3)}$$

$\frac{e^{-2x+1}}{e} \to e^{-2x+1-1} = 3$

$$e^{-2x} = 3 \qquad \text{Divide by } e;\ \frac{a^m}{a^n} = a^{m-n}. \text{ (Section 1.3)}$$

$$\ln e^{-2x} = \ln 3 \qquad \text{Take the natural logarithm on each side.}$$

$$-2x \ln e = \ln 3 \qquad \text{Power property}$$

$$-2x = \ln 3 \qquad \ln e = 1$$

$$x = -\frac{1}{2} \ln 3 \qquad \text{Multiply by } -\tfrac{1}{2}.$$

$$x \approx -0.549 \qquad \text{Use a calculator.}$$

The solution set is $\{-0.549\}$.

Very Important

HOMEWORK 2 Solving an Exponential Equation Quadratic in Form

Solve $e^{2x} - 4e^x + 3 = 0$. Give exact value(s) for x.

Logarithmic Equations The following equations involve logarithms of variable expressions.

EXAMPLE 3 Solving Logarithmic Equations

Solve each equation. Give exact values.

(a) $7 \ln x = 28$ (b) $\log_2(x^3 - 19) = 3$

لحل المعادلة اللوغاريتمية:
لدينا شكلين:
١- log في طرف والطرف الآخر عدد.
لحل هذه المعادلة نحولها إلى الشكل الأسي
٢- log في طرف وفي الطرف الآخر log من نفس الBase.
لحل هذه المعادلة نساوي ما بداخل الـ log في الطرف الأيمن بما في داخل الـ log في الطرف الأيسر.

SOLUTION

(a)

$$7 \ln x = 28$$

$$\ln x = 4 \qquad \text{Divide by 7.}$$

$$x = e^4 \qquad \text{Write the natural logarithm in exponential form.}$$

The solution set is $\{e^4\}$.

(b)

$$\log_2(x^3 - 19) = 3$$

$$x^3 - 19 = 2^3 \qquad \text{Write in exponential form.}$$

$$x^3 - 19 = 8 \qquad \text{Apply the exponent.}$$

$$x^3 = 27 \qquad \text{Add 19.}$$

$$x = \sqrt[3]{27} \qquad \text{Take cube roots.}$$

$$x = 3 \qquad \sqrt[3]{27} = 3$$

The solution set is $\{3\}$.

HOMEWORK 3 Solving a Logarithmic Equation

Solve $\log(x + 6) - \log(x + 2) = \log x$. Give exact value(s).

* في المعادلات اللوغاريتمية لازم نسوي check للحل

CAUTION Recall that the domain of $y = \log_a x$ is $(0, \infty)$. For this reason, ***it is always necessary to check that proposed solutions of a logarithmic equation result in logarithms of positive numbers in the original equation.***

EXAMPLE 4 **Solving a Logarithmic Equation**

Solve $\log_2[(3x-7)(x-4)] = 3$. Give exact value(s).

SOLUTION $\log_2[(3x-7)(x-4)] = 3$

$$(3x-7)(x-4) = 2^3 \quad \text{Write in exponential form.}$$
$$3x^2 - 19x + 28 = 8 \quad \text{Multiply. (Section 1.3)}$$
$$3x^2 - 19x + 20 = 0 \quad \text{Standard form}$$
$$(3x-4)(x-5) = 0 \quad \text{Factor.}$$
$$3x - 4 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{Zero-factor property}$$
$$x = \frac{4}{3} \quad \text{or} \quad x = 5 \quad \text{Solve for } x.$$

A check is necessary to be sure that the argument of the logarithm in the given equation is positive. In both cases, the product $(3x-7)(x-4)$ leads to 8, and $\log_2 8 = 3$ is true. The solution set is $\left\{\frac{4}{3}, 5\right\}$.

HOMEWORK 4 **Solving a Logarithmic Equation**

Solve $\log(3x+2) + \log(x-1) = 1$. Give exact value(s).

NOTE We could have used the definition of logarithm in **Homework 4** by first writing

$$\log(3x+2) + \log(x-1) = 1 \quad \text{Equation from Home Work 4}$$
$$\log_{10}[(3x+2)(x-1)] = 1 \quad \text{Product property}$$
$$(3x+2)(x-1) = 10^1, \quad \text{Definition of logarithm (Section 5.3)}$$

and then continuing as shown on the preceding page.

EXAMPLE 5 **Solving a Base *e* Logarithmic Equation**

Solve $\ln e^{\ln x} - \ln(x-3) = \ln 2$. Give exact value(s).

SOLUTION This logarithmic equation differs from those in **Example 4 and Homework 4** because the expression on the right side involves a logarithm.

$$\ln e^{\ln x} - \ln(x-3) = \ln 2$$
$$\ln x - \ln(x-3) = \ln 2 \quad e^{\ln x} = x \text{ (Section 5.3)}$$
$$\ln \frac{x}{x-3} = \ln 2 \quad \text{Quotient property}$$
$$\frac{x}{x-3} = 2 \quad \text{Property of logarithms}$$
$$x = 2(x-3) \quad \text{Multiply by } x-3.$$
$$x = 2x - 6 \quad \text{Distributive property}$$
$$6 = x \quad \text{Solve for } x.$$

Check that the solution set is $\{6\}$.

Solving Exponential or Logarithmic Equations

To solve an exponential or logarithmic equation, change the given equation into one of the following forms, where a and b are real numbers, $a > 0$ and $a \neq 1$, and follow the guidelines.

1. $a^{f(x)} = b$
   Solve by taking logarithms on both sides.
2. $\log_a f(x) = b$
   Solve by changing to exponential form $a^b = f(x)$.
3. $\log_a f(x) = \log_a g(x)$
   The given equation is equivalent to the equation $f(x) = g(x)$. Solve algebraically.
4. In a more complicated equation, such as
   $$e^{2x+1} \cdot e^{-4x} = 3e$$
   in **Example 2(b),** it may be necessary to first solve for $a^{f(x)}$ or $\log_a f(x)$ and then solve the resulting equation using one of the methods given above.
5. Check that each proposed solution is in the domain.

Applications and Models

HOMEWORK 5 **Applying an Exponential Equation to the Strength of a Habit**

The strength of a habit is a function of the number of times the habit is repeated. If N is the number of repetitions and H is the strength of the habit, then, according to psychologist C.L. Hull,

$$H = 1000(1 - e^{-kN}),$$

where k is a constant. Solve this equation for k.

EXAMPLE 6 **Modeling Coal Consumption in the U.S.**

The table gives U.S. coal consumption (in quadrillions of British thermal units, or *quads*) for several years. The data can be modeled by the function

$$f(t) = 24.92 \ln t - 93.31, \quad t \geq 80,$$

where t is the number of years after 1900, and $f(t)$ is in quads.

Year	Coal Consumption (in quads)
1980	15.42
1985	17.48
1990	19.17
1995	20.09
2000	22.58
2005	22.80
2008	22.39

Source: U.S. Energy Information Administration.

(a) Approximately what amount of coal was consumed in the United States in 2003? How does this figure compare to the actual figure of 22.32 quads?

(b) If this trend continues, approximately when will annual consumption reach 25 quads?

SOLUTION

(a) The year 2003 is represented by $t = 2003 - 1900 = 103$.

$$f(103) = 24.92 \ln 103 - 93.31 \quad \text{Let } t = 103.$$
$$\approx 22.19$$

Based on this model, 22.19 quads were used in 2003. This figure is very close to the actual amount of 22.32 quads.

(b) Replace $f(t)$ with 25, and solve for t.

$$25 = 24.92 \ln t - 93.31 \quad f(t) = 25 \text{ in the given model.}$$

$$118.31 = 24.92 \ln t \quad \text{Add 93.31.}$$

$$\ln t = \frac{118.31}{24.92} \quad \text{Divide by 24.92. Rewrite.}$$

$$t = e^{118.31/24.92} \quad \text{Write in exponential form.}$$

$$t \approx 115.3$$

Add 115 to 1900 to get 2015. Based on this model, annual consumption will reach 25 quads in 2015.

5.4 Exercises

Concept Check An exponential equation such as

$$5^x = 9$$

can be solved for its exact solution using the meaning of logarithm and the change-of-base theorem. Since x is the exponent to which 5 must be raised in order to obtain 9, the exact solution is

$$\log_5 9, \quad or \quad \frac{\log 9}{\log 5}, \quad or \quad \frac{\ln 9}{\ln 5}.$$

For each equation, give the exact solution in three forms similar to the forms explained above.

1. $7^x = 19$
2. $\left(\frac{1}{2}\right)^x = 12$

Solve each exponential equation. In Exercises 3–14, express irrational solutions as decimals correct to the nearest thousandth. In Exercises 15–17, express solutions in exact form. ***See Examples 1–2 and Homework 1–2.***

3. $3^x = 7$
4. $\left(\frac{1}{2}\right)^x = 5$
5. $0.8^x = 4$
6. $4^{x-1} = 3^{2x}$
7. $6^{x+1} = 4^{2x-1}$
8. $e^{x^2} = 100$
9. $e^{3x-7} \cdot e^{-2x} = 4e$
10. $\left(\frac{1}{3}\right)^x = -3$
11. $0.05(1.15)^x = 5$
12. $3(2)^{x-2} + 1 = 100$
13. $2(1.05)^x + 3 = 10$
14. $5(1.015)^{x-1980} = 8$
15. $e^{2x} - 6e^x + 8 = 0$
16. $2e^{2x} + e^x = 6$
17. $5^{2x} + 3(5^x) = 28$

Solve each logarithmic equation. Express all solutions in exact form. ***See Examples 3–5.***

18. $5 \ln x = 10$
19. $\ln(4x) = 1.5$
20. $\log(2 - x) = 0.5$
21. $\log_6(2x + 4) = 2$
22. $\log_4(x^3 + 37) = 3$
23. $\ln x + \ln x^2 = 3$
24. $\log_3[(x + 5)(x - 3)] = 2$
25. $\log_2[(2x + 8)(x + 4)] = 5$

26. $\log x + \log(x + 15) = 2$

27. $\log(x + 25) = \log(x + 10) + \log 4$

28. $\log(x - 10) - \log(x - 6) = \log 2$

29. $\ln(5 - x) + \ln(-3 - x) = \ln(1 - 8x)$

30. $\log_8(x + 2) + \log_8(x + 4) = \log_8 8$

31. $\log_2(x^2 - 100) - \log_2(x + 10) = 1$

32. $\log x + \log(x - 21) = \log 100$

33. $\log(9x + 5) = 3 + \log(x + 2)$

34. $\ln(4x - 2) - \ln 4 = -\ln(x - 2)$

35. $\log_5(x + 2) + \log_5(x - 2) = 1$

36. $\log_2(2x - 3) + \log_2(x + 1) = 1$

37. $\ln e^x - 2 \ln e = \ln e^4$

38. $\log_2(\log_2 x) = 1$

39. $\log x^2 = (\log x)^2$

40. ***Concept Check*** Suppose you overhear the following statement: "I must reject any negative proposed solution when I solve an equation involving logarithms." Is this correct? Why or why not?

Solve each equation for the indicated variable. Use logarithms with the appropriate bases. ***See Home Work 5.***

41. $p = a + \dfrac{k}{\ln x}$, for x

42. $T = T_0 + (T_1 - T_0)10^{-kt}$, for t

43. $I = \dfrac{E}{R}(1 - e^{-Rt/2})$, for t

44. $y = A + B(1 - e^{-Cx})$, for x

45. $\log A = \log B - C \log x$, for A

46. $A = P\left(1 + \dfrac{r}{n}\right)^{tn}$, for t

Glossary

argument In the expression $\log_a x$, x is the argument.

المتغير المطلق في التعبير اللوغاريثمي $\log_a x$، تكون x هي المتغير المطلق.

base of a logarithm In the expression $\log_a x$, a is the base.

قاعدة لوغاريتم في التعبير اللوغاريتمي مثل $\log_a x$، تكون a هي القاعدة.

compound amount In an investment paying compound interest, the compound amount is the balance *after* interest has been earned. (The compound amount is sometimes called the *future value*.)

المبلغ المركب في سداد استثمار لأي فائدة مركبة؛ فإن المبلغ المركب هو *الرصيد بعد احتساب الفائدة*. (يطلق على المبلغ المركب أحيانًا *القيمة المستقبلية*.)

continuous compounding Continuous compounding of money involves the computation of interest as the frequency of compounding approaches infinity, leading to the formula $A = Pe^{rt}$.

التركيب المستمر للفوائد ينطوي التركيب المستمر للأموال على احتساب الفائدة مع اقتراب تكرار التركيب إلى حد اللانهاية ؛ مما يؤدي إلى الصيغة $A = Pe^{rt}$

exponential equation An exponential equation is an equation with a variable in an exponent.

المعادلة الأسية أي معادلة أسية هي معادلة يكون أسّها متغيرًا.

exponential function If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ defines the exponential function with base a.

الدالة الأسية إذا كانت $a > 0$ و $a \neq 1$ فإن $f(x) = a^x$ تحدد الدالة الأسية بالقاعدة a.

future value In an investment paying compound interest, the future value is the balance *after* interest has been earned. (The future value is sometimes called the *compound amount*.)

القيمة الآجلة في سداد استثمار لأي فائدة مركبة؛ فإن القيمة الآجلة هي *الرصيد بعد احتساب* الفائدة. (يطلق على القيمة الآجلة في بعض الأحيان *المبلغ المركب*).

inverse function Let f be a one-to-one function. Then g is the inverse function of f if $(f \circ g)(x) = x$ for every x in the domain of g, and $(g \circ f)(x) = x$ for every x in the domain of f.

الدالة العكسية اجعل الدالة f دالة الواحد لواحد. لذا فإن g هي الدالة العكسية للدالة f إذا كان $(f \circ g)(x) = x$ لكل x في نطاق g، و $(g \circ f)(x) = x$ لكل x في النطاق f.

6.1 Angles

- Basic Terminology
- Degree Measure
- Standard Position
- Coterminal Angles

Basic Terminology Two distinct points A and B determine a line called **line AB**. The portion of the line between A and B, including points A and B themselves, is **line segment AB**, or simply **segment AB**. The portion of line AB that starts at A and continues through B, and on past B, is the **ray AB**. Point A is the **endpoint of the ray.** See **Figure 1.**

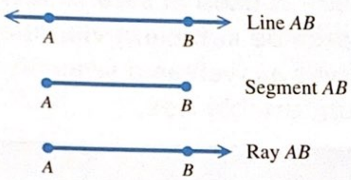


Figure 1

In trigonometry, an **angle** consists of two rays in a plane with a common endpoint, or two line segments with a common endpoint. These two rays (or segments) are the **sides** of the angle, and the common endpoint is the **vertex** of the angle. Associated with an angle is its measure, generated by a rotation about the vertex. See **Figure 2.** This measure is determined by rotating a ray starting at one side of the angle, the **initial side,** to the position of the other side, the **terminal side.** ***A counterclockwise rotation generates a positive measure, and a clockwise rotation generates a negative measure*** The rotation can consist of more than one complete revolution.

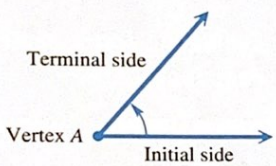


Figure 2

Figure 3 shows two angles, one **positive** and one **negative.**

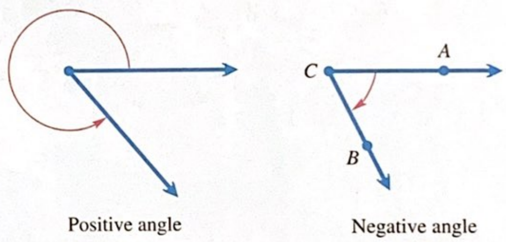
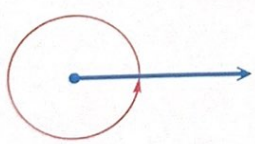


Figure 3

An angle can be named by using the name of its vertex. For example, the angle on the right in **Figure 3** can be named angle C. Alternatively, an angle can be named using three letters, with the vertex letter in the middle. Thus, the angle on the right also could be named angle ACB or angle BCA.

Degree Measure The most common unit for measuring angles is the **degree.** Degree measure was developed by the Babylonians 4000 yr ago. To use degree measure, we assign 360 degrees to a complete rotation of a ray.* In **Figure 4,** notice that the terminal side of the angle corresponds to its initial side when it makes a complete rotation.



A complete rotation of a ray gives an angle whose measure is $360°$. $\frac{1}{360}$ of a complete rotation gives an angle whose measure is $1°$.

Figure 4

$$\text{One degree, written } 1°, \text{ represents } \frac{1}{360} \text{ of a rotation.}$$

Therefore, $90°$ represents $\frac{90}{360} = \frac{1}{4}$ of a complete rotation, and $180°$ represents $\frac{180}{360} = \frac{1}{2}$ of a complete rotation.

An angle measuring between $0°$ and $90°$ is an **acute angle.** An angle measuring exactly $90°$ is a **right angle.** The symbol ⅂ is often used at the vertex of a right angle to denote the $90°$ measure. An angle measuring more than $90°$ but less than $180°$ is an **obtuse angle,** and an angle of exactly $180°$ is a **straight angle.**

*The Babylonians were the first to subdivide the circumference of a circle into 360 parts. There are various theories about why the number 360 was chosen. One is that it is approximately the number of days in a year, and it has many divisors, which makes it convenient to work with.

In **Figure 5,** we use the **Greek letter *θ* (theta)*** to name each angle.

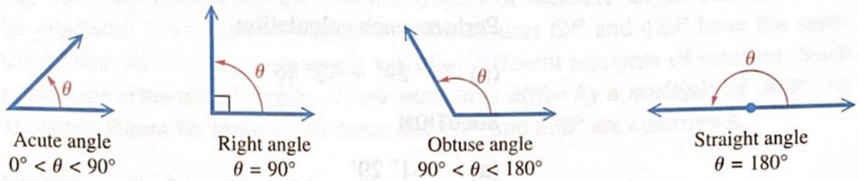


Figure 5

If the sum of the measures of two positive angles is 90°, the angles are **complementary** and the angles are **complements** of each other. Two positive angles with measures whose sum is 180° are **supplementary,** and the angles are **supplements.**

EXAMPLE 1 **Finding the Complement and the Supplement of an Angle**

For an angle measuring 40°, find the measure of **(a)** its complement and **(b)** its supplement.

SOLUTION

(a) To find the measure of its complement, subtract the measure of the angle from 90°.

$$90° - 40° = 50° \quad \text{Complement of } 40°$$

(b) To find the measure of its supplement, subtract the measure of the angle from 180°.

$$180° - 40° = 140° \quad \text{Supplement of } 40°$$

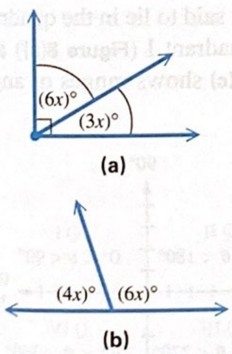


Figure 6

HOMEWORK 1 **Finding Measures of Complementary and Supplementary Angles**

Find the measure of each marked angle in **Figure 6.**

The measure of angle A in **Figure 7** is 35°. This measure is often expressed by saying that $\boldsymbol{m}(\textbf{angle } \boldsymbol{A})$ is 35°, where $m(\text{angle } A)$ is read **"the measure of angle *A*."** It is convenient, however, to abbreviate the symbolism $m(\text{angle } A) = 35°$ as $A = 35°$.

Traditionally, portions of a degree have been measured with minutes and seconds. One **minute,** written **1′**, is $\frac{1}{60}$ of a degree.

$$1' = \frac{1}{60}^{\circ} \quad \text{or} \quad 60' = 1°$$

One **second, 1″**, is $\frac{1}{60}$ of a minute.

$$1'' = \frac{1}{60}' = \frac{1}{3600}^{\circ} \quad \text{or} \quad 60'' = 1'$$

The measure 12° 42′ 38″ represents 12 degrees, 42 minutes, 38 seconds.

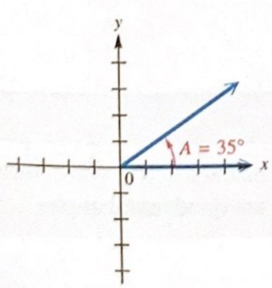


Figure 7

*In addition to θ (theta), other Greek letters such as α (alpha) and β (beta) are often used.

EXAMPLE 2 Calculating with Degrees, Minutes, and Seconds

Perform each calculation.

(a) $51°\,29' + 32°\,46'$ **(b)** $90° - 73°\,12'$

SOLUTION

(a)
$$\begin{array}{r} 51°\,29' \\ +\;32°\,46' \\ \hline 83°\,75' \end{array}$$
Add degrees and minutes separately.

The sum $83°\,75'$ can be rewritten as follows.

$$\begin{aligned} 83°\,75' &= 83° + 1°\,15' && 75' = 60' + 15' = 1°\,15' \\ &= 84°\,15' && \text{Add.} \end{aligned}$$

(b)
$$\begin{array}{r} 89°\,60' \\ -\;73°\,12' \\ \hline 16°\,48' \end{array}$$
Write $90°$ as $89°\,60'$.

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HOMEWORK 2 Converting between Decimal Degrees and Degrees, Minutes, and Seconds

(a) Convert $74°\,08'\,14''$ to decimal degrees to the nearest thousandth.

(b) Convert $34.817°$ to degrees, minutes, and seconds to the nearest second.

Standard Position An angle is in **standard position** if its vertex is at the origin and its initial side lies on the positive x-axis. The angles in **Figures 8(a) and 8(b)** are in standard position. An angle in standard position is said to lie in the quadrant in which its terminal side lies. An acute angle is in quadrant I (**Figure 8(a)**) and an obtuse angle is in quadrant II (**Figure 8(b)**). **Figure 8(c)** shows ranges of angle measures for each quadrant when $0° < \theta < 360°$.

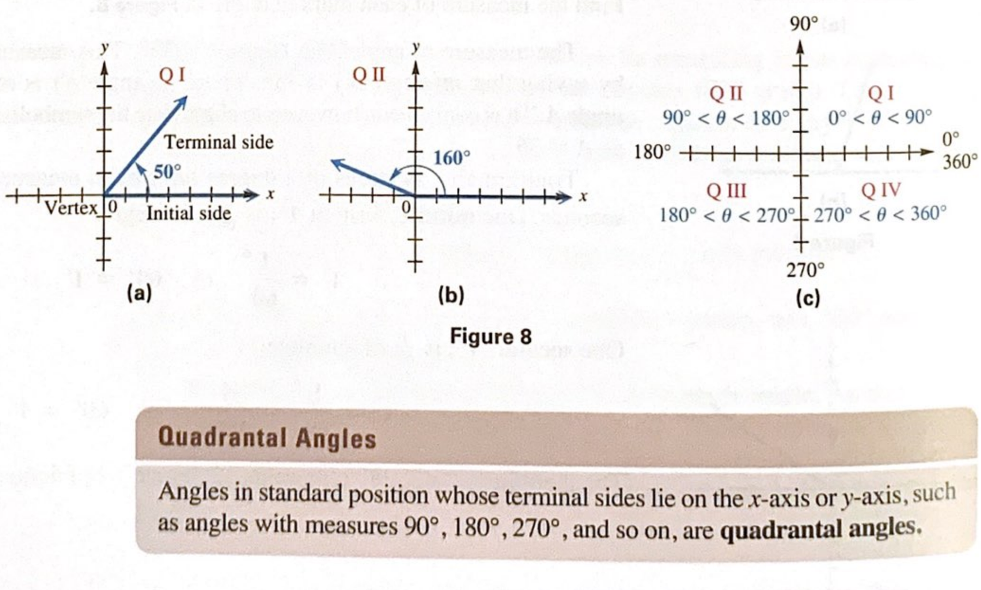


Figure 8

Quadrantal Angles

Angles in standard position whose terminal sides lie on the x-axis or y-axis, such as angles with measures $90°$, $180°$, $270°$, and so on, are **quadrantal angles.**

Coterminal Angles A complete rotation of a ray results in an angle measuring 360°. By continuing the rotation, angles of measure larger than 360° can be produced. The angles in **Figure 9** with measures 60° and 420° have the same initial side and the same terminal side, but different amounts of rotation. Such angles are **coterminal angles.** ***Their measures differ by a multiple of 360°.*** As shown in **Figure 10,** angles with measures 110° and 830° are coterminal.

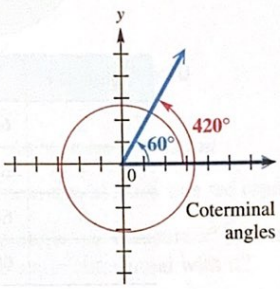


Figure 9

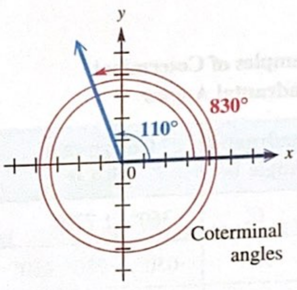


Figure 10

EXAMPLE 3 Finding Measures of Coterminal Angles

Find the angles of least positive measure that are coterminal with each angle.

(a) 908° **(b)** −75° **(c)** −800°

SOLUTION

(a) Subtract 360° as many times as needed to obtain an angle with measure greater than 0° but less than 360°. Since

$$908° - 2 \cdot 360° = 188°,$$

an angle of 188° is coterminal with an angle of 908°. See **Figure 11.**

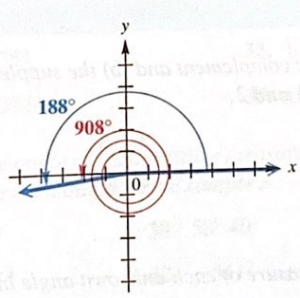


Figure 11

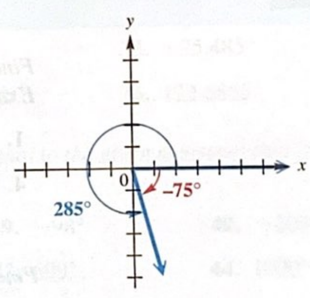


Figure 12

(b) See **Figure 12.** Use a rotation of

$$360° + (-75°) = 285°.$$

(c) The least integer multiple of 360° greater than 800° is

$$360° \cdot 3 = 1080°.$$

Add 1080° to −800° to obtain

$$1080° + (-800°) = 280°.$$

Sometimes it is necessary to find an expression that will generate all angles coterminal with a given angle. For example, we can obtain any angle coterminal with 60° by adding an integer multiple of 360° to 60°. Let n represent any integer. Then the following expression represents all such coterminal angles.

$$60° + n \cdot 360° \quad \text{Angles coterminal with } 60°$$

The table below shows a few possibilities.

Value of n	Angle Coterminal with 60°
2	$60° + 2 \cdot 360° = 780°$
1	$60° + 1 \cdot 360° = 420°$
0	$60° + 0 \cdot 360° = 60°$ (the angle itself)
−1	$60° + (-1) \cdot 360° = -300°$

Examples of Coterminal Quadrantal Angles

Quadrantal Angle θ	Coterminal with θ
0°	$\pm 360°, \pm 720°$
90°	$-630°, -270°, 450°$
180°	$-180°, 540°, 900°$
270°	$-450°, -90°, 630°$

The table in the margin shows some examples of coterminal quadrantal angles.

HOMEWORK 3 Analyzing the Revolutions of a CD Player

CD players always spin at the same speed. Suppose a player makes 480 revolutions per min. Through how many degrees will a point on the edge of a CD move in 2 sec?

6.1 Exercises

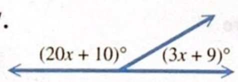
Find ***(a)*** *the complement and* ***(b)*** *the supplement of an angle with the given measure.* ***See Examples 1 and 2.***

1. 60° **2.** 18° **3.** 89°

4. 10° **5.** 39° 50′ **6.** 50° 40′ 50″

Find the measure of each unknown angle in Exercises 7–11.

7.



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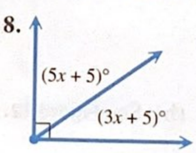

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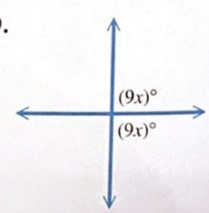

10. supplementary angles with measures $6x - 4$ and $8x - 12$ degrees

11. complementary angles with measures $3x - 5$ and $6x - 40$ degrees

12. *Concept Check* What is the measure of an angle that is its own supplement?

Find the measure of the smaller angle formed by the hands of a clock at the following time.

13.

14 9:45 **15.** 6:10

Concept Check *Answer each question.*

16. If an angle measures $x°$, how can we represent its supplement?

17. If a negative angle has measure $x°$ between $0°$ and $-60°$, how can we represent the first positive angle coterminal with it?

Perform each calculation. ***See Example 2.***

18. $75° 15' + 83° 32'$

19. $110° 25' + 32° 55'$

20. $47° 23' - 73° 48'$

21. $90° - 17° 13'$

22. $180° - 124° 51'$

23. $55° 30' + 12° 44' - 8° 15'$

24. $90° - 36° 18' 47''$

Convert each angle measure to decimal degrees. If applicable, round to the nearest thousandth of a degree.

25. $82° 30'$

26. $133° 45'$

27. $-70° 48'$

28. $38° 42' 00''$

29. $34° 51' 35''$

30. $165° 51' 09''$

Convert each angle measure to degrees, minutes, and seconds. Round answers to the nearest second, if applicable.

31. $46.75°$

32. $174.255°$

33. $-25.485°$

34. $59.0854°$

35. $102.3771°$

36. $122.6853°$

Find the angle of least positive measure (not equal to the given measure) that is coterminal with each angle. ***See Example 3.***

37. $86°$

38. $58° 40'$

39. $-98°$

40. $-203°$

41. $541°$

42. $-541°$

43. $699°$

44. $1000°$

45. $8440°$

46. $-8440°$

Give two positive and two negative angles that are coterminal with the given quadrantal angle.

47. $180°$

48. $270°$

Give an expression that generates all angles coterminal with each angle. Let n represent any integer.

49. $45°$

50. $225°$

51. $-180°$

52. $360°$

الدوال المثلثية.

6.2 Trigonometric Functions

- Trigonometric Functions
- Quadrantal Angles
- Reciprocal Identities
- Signs and Ranges of Function Values
- Pythagorean Identities
- Quotient Identities

Trigonometric Functions To define the six **trigonometric functions,** we start with an angle θ in standard position and choose any point P having coordinates (x, y) on the terminal side of angle θ. (The point P must not be the vertex of the angle.) See **Figure 13** on the next page. A perpendicular from point P to the x-axis at point Q determines a right triangle, having vertices at O, P, and Q. We find the distance r from $P(x, y)$ to the origin, $(0, 0)$, using the distance formula.

$$r = \sqrt{(x-0)^2 + (y-0)^2}$$

$$r = \sqrt{x^2 + y^2}$$

Notice that $r > 0$ since this is the undirected distance.

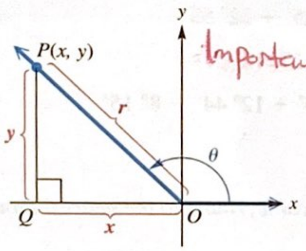

Figure 13

Important

The six trigonometric functions of angle θ are **sine, cosine, tangent, cotangent, secant,** and **cosecant,** abbreviated **sin, cos, tan, cot, sec,** and **csc.**

Trigonometric Functions

Let (x, y) be a point other than the origin on the terminal side of an angle θ in standard position. The distance from the point to the origin is $r = \sqrt{x^2 + y^2}$. The six trigonometric functions of θ are defined as follows.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

EXAMPLE 1 Finding Function Values of an Angle

The terminal side of an angle θ in standard position passes through the point $(8, 15)$. Find the values of the six trigonometric functions of angle θ.

SOLUTION **Figure 14** shows angle θ and the triangle formed by dropping a perpendicular from the point $(8, 15)$ to the x-axis. The point $(8, 15)$ is 8 units to the right of the y-axis and 15 units above the x-axis, so $x = 8$ and $y = 15$. Now use $r = \sqrt{x^2 + y^2}$.

$$r = \sqrt{8^2 + 15^2} = \sqrt{64 + 225} = \sqrt{289} = 17$$

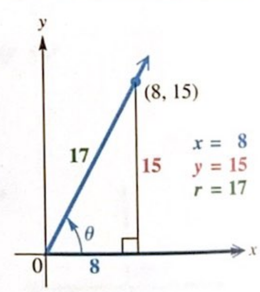

Figure 14

We can now find the values of the six trigonometric functions of angle θ.

$$\sin\theta = \frac{y}{r} = \frac{15}{17} \qquad \cos\theta = \frac{x}{r} = \frac{8}{17} \qquad \tan\theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{15} \qquad \sec\theta = \frac{r}{x} = \frac{17}{8} \qquad \cot\theta = \frac{x}{y} = \frac{8}{15}$$

HOMEWORK 1 **Finding Function Values of an Angle**

The terminal side of an angle θ in standard position passes through the point $(-3, -4)$. Find the values of the six trigonometric functions of angle θ.

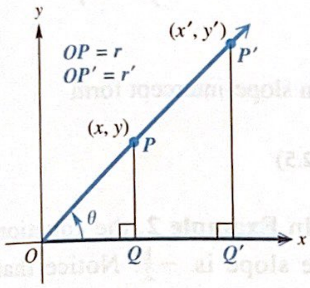

Figure 15

We can find the six trigonometric functions using *any* point other than the origin on the terminal side of an angle. To see why any point can be used, refer to **Figure 15,** which shows an angle θ and two distinct points on its terminal side. Point P has coordinates (x, y), and point P' (read **"P-prime"**) has coordinates (x', y'). Let r be the length of the hypotenuse of triangle OPQ, and let r' be the length of the hypotenuse of triangle $OP'Q'$. Since corresponding sides of similar triangles are proportional,

$$\frac{y}{r} = \frac{y'}{r'},$$

so $\sin\theta = \frac{y}{r}$ is the same no matter which point is used to find it. A similar result holds for the other five trigonometric functions.

We can also find the trigonometric function values of an angle if we know the equation of the line coinciding with the terminal ray. Recall from algebra that the graph of the equation

$$Ax + By = 0$$

is a line that passes through the origin. If we restrict x to have only nonpositive or only nonnegative values, we obtain as the graph a ray with endpoint at the origin. For example, the graph of $x + 2y = 0$, $x \geq 0$, shown in **Figure 16,** is a ray that can serve as the terminal side of an angle θ in standard position. By choosing a point on the ray, we can find the trigonometric function values of the angle.

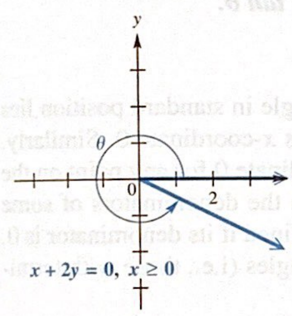

Figure 16

EXAMPLE 2 **Finding Function Values of an Angle**

Find the six trigonometric function values of the angle θ in standard position, if the terminal side of θ is defined by $x + 2y = 0$, $x \geq 0$.

SOLUTION The angle is shown in **Figure 17**. We can use *any* point except $(0, 0)$ on the terminal side of θ to find the trigonometric function values. We choose $x = 2$ and find the corresponding y-value.

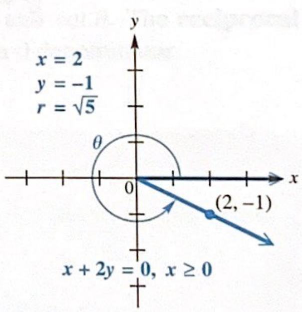

Figure 17

$$\begin{aligned} x + 2y &= 0, \quad x \geq 0 \\ 2 + 2y &= 0 && \text{Let } x = 2. \\ 2y &= -2 && \text{Subtract 2.} \\ y &= -1 && \text{Divide by 2.} \end{aligned}$$

The point $(2, -1)$ lies on the terminal side, and the corresponding value of r is $r = \sqrt{2^2 + (-1)^2} = \sqrt{5}$. Now we use the definitions of the trigonometric functions.

$$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{5}} = \frac{-1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$$

Multiply by $\frac{\sqrt{5}}{\sqrt{5}}$, which equals 1, to rationalize the denominators.

$$\cos\theta = \frac{x}{r} = \frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{2} = -\frac{1}{2}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{-1} = -\sqrt{5} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{2} \qquad \cot\theta = \frac{x}{y} = \frac{2}{-1} = -2$$

Recall that when the equation of a line is written in slope-intercept form

$$y = mx + b, \quad \text{(Section 2.5)}$$

the coefficient m of x is the slope of the line. In **Example 2,** the equation $x + 2y = 0$ can be written as $y = -\frac{1}{2}x$, so the slope is $-\frac{1}{2}$. Notice that $\tan\theta = -\frac{1}{2}$.

In general, it is true that $m = \tan\theta$.

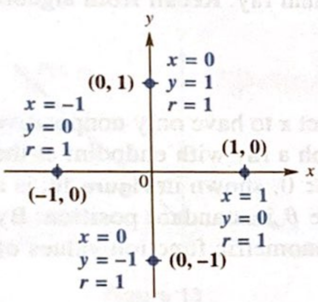

Figure 18

Quadrantal Angles If the terminal side of an angle in standard position lies along the y-axis, any point on this terminal side has x-coordinate 0. Similarly, an angle with terminal side on the x-axis has y-coordinate 0 for any point on the terminal side. Since the values of x and y appear in the denominators of some trigonometric functions, and since a fraction is undefined if its denominator is 0, some trigonometric function values of quadrantal angles (i.e., those with terminal side on an axis) are undefined.

When determining trigonometric function values of quadrantal angles, **Figure 18** can help find the ratios. Because *any* point on the terminal side can be used, it is convenient to choose the point one unit from the origin, with $r = 1$.

To find the function values of a quadrantal angle, determine the position of the terminal side, choose the one of these four points that lies on this terminal side, and then use the definitions involving x, y, and r.

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HOMEWORK 2 Finding Function Values of Quadrantal Angles

Find the values of the six trigonometric functions for each angle.

(a) an angle of 90°

(b) an angle θ in standard position with terminal side through $(-3, 0)$

The conditions under which the trigonometric function values of quadrantal angles are undefined are summarized here.

Conditions for Undefined Function Values

Identify the terminal side of a quadrantal angle.

- If the terminal side of the quadrantal angle lies along the y-axis, then the tangent and secant functions are undefined.
- If the terminal side of the quadrantal angle lies along the x-axis, then the cotangent and cosecant functions are undefined.

The function values of some commonly used quadrantal angles, 0°, 90°, 180°, 270°, and 360°, are summarized in the table. They can be determined when needed by using **Figure 18** and the method of **Homework 2(a).**

For other quadrantal angles such as $-90°$, $-270°$, and $450°$, first determine the coterminal angle that lies between 0° and 360°, and then refer to the table entries for that particular angle. For example, the function values of a $-90°$ angle would correspond to those of a 270° angle.

Function Values of Quadrantal Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0	1	0	Undefined	1	Undefined
90°	1	0	Undefined	0	Undefined	1
180°	0	-1	0	Undefined	-1	Undefined
270°	-1	0	Undefined	0	Undefined	-1
360°	0	1	0	Undefined	1	Undefined

Reciprocal Identities Identities are equations that are true for all values of the variables for which all expressions are defined.

$$(x+y)^2 = x^2 + 2xy + y^2 \qquad 2(x+3) = 2x + 6 \quad \text{Identities}$$

Recall the definition of a reciprocal: the **reciprocal** of the nonzero number x is $\frac{1}{x}$. For example, the reciprocal of 2 is $\frac{1}{2}$, and the reciprocal of $\frac{8}{11}$ is $\frac{11}{8}$. There is no reciprocal for 0.

The definitions of the trigonometric functions earlier in this section were written so that functions in the same column were reciprocals of each other. Since $\sin\theta = \frac{y}{r}$ and $\csc\theta = \frac{r}{y}$,

$$\sin\theta = \frac{1}{\csc\theta} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta}, \quad \text{provided } \sin\theta \neq 0.$$

Also, $\cos\theta$ and $\sec\theta$ are reciprocals, as are $\tan\theta$ and $\cot\theta$. The **reciprocal identities** hold for any angle θ that does not lead to a 0 denominator.

Reciprocal Identities

For all angles θ for which both functions are defined, the following identities hold.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

The reciprocal identities can be written in different forms. For example,

$$\sin\theta = \frac{1}{\csc\theta} \quad \text{can be written} \quad \csc\theta = \frac{1}{\sin\theta}, \quad \text{or} \quad (\sin\theta)(\csc\theta) = 1.$$

EXAMPLE 3 Using the Reciprocal Identities

Find each function value.

(a) $\cos\theta$, given that $\sec\theta = \frac{5}{3}$ **(b)** $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

SOLUTION

(a) Since $\cos\theta$ is the reciprocal of $\sec\theta$,

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{5}{3}} = 1 \div \frac{5}{3} = 1 \cdot \frac{3}{5} = \frac{3}{5}.$$ Simplify the complex fraction.

(b) $\sin\theta = \dfrac{1}{-\frac{\sqrt{12}}{2}}$ $\quad$ $\sin\theta = \frac{1}{\csc\theta}$ and $\csc\theta = -\frac{\sqrt{12}}{2}$

$= -\dfrac{2}{\sqrt{12}}$ $\quad$ Simplify the complex fraction as in part (a).

$= -\dfrac{2}{2\sqrt{3}}$ $\quad$ $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$

$= -\dfrac{1}{\sqrt{3}}$ $\quad$ Divide out the common factor 2.

$= -\dfrac{1}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ $\quad$ Rationalize the denominator.

$= -\dfrac{\sqrt{3}}{3}$ $\quad$ Multiply.

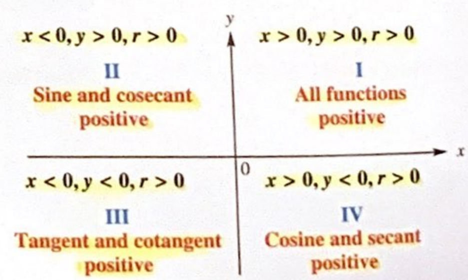
Signs and Ranges of Function Values In the definitions of the trigonometric functions, r is the distance from the origin to the point (x, y). This distance is undirected, so $r > 0$. If we choose a point (x, y) in quadrant I, then both x and y will be positive, and the values of all six functions will be positive.

A point (x, y) in quadrant II satisfies $x < 0$ and $y > 0$. This makes the values of sine and cosecant positive for quadrant II angles, while the other four functions take on negative values. Similar results can be obtained for the other quadrants.

This important information is summarized here.

Signs of Function Values

θ in Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−



HOMEWORK 3 Determining Signs of Functions of Nonquadrantal Angles

Determine the signs of the trigonometric functions of an angle in standard position with the given measure.

(a) 87° **(b)** 300° **(c)** −200°

NOTE Because numbers that are reciprocals always have the same sign, the sign of a function value automatically determines the sign of the reciprocal function value.

EXAMPLE 4 Identifying the Quadrant of an Angle

Identify the quadrant (or possible quadrants) of an angle θ that satisfies the given conditions.

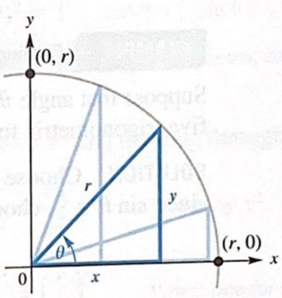
(a) $\sin\theta > 0, \tan\theta < 0$ **(b)** $\cos\theta < 0, \sec\theta < 0$

SOLUTION

(a) Since $\sin\theta > 0$ in quadrants I and II and $\tan\theta < 0$ in quadrants II and IV, both conditions are met only in quadrant II.

(b) The cosine and secant functions are both negative in quadrants II and III, so in this case θ could be in either of these two quadrants.

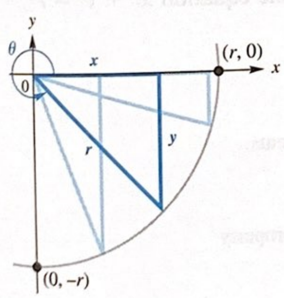
Figure 19(a) shows an angle θ as it increases in measure from near 0° toward 90°. In each case, the value of r is the same. As the measure of the angle increases, y increases but never exceeds r, so $y \le r$. Dividing both sides by the positive number r gives $\frac{y}{r} \le 1$.



(a)

Figure 19

In a similar way, angles in quadrant IV as in **Figure 19(b)** suggest that



(b)

Figure 19

$$-1 \le \frac{y}{r},$$

so $$-1 \le \frac{y}{r} \le 1$$

and $$\mathbf{-1 \le \sin\theta \le 1.} \quad \frac{y}{r} = \sin\theta \text{ for any angle } \theta.$$

Similarly, $$\mathbf{-1 \le \cos\theta \le 1.}$$

The tangent of an angle is defined as $\frac{y}{x}$. It is possible that $x < y$, $x = y$, or $x > y$. Thus, $\frac{y}{x}$ can take any value, so **$\tan\theta$ can be any real number, as can $\cot\theta$.**

The functions $\sec\theta$ and $\csc\theta$ are reciprocals of the functions $\cos\theta$ and $\sin\theta$, respectively, making

$$\mathbf{\sec\theta \le -1} \quad \text{or} \quad \mathbf{\sec\theta \ge 1} \quad \text{and} \quad \mathbf{\csc\theta \le -1} \quad \text{or} \quad \mathbf{\csc\theta \ge 1.}$$

In summary, the ranges of the trigonometric functions are as follows.

Ranges of Trigonometric Functions

Trigonometric Function of θ	Range (Set-Builder Notation)	Range (Interval Notation)
$\sin\theta, \cos\theta$	$\{y \mid \lvert y\rvert \le 1\}$	$[-1, 1]$
$\tan\theta, \cot\theta$	$\{y \mid y \text{ is a real number}\}$	$(-\infty, \infty)$
$\sec\theta, \csc\theta$	$\{y \mid \lvert y\rvert \ge 1\}$	$(-\infty, -1] \cup [1, \infty)$

HOMEWORK 4 Deciding Whether a Value Is in the Range of a Trigonometric Function

Decide whether each statement is *possible* or *impossible*.

(a) $\sin\theta = 2.5$ **(b)** $\tan\theta = 110.47$ **(c)** $\sec\theta = 0.6$

The six trigonometric functions are defined in terms of x, y, and r, where the Pythagorean theorem shows that

$$r^2 = x^2 + y^2 \quad \text{and} \quad r > 0.$$

With these relationships, knowing the value of only one function and the quadrant in which the angle lies makes it possible to find the values of the other trigonometric functions.

EXAMPLE 5 Finding All Function Values Given One Value and the Quadrant

Suppose that angle θ is in quadrant II and $\sin\theta = \frac{2}{3}$. Find the values of the other five trigonometric functions.

SOLUTION Choose any point on the terminal side of angle θ. For simplicity, since $\sin\theta = \frac{y}{r}$, choose the point with $r = 3$.

$$\sin\theta = \frac{2}{3} \quad \text{Given value}$$

$$\frac{y}{r} = \frac{2}{3} \quad \text{Substitute } \tfrac{y}{r} \text{ for } \sin\theta.$$

Since $\frac{y}{r} = \frac{2}{3}$ and $r = 3$, then $y = 2$. To find x, use the equation $x^2 + y^2 = r^2$.

$$x^2 + y^2 = r^2$$

$$x^2 + 2^2 = 3^2 \quad \text{Substitute.}$$

$$x^2 + 4 = 9 \quad \text{Apply exponents.}$$

$$x^2 = 5 \quad \text{Subtract 4.}$$

Remember *both* roots.

$$x = \sqrt{5} \quad \text{or} \quad x = -\sqrt{5} \quad \text{Square root property}$$

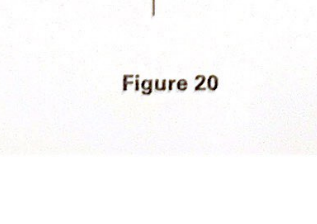


Figure 20

Since θ is in quadrant II, x must be negative. Choose $x = -\sqrt{5}$ so that the point $(-\sqrt{5}, 2)$ is on the terminal side of θ. See **Figure 20.** Now we can find the values of the remaining trigonometric functions.

$$\cos\theta = \frac{x}{r} = \frac{-\sqrt{5}}{3} = -\frac{\sqrt{5}}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{3}{-\sqrt{5}} = -\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{2}{-\sqrt{5}} = -\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

These have rationalized denominators.

$$\cot\theta = \frac{x}{y} = \frac{-\sqrt{5}}{2} = -\frac{\sqrt{5}}{2}$$

$$\csc\theta = \frac{r}{y} = \frac{3}{2}$$

Pythagorean Identities We derive three new identities from the relationship $x^2 + y^2 = r^2$.

$$x^2 + y^2 = r^2 \quad \text{Equation of the Pythagorean theorem}$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2} \quad \text{Divide by } r^2.$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1 \quad \text{Power rule for exponents; } \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \quad \cos\theta = \frac{x}{r},\ \sin\theta = \frac{y}{r}$$

$$\mathbf{\sin^2\theta + \cos^2\theta = 1} \quad \text{Apply exponents; commutative property}$$

Starting again with $x^2 + y^2 = r^2$ and dividing through by x^2 gives the following.

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2} \quad \text{Divide by } x^2.$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2 \quad \text{Power rule for exponents}$$

$$1 + (\tan\theta)^2 = (\sec\theta)^2 \quad \tan\theta = \frac{y}{x},\ \sec\theta = \frac{r}{x}$$

$$\mathbf{\tan^2\theta + 1 = \sec^2\theta} \quad \text{Apply exponents; commutative property}$$

Similarly, dividing through by y^2 leads to another identity.

$$\mathbf{1 + \cot^2\theta = \csc^2\theta}$$

These three identities are the **Pythagorean identities** since the original equation that led to them, $x^2 + y^2 = r^2$, comes from the Pythagorean theorem.

Pythagorean Identities

For all angles θ for which the function values are defined, the following identities hold.

Important

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

As before, we have given only one form of each identity. However, algebraic transformations produce equivalent identities. For example, by subtracting $\sin^2\theta$ from both sides of

$$\sin^2\theta + \cos^2\theta = 1,$$

we obtain an equivalent identity.

$$\cos^2\theta = 1 - \sin^2\theta \quad \text{Alternative form}$$

It is important to be able to transform these identities quickly and also to recognize their equivalent forms.

LOOKING AHEAD TO CALCULUS

The reciprocal, Pythagorean, and quotient identities are used in calculus to find derivatives and integrals of trigonometric functions. A standard technique of integration called **trigonometric substitution** relies on the Pythagorean identities.

Quotient Identities Consider the quotient of $\sin\theta$ and $\cos\theta$, for $\cos\theta \neq 0$.

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} \div \frac{x}{r} = \frac{y}{r} \cdot \frac{r}{x} = \frac{y}{x} = \tan\theta$$

Similarly, $\frac{\cos\theta}{\sin\theta} = \cot\theta$, for $\sin\theta \neq 0$. Thus, we have the **quotient identities.**

Quotient Identities

For all angles θ for which the denominators are not zero, the following identities hold.

Important ← $\quad \dfrac{\sin\theta}{\cos\theta} = \tan\theta \qquad \dfrac{\cos\theta}{\sin\theta} = \cot\theta$

HOMEWORK 5 Using Identities to Find Function Values

Find $\sin\theta$ and $\tan\theta$, given that $\cos\theta = -\frac{\sqrt{3}}{4}$ and $\sin\theta > 0$.

CAUTION ***Be careful to choose the correct sign when taking square roots.***

EXAMPLE 6 Using Identities to Find Function Values

Find $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \frac{4}{3}$ and θ is in quadrant III.

SOLUTION Since θ is in quadrant III, $\sin\theta$ and $\cos\theta$ will both be negative. It is tempting to say that since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\tan\theta = \frac{4}{3}$, then $\sin\theta = -4$ and $\cos\theta = -3$. This is *incorrect,* because $\sin\theta$ and $\cos\theta$ must be in the interval $[-1, 1]$.

We use the Pythagorean identity $\tan^2\theta + 1 = \sec^2\theta$ to find $\sec\theta$, and then the reciprocal identity $\cos\theta = \frac{1}{\sec\theta}$ to find $\cos\theta$.

$$\tan^2\theta + 1 = \sec^2\theta \quad \text{Pythagorean identity}$$

$$\left(\frac{4}{3}\right)^2 + 1 = \sec^2\theta \quad \tan\theta = \frac{4}{3}$$

$$\frac{16}{9} + 1 = \sec^2\theta \quad \text{Square } \frac{4}{3}.$$

Be careful to choose the correct sign here.

$$\frac{25}{9} = \sec^2\theta \quad \text{Add.}$$

$$-\frac{5}{3} = \sec\theta \quad \text{Choose the negative square root since } \sec\theta \text{ is negative when } \theta \text{ is in quadrant III.}$$

$$-\frac{3}{5} = \cos\theta \quad \text{Secant and cosine are reciprocals.}$$

Since $\sin^2\theta = 1 - \cos^2\theta$,

$$\sin^2\theta = 1 - \left(-\frac{3}{5}\right)^2 \quad \cos\theta = -\tfrac{3}{5}$$

$$\sin^2\theta = 1 - \frac{9}{25} \quad \text{Square } -\tfrac{3}{5}.$$

$$\sin^2\theta = \frac{16}{25} \quad \text{Subtract.}$$

Again, be careful.

$$\sin\theta = -\frac{4}{5}. \quad \text{Choose the negative square root.}$$

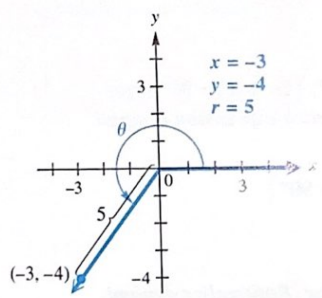


Figure 21

NOTE **Example 6** can also be worked by sketching θ in standard position in quadrant III, finding r to be 5, and then using the definitions of $\sin\theta$ and $\cos\theta$ in terms of x, y, and r. See **Figure 21**.

When using this method, be sure to choose the correct signs for x and y as determined by the quadrant in which the terminal side of θ lies. This is analogous to choosing the correct signs after applying the Pythagorean identities.

6.2 Exercises

Concept Check Sketch an angle θ in standard position such that θ has the least positive measure, and the given point is on the terminal side of θ. Then find the values of the six trigonometric functions for each angle. Rationalize denominators when applicable. See Example 1.

1. $(-12, -5)$ **2.** $(-4, -3)$ **3.** $(15, -8)$ **4.** $(-24, -7)$

5. $(0, 5)$ **6.** $(-5, 0)$ **7.** $(-1, \sqrt{3})$ **8.** $(-2\sqrt{3}, 2)$

9. *Concept Check* How is the value of r interpreted geometrically in the definitions of the sine, cosine, secant, and cosecant functions?

Concept Check Suppose that the point (x, y) is in the indicated quadrant. Decide whether the given ratio is positive or negative. Recall that $r = \sqrt{x^2 + y^2}$. (Hint: Drawing a sketch may help.)

10. III, $\frac{y}{r}$ **11.** IV, $\frac{x}{y}$ **12.** III, $\frac{x}{r}$ **13.** IV, $\frac{y}{r}$

14. II, $\frac{y}{x}$ **15.** III, $\frac{x}{y}$ **16.** III, $\frac{r}{y}$ **17.** I, $\frac{y}{x}$

In Exercises 18–21, an equation of the terminal side of an angle θ in standard position is given with a restriction on x. Sketch the least positive such angle θ, and find the values of the six trigonometric functions of θ. ***See Example 2.***

18. $3x + 5y = 0, x \geq 0$

19. $-5x - 3y = 0, x \leq 0$

20. $x - y = 0, x \geq 0$

21. $\sqrt{3}x + y = 0, x \leq 0$

To work Exercises 22–30, begin by reproducing the graph in **Figure 18** *Keep in mind that for each of the four points labeled in the figure, $r = 1$. For each quadrantal angle, identify the appropriate values of x, y, and r to find the indicated function value. If it is undefined, say so.*

22. $\sin 90°$

23. $\cot 90°$

24. $\csc 270°$

25. $\cos(-90°)$

26. $\tan 450°$

27. $\sec(-540°)$

28. $\cos 1800°$

29. $\cot 1800°$

30. $\tan 1800°$

Use the trigonometric function values of quadrantal angles given in this section to evaluate each expression. An expression such as $\cot^2 90°$ means $(\cot 90°)^2$, which is equal to $0^2 = 0$.

31. $\tan 0° - 6 \sin 90°$

32. $4 \csc 270° + 3 \cos 180°$

33. $2 \sec 0° + 4 \cot^2 90° + \cos 360°$

34. $-3 \sin^4 90° + 4 \cos^3 180°$

35. $\cos^2(-180°) + \sin^2(-180°)$

If n is an integer, $n \cdot 180°$ represents an integer multiple of 180°, $(2n + 1) \cdot 90°$ represents an odd integer multiple of 90°, and so on. Decide whether each expression is equal to 0, 1, *or* −1 *or is* undefined.

36. $\sin[n \cdot 180°]$

37. $\tan[(2n + 1) \cdot 90°]$

38. $\cot[n \cdot 180°]$

Use the appropriate reciprocal identity to find each function value. Rationalize denominators when applicable. ***See Example 3.***

39. $\sec \theta$, given that $\cos \theta = \frac{5}{8}$

40. $\csc \theta$, given that $\sin \theta = -\frac{8}{43}$

41. $\cot \theta$, given that $\tan \theta = 18$

42. $\sin \theta$, given that $\csc \theta = \frac{\sqrt{24}}{3}$

43. $\cos \theta$, given that $\sec \theta = 9.80425133$

Determine the signs of the trigonometric functions of an angle in standard position with the given measure.

44. 84°

45. 195°

46. 125°

47. −15°

48. 1005°

Identify the quadrant (or possible quadrants) of an angle θ that satisfies the given conditions. ***See Example 4.***

49. $\cos \theta > 0, \sec \theta > 0$

50. $\sin \theta > 0, \tan \theta > 0$

51. $\cos \theta < 0, \sin \theta < 0$

52. $\csc \theta > 0, \cot \theta > 0$

53. $\cot \theta < 0, \sec \theta < 0$

54. $\tan \theta < 0, \cot \theta < 0$

Decide whether each statement is possible *or* impossible *for some angle θ.*

55. $\sin \theta = 3$

56. $\cos \theta = -0.56$

57. $\cot \theta = 0.93$

58. $\sec \theta = -0.9$

59. $\csc \theta = -100$

Use identities to solve each of the following. ***See Examples 5 and 6.***

60. Find $\sin\theta$, given that $\cos\theta = \frac{4}{5}$ and θ is in quadrant IV.

61. Find $\sec\theta$, given that $\tan\theta = \frac{\sqrt{7}}{3}$ and θ is in quadrant III.

62. Find $\cot\theta$, given that $\csc\theta = -2$ and θ is in quadrant III.

6.3 Evaluating Trigonometric Functions

- Right-Triangle-Based Definitions of the Trigonometric Functions
- Cofunctions
- Trigonometric Function Values of Special Angles
- Reference Angles
- Special Angles as Reference Angles
- Finding Angle Measures

Right-Triangle-Based Definitions of the Trigonometric Functions We used angles in standard position to define the trigonometric functions in **Section 6.2.** There is another way to approach them: As ratios of the lengths of the sides of right triangles.

Figure 22 shows an acute angle A in standard position. The definitions of the trigonometric function values of angle A require x, y, and r. As drawn in **Figure 22** x and y are the lengths of the two legs of the right triangle ABC, and r is the length of the hypotenuse.

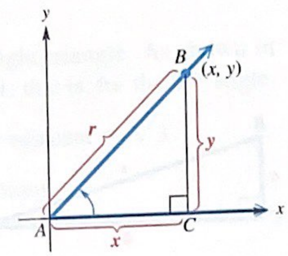


Figure 22

The side of length y is called the **side opposite** angle A, and the side of length x is called the **side adjacent** to angle A. We use the lengths of these sides to replace x and y in the definitions of the trigonometric functions, and the length of the hypotenuse to replace r, to get the following right-triangle-based definitions.

Very important

Right-Triangle-Based Definitions of Trigonometric Functions

Let A represent any acute angle in standard position.

$$\sin A = \frac{y}{r} = \frac{\text{side opposite } A}{\text{hypotenuse}} \qquad \csc A = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{side opposite } A}$$

$$\cos A = \frac{x}{r} = \frac{\text{side adjacent to } A}{\text{hypotenuse}} \qquad \sec A = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{side adjacent to } A}$$

$$\tan A = \frac{y}{x} = \frac{\text{side opposite } A}{\text{side adjacent to } A} \qquad \cot A = \frac{x}{y} = \frac{\text{side adjacent to } A}{\text{side opposite } A}$$

NOTE We will sometimes shorten wording like "side opposite A" to just "side opposite" when the meaning is obvious.

EXAMPLE 1 Finding Trigonometric Function Values of an Acute Angle

Find the sine, cosine, and tangent values for angles A and B in the right triangle in **Figure 23.**

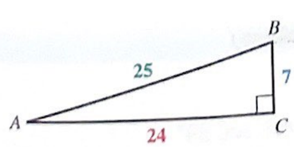


Figure 23

SOLUTION The length of the side opposite angle A is 7, the length of the side adjacent to angle A is 24, and the length of the hypotenuse is 25.

$$\sin A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{7}{25} \qquad \cos A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{24}{25} \qquad \tan A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{7}{24}$$

The length of the side opposite angle B is 24, and the length of the side adjacent to B is 7.

$$\sin B = \frac{24}{25} \qquad \cos B = \frac{7}{25} \qquad \tan B = \frac{24}{7}$$

Use the relationships given in the box.

Now Try Exercise 1.

NOTE Because the cosecant, secant, and cotangent ratios are the reciprocals of the sine, cosine, and tangent values, respectively, in **Example 1,**

$$\csc A = \frac{25}{7}, \quad \sec A = \frac{25}{24}, \quad \cot A = \frac{24}{7}, \quad \csc B = \frac{25}{24},$$

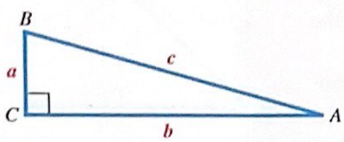
$$\sec B = \frac{25}{7}, \quad \text{and} \quad \cot B = \frac{7}{24}.$$

Cofunctions In **Example 1,** notice that $\sin A = \cos B$ and $\cos A = \sin B$. Such relationships are always true for the two acute angles of a right triangle. **Figure 24** shows a right triangle with acute angles A and B and a right angle at C. The length of the side opposite angle A is a, and the length of the side opposite angle B is b. The length of the hypotenuse is c.

By the preceding definitions, $\sin A = \frac{a}{c}$. Also, $\cos B = \frac{a}{c}$. Thus,

$$\sin A = \frac{a}{c} = \cos B.$$

Similarly, $\tan A = \frac{a}{b} = \cot B$ and $\sec A = \frac{c}{b} = \csc B$.



Whenever we use A, B, and C to name angles in a right triangle, C will be the right angle.

Figure 24

Since the sum of the three angles in any triangle is 180° and angle C equals 90°, angles A and B must have a sum of $180° - 90° = 90°$. As mentioned in **Section 6.1,** angles with a sum of 90° are complementary angles. Since angles A and B are complementary and $\sin A = \cos B$, the functions sine and cosine are **cofunctions.** Tangent and cotangent are also cofunctions, as are secant and cosecant. And since the angles A and B are complementary, $A + B = 90°$, or $B = 90° - A$, giving the following.

$$\sin A = \cos B = \cos(90° - A)$$

Similar **cofunction identities** are true for the other trigonometric functions.

Cofunction Identities

For any acute angle A, cofunction values of complementary angles are equal.

$$\sin A = \cos(90° - A) \quad \sec A = \csc(90° - A) \quad \tan A = \cot(90° - A)$$

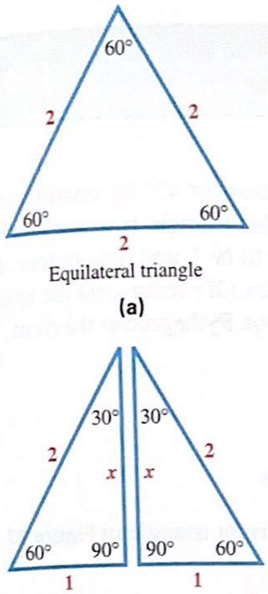
$$\cos A = \sin(90° - A) \quad \csc A = \sec(90° - A) \quad \cot A = \tan(90° - A)$$

HOMEWORK 1 Writing Functions in Terms of Cofunctions

Write each function in terms of its cofunction.

(a) cos 52° **(b)** tan 71° **(c)** sec 24°

Now Try Exercises 25 and 27.



Equilateral triangle

(a)

30°–60° right triangle

(b)

Figure 25

Trigonometric Function Values of Special Angles Certain special angles, such as 30°, 45°, and 60°, occur so often in trigonometry and in more advanced mathematics that they deserve special study. We start with an equilateral triangle, a triangle with all sides of equal length. Each angle of such a triangle measures 60°. Although the results we will obtain are independent of the length, for convenience we choose the length of each side to be 2 units. See **Figure 25(a)**

Bisecting one angle of this equilateral triangle leads to two right triangles, each of which has angles of 30°, 60°, and 90°, as shown in **Figure 25(b)**. An angle bisector of an equilateral triangle also bisects the opposite side; therefore, the shorter leg has length 1. Let x represent the length of the longer leg.

$$2^2 = 1^2 + x^2 \quad \text{Pythagorean theorem}$$

$$4 = 1 + x^2 \quad \text{Apply the exponents.}$$

$$3 = x^2 \quad \text{Subtract 1 from each side.}$$

$$\sqrt{3} = x \quad \text{Square root property; choose the positive root.}$$

Figure 26 summarizes our results using a 30°–60° right triangle. As shown in the figure, the side opposite the 30° angle has length 1; that is, for the 30° angle,

$$\text{hypotenuse} = 2, \quad \text{side opposite} = 1, \quad \text{side adjacent} = \sqrt{3}.$$

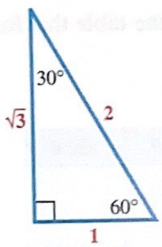


Figure 26

Now we use the definitions of the trigonometric functions.

$$\sin 30^\circ = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Rationalize the denominator.

$$\csc 30^\circ = \frac{2}{1} = 2$$

$$\sec 30^\circ = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 30^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$$

EXAMPLE 2 Finding Trigonometric Function Values for 60°

Find the six trigonometric function values for a 60° angle.

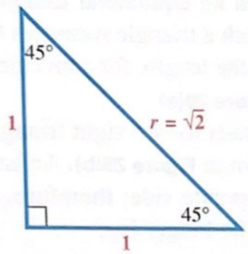
SOLUTION Refer to **Figure 26** to find the following ratios.

$$\sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{1}{2} \qquad \tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\csc 60^\circ = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sec 60^\circ = \frac{2}{1} = 2 \qquad \cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

✔ ***Now Try Exercises 29, 31, and 33.***

NOTE The results in **Example 2** can also be found using the fact that cofunction values of complementary angles are equal.



45°–45° right triangle

Figure 27

We find the values of the trigonometric functions for 45° by starting with a 45°–45° right triangle, as shown in **Figure 27.** This triangle is isosceles. For simplicity, we choose the lengths of the equal sides to be 1 unit. (As before, the results are independent of the length of the equal sides.) If r represents the length of the hypotenuse, then we can find its value using the Pythagorean theorem.

$$1^2 + 1^2 = r^2 \quad \text{Pythagorean theorem}$$

$$2 = r^2 \quad \text{Simplify.}$$

$$\sqrt{2} = r \quad \text{Choose the positive root.}$$

Now we use the measures indicated on the 45°–45° right triangle in **Figure 27.**

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \tan 45^\circ = \frac{1}{1} = 1$$

$$\csc 45^\circ = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \sec 45^\circ = \frac{\sqrt{2}}{1} = \sqrt{2} \qquad \cot 45^\circ = \frac{1}{1} = 1$$

Function values for 30°, 45°, and 60° are summarized in the table that follows.

Function Values of Special Angles

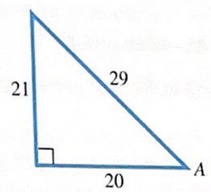
θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

NOTE You will be able to reproduce this table quickly if you learn the values of sin 30°, sin 45°, and sin 60°. Then you can complete the rest of the table using the reciprocal, cofunction, and quotient identities.

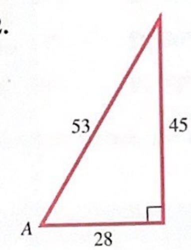
6.3 Exercises

Find exact values or expressions for sin A, cos A, *and* tan A. ***See Example 1.***

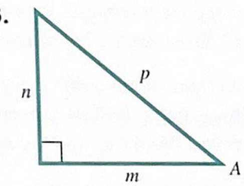
1.



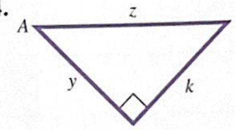
2.



3.



4.



Concept Check *For each trigonometric function in Column I, choose its value from Column II.*

I		II		
5. sin 30°	**6.** cos 45°	**A.** $\sqrt{3}$	**B.** 1	**C.** $\frac{1}{2}$
7. tan 45°	**8.** sec 60°	**D.** $\frac{\sqrt{3}}{2}$	**E.** $\frac{2\sqrt{3}}{3}$	**F.** $\frac{\sqrt{3}}{3}$
9. csc 60°	**10.** cot 30°	**G.** 2	**H.** $\frac{\sqrt{2}}{2}$	**I.** $\sqrt{2}$

Suppose ABC is a right triangle with sides of lengths a, b, and c and right angle at C. (See **Figure 24**.*) Find the unknown side length using the Pythagorean theorem, and then find the values of the six trigonometric functions for angle B. Rationalize denominators when applicable.*

11. $a = 5,\ b = 12$ **12.** $a = 3,\ b = 4$ **13.** $a = 6,\ c = 7$

14. $b = 7,\ c = 12$ **15.** $a = 3,\ c = 10$ **16.** $b = 8,\ c = 11$

17. $a = 1,\ c = 2$ **18.** $a = \sqrt{2},\ c = 2$ **19.** $b = 2,\ c = 5$

20. ***Concept Check*** Give a summary of the six cofunction relationships.

Write each function in terms of its cofunction. Assume that all angles in which an unknown appears are acute angles.

21. cos 30° **22.** sin 45° **23.** csc 60° **24.** cot 73°

25. sec 39° **26.** tan 25.4° **27.** sin 38.7° **28.** csc 49.9°

For each expression, give the exact value. ***See Example 2.***

29. tan 30° **30.** cot 30° **31.** sin 30° **32.** cos 30°

33. sec 30° **34.** csc 30° **35.** csc 45° **36.** sec 45°

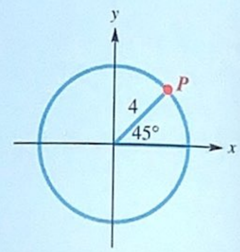
37. cos 45° **38.** cot 45° **39.** tan 45° **40.** sin 45°

41. sin 60° **42.** cos 60° **43.** tan 60° **44.** csc 60°

Relating Concepts

For individual or collaborative investigation *(Exercises 45–48)*

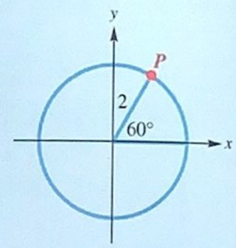
The figure shows a 45° central angle in a circle with radius 4 units. To find the coordinates of point P on the circle, ***work Exercises 45–48 in order.***



45. Sketch a line segment from P perpendicular to the x-axis.

46. Use the trigonometric ratios for a 45° angle to label the sides of the right triangle you sketched in **Exercise 45.**

47. Which sides of the right triangle give the coordinates of point P? What are the coordinates of P?



48. The figure at the right shows a 60° central angle in a circle of radius 2 units. Follow the same procedure as in **Exercises 45–47** to find the coordinates of P in the figure.

Concept Check *Work each problem.*

49. Find the equation of the line that passes through the origin and makes a 30° angle with the x-axis.

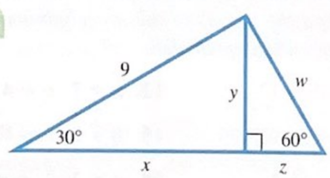
50. Find the equation of the line that passes through the origin and makes a 60° angle with the x-axis.

51. What angle does the line $y = \sqrt{3}x$ make with the positive x-axis?

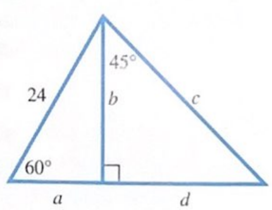
52. What angle does the line $y = \frac{\sqrt{3}}{3}x$ make with the positive x-axis?

Find the exact value of each part labeled with a variable in each figure.

53.



54.



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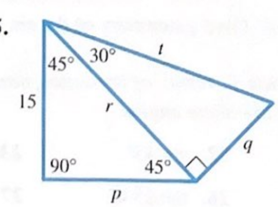
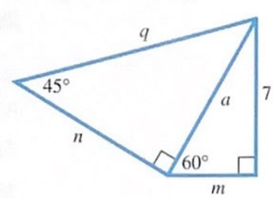
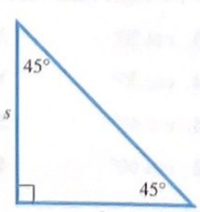


56.

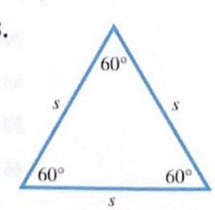


Find a formula for the area of each figure in terms of s.

57.



58.



Match each angle in Column I with its reference angle in Column II. Choices may be used once, more than once, or not at all.

I		II	
59. 98°	**60.** 212°	**A.** 45°	**B.** 60°
61. −135°	**62.** −60°	**C.** 82°	**D.** 30°
63. 750°	**64.** 480°	**E.** 38°	**F.** 32°

Complete the table with exact trigonometric function values. Do not use a calculator. ***See Example 3.***

	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
65.	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$			$\frac{2\sqrt{3}}{3}$	2
66.	45°			1	1		
67.	60°		$\frac{1}{2}$	$\sqrt{3}$		2	
68.	120°	$\frac{\sqrt{3}}{2}$		$-\sqrt{3}$			$\frac{2\sqrt{3}}{3}$
69.	135°	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$			$-\sqrt{2}$	$\sqrt{2}$
70.	150°		$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$			2
71.	210°	$-\frac{1}{2}$		$\frac{\sqrt{3}}{3}$	$\sqrt{3}$		−2
72.	240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$			−2	$-\frac{2\sqrt{3}}{3}$

7.1 Limits of Functions

- One-Sided Limits
- Rules for Calculating Limits
- The Squeeze Theorem

In order to speak meaningfully about rates of change, tangent lines, and areas bounded by curves, we have to investigate the process of finding limits. Indeed, the concept of *limit* is the cornerstone on which the development of calculus rests. Before we try to give a definition of a limit, let us look at more examples.

$f(x) = \frac{x^2-1}{x-1}$ near $x=1$

$\frac{(x-1)(x+1)}{x-1} = x+1$

$\lim_{x\to 1} = x+1$

EXAMPLE 1

Describe the behavior of the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ near $x = 1$.

SOLUTION Note that $f(x)$ is defined for all real numbers x except $x = 1$. (We can't divide by zero.) For any $x \neq 1$ we can simplify the expression for $f(x)$ by factoring the numerator and canceling common factors:

$$f(x) = \frac{(\cancel{x-1})(x+1)}{\cancel{x-1}} = x + 1 \quad \text{for} \quad x \neq 1.$$

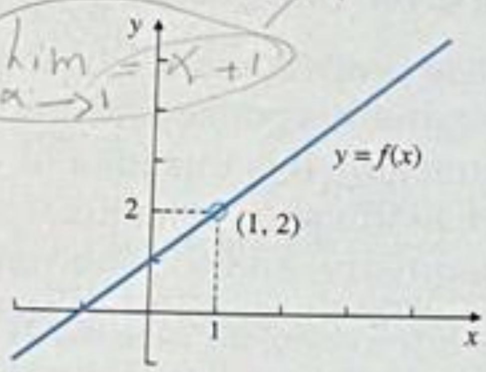


Figure 1 The graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$

The graph of f is the line $y = x + 1$ with one point removed, namely, the point $(1, 2)$. This removed point is shown as a "hole" in the graph in **Figure 1**. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1. Therefore, we say that $f(x)$ approaches arbitrarily close to 2 as x approaches 1, or, more simply, $f(x)$ approaches *the limit* 2 as x approaches 1. We write this as

$$\lim_{x \to 1} f(x) = 2 \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

HOMEWORK 1

What happens to the function $g(x) = (1 + x^2)^{1/x^2}$ as x approaches zero?

The examples above suggest the following *informal* definition of limit.

Definition 1 An informal definition of limit

If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a, but not equal to a, we say that the function f approaches the **limit** L as x approaches a, and we write

$$\lim_{x \to a} f(x) = L.$$

This definition is *informal* because phrases such as *close as we want* and *close enough* are imprecise; their meaning depends on the context. To a machinist manufacturing a piston, *close enough* may mean *within a few thousandths of an inch*. To an astronomer studying distant galaxies, *close enough* may mean *within a few thousand light-years*. The definition should be clear enough, however, to enable us to recognize and evaluate limits of specific functions. A more precise "formal" definition, given in Section 7.4, is needed if we want to *prove* theorems about limits like Theorems 2–4, stated later in this section.

EXAMPLE 2

Find (a) $\lim_{x \to a} x$ and (b) $\lim_{x \to a} c$ (where c is a constant).

SOLUTION In words, part (a) asks: "What does x approach as x approaches a?" The answer is surely a.

$$\lim_{x \to a} x = a.$$

Similarly, part (b) asks: "What does c approach as x approaches a?" The answer here is that c approaches c; you can't get any closer to c than by *being* c.

$$\lim_{x \to a} c = c.$$

Example 2 shows that $\lim_{x \to a} f(x)$ can *sometimes* be evaluated by just calculating $f(a)$. This will be the case if $f(x)$ is defined in an open interval containing $x = a$ and the graph of f passes unbroken through the point $(a, f(a))$. The next example shows various ways algebraic manipulations can be used to evaluate $\lim_{x \to a} f(x)$ in situations where $f(a)$ is undefined. This usually happens when $f(x)$ is a fraction with denominator equal to 0 at $x = a$.

HOMEWORK 2

Evaluate:

(a) $\lim_{x \to -2} \dfrac{x^2 + x - 2}{x^2 + 5x + 6}$, (b) $\lim_{x \to a} \dfrac{\frac{1}{x} - \frac{1}{a}}{x - a}$, and (c) $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x^2 - 16}$.

$\dfrac{-2^2 + -2(-2)}{-2^2 + 5(-2) + 6}$

$\dfrac{(x-1)(x+2)}{(x+3)(x+2)}$

$\dfrac{x-1}{x+3} = \dfrac{-2-1}{-2+3}$

A function f may be defined on both sides of $x = a$ but still not have a limit at $x = a$. For example, the function $f(x) = 1/x$ has no limit as x approaches 0. As can be seen in **Figure 2(a)**, the values $1/x$ grow ever larger in absolute value as x approaches 0; there is no single number L that they approach.

The following example shows that even if $f(x)$ is defined at $x = a$, the limit of $f(x)$ as x approaches a may not be equal to $f(a)$.

BEWARE!

Always be aware that the existence of $\lim_{x \to a} f(x)$ does not require that $f(a)$ exist and does not depend on $f(a)$ even if $f(a)$ does exist. It depends only on the values of $f(x)$ for x *near but not equal to a*.

EXAMPLE 3

Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$ (See **Figure 2(b)**.) Then

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x = 2, \quad \text{although} \quad g(2) = 1.$$

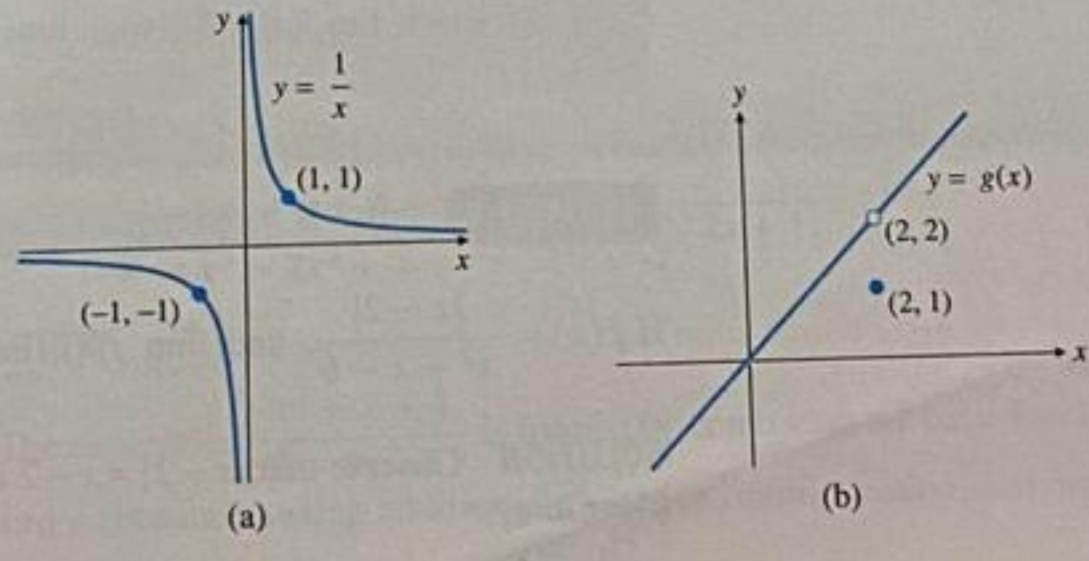


Figure 2

(a) $\lim_{x \to 0} \dfrac{1}{x}$ does not exist

(b) $\lim_{x \to 2} g(x) = 2$, but $g(2) = 1$

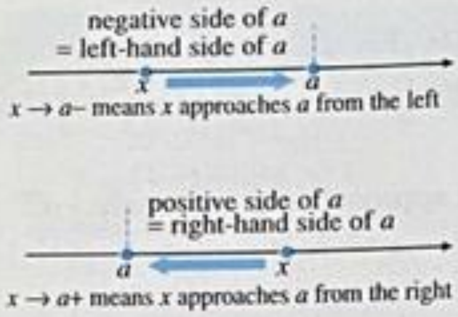


Figure 3 One-sided approach

One-Sided Limits Limits are *unique*; if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$. Although a function f can only have one limit at any particular point, it is, nevertheless, useful to be able to describe the behavior of functions that approach different numbers as x approaches a from one side or the other. (See **Figure 3**.)

Definition 2 Informal definition of left and right limits

If $f(x)$ is defined on some interval (b, a) extending to the left of $x = a$, and if we can ensure that $f(x)$ is as close as we want to L by taking x to the left of a and close enough to a, then we say $f(x)$ has **left limit** L at $x = a$, and we write

$$\lim_{x \to a-} f(x) = L.$$

If $f(x)$ is defined on some interval (a, b) extending to the right of $x = a$, and if we can ensure that $f(x)$ is as close as we want to L by taking x to the right of a and close enough to a, then we say $f(x)$ has **right limit** L at $x = a$, and we write

$$\lim_{x \to a+} f(x) = L.$$

Note the use of the suffix + to denote approach from the right (the *positive* side) and the suffix – to denote approach from the left (the *negative* side).

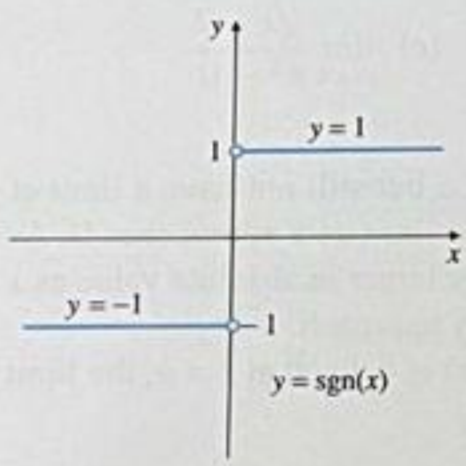


Figure 4 $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist, because $\lim_{x \to 0-} \operatorname{sgn}(x) = -1$, $\lim_{x \to 0+} \operatorname{sgn}(x) = 1$

HOMEWORK 3

The signum function $\operatorname{sgn}(x) = x/|x|$ (see **Figure 4**) has left limit -1 and right limit 1 at $x = 0$:

$$\lim_{x \to 0-} \operatorname{sgn}(x) = -1 \quad \text{and} \quad \lim_{x \to 0+} \operatorname{sgn}(x) = 1$$

because the values of $\operatorname{sgn}(x)$ approach -1 (they *are* -1) if x is negative and approaches 0, and they approach 1 if x is positive and approaches 0. Since these left and right limits are not equal, $\lim_{x \to 0} \operatorname{sgn}(x)$ *does not exist*.

As suggested in Home Work 3, the relationship between ordinary (two-sided) limits and one-sided limits can be stated as follows:

Theorem 1: Relationship between one-sided and two-sided limits

A function $f(x)$ has limit L at $x = a$ if and only if it has both left and right limits there and these one-sided limits are both equal to L:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = L.$$

EXAMPLE 4

If $f(x) = \dfrac{|x-2|}{x^2+x-6}$, find: $\lim_{x \to 2+} f(x)$, $\lim_{x \to 2-} f(x)$, and $\lim_{x \to 2} f(x)$.

SOLUTION Observe that $|x-2| = x-2$ if $x > 2$, and $|x-2| = -(x-2)$ if $x < 2$. Therefore,

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} \frac{x-2}{x^2+x-6} = \lim_{x\to 2+} \frac{x-2}{(x-2)(x+3)} = \lim_{x\to 2+} \frac{1}{x+3} = \frac{1}{5},$$

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} \frac{-(x-2)}{x^2+x-6} = \lim_{x\to 2-} \frac{-(x-2)}{(x-2)(x+3)} = \lim_{x\to 2-} \frac{-1}{x+3} = -\frac{1}{5}.$$

Since $\lim_{x\to 2-} f(x) \neq \lim_{x\to 2+} f(x)$, the limit $\lim_{x\to 2} f(x)$ does not exist.

HOMEWORK 4

What one-sided limits does $g(x) = \sqrt{1-x^2}$ have at $x=-1$ and $x=1$?

Rules for Calculating Limits The following theorems make it easy to calculate limits and one-sided limits of many kinds of functions when we know some elementary limits. We will not prove the theorems here. (See Section 7.4.)

Theorem 2: Limit Rules

If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and k is a constant, then

1. **Limit of a sum:** $\lim_{x\to a} [f(x)+g(x)] = L+M$
2. **Limit of a difference:** $\lim_{x\to a} [f(x)-g(x)] = L-M$
3. **Limit of a product:** $\lim_{x\to a} f(x)g(x) = LM$
4. **Limit of a multiple:** $\lim_{x\to a} kf(x) = kL$
5. **Limit of a quotient:** $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$.

If m is an integer and n is a positive integer, then

6. **Limit of a power:** $\lim_{x\to a} [f(x)]^{m/n} = L^{m/n}$, provided $L>0$ if n is even, and $L \neq 0$ if $m<0$.

If $f(x) \le g(x)$ on an interval containing a in its interior, then

7. **Order is preserved:** $L \le M$

Rules 1–6 are also valid for right limits and left limits. So is Rule 7, under the assumption that $f(x) \le g(x)$ on an open interval extending in the appropriate direction from a.

In words, rule 1 of Theorem 2 says that the limit of a sum of functions is the sum of their limits. Similarly, rule 5 says that the limit of a quotient of two functions is the quotient of their limits, provided that the limit of the denominator is not zero. Try to state the other rules in words.

We can make use of the limits (a) $\lim_{x\to a} c = c$ (where c is a constant) and (b) $\lim_{x\to a} x = a$, from Example 2, together with parts of Theorem 2 to calculate limits of many combinations of functions.

EXAMPLE 5

Find: (a) $\lim_{x\to a} \dfrac{x^2+x+4}{x^3-2x^2+7}$ and (b) $\lim_{x\to 2} \sqrt{2x+1}$.

SOLUTION

(a) The expression $\dfrac{x^2+x+4}{x^3-2x^2+7}$ is formed by combining the basic functions x and c (constant) using addition, subtraction, multiplication, and division.

Theorem 2 assures us that the limit of such a combination is the same combination of the limits a and c of the basic functions, provided the denominator does not have limit zero. Thus,

$$\lim_{x\to a} \frac{x^2+x+4}{x^3-2x^2+7} = \frac{a^2+a+4}{a^3-2a^2+7} \qquad \text{provided } a^3-2a^2+7\neq 0.$$

(b) The same argument as in (a) shows that $\lim_{x\to 2}(2x+1) = 2(2)+1 = 5$. Then the Power Rule (rule 6 of Theorem 2) assures us that

$$\lim_{x\to 2}\sqrt{2x+1} = \sqrt{5}.$$

The following result is an immediate corollary of Limit Rules.

Theorem 3: Limits of Polynomials and Rational Functions

1. If $P(x)$ is a polynomial and a is any real number, then

$$\lim_{x\to a} P(x) = P(a).$$

2. If $P(x)$ and $Q(x)$ are polynomials and $Q(a)\neq 0$, then

$$\lim_{x\to a}\frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

The Squeeze Theorem The following theorem will enable us to calculate some very important limits in subsequent chapters. It is called the *Squeeze Theorem* because it refers to a function g whose values are squeezed between the values of two other functions f and h that have the same limit L at a point a. Being trapped between the values of two functions that approach L, the values of g must also approach L. (See **Figure 5**.)

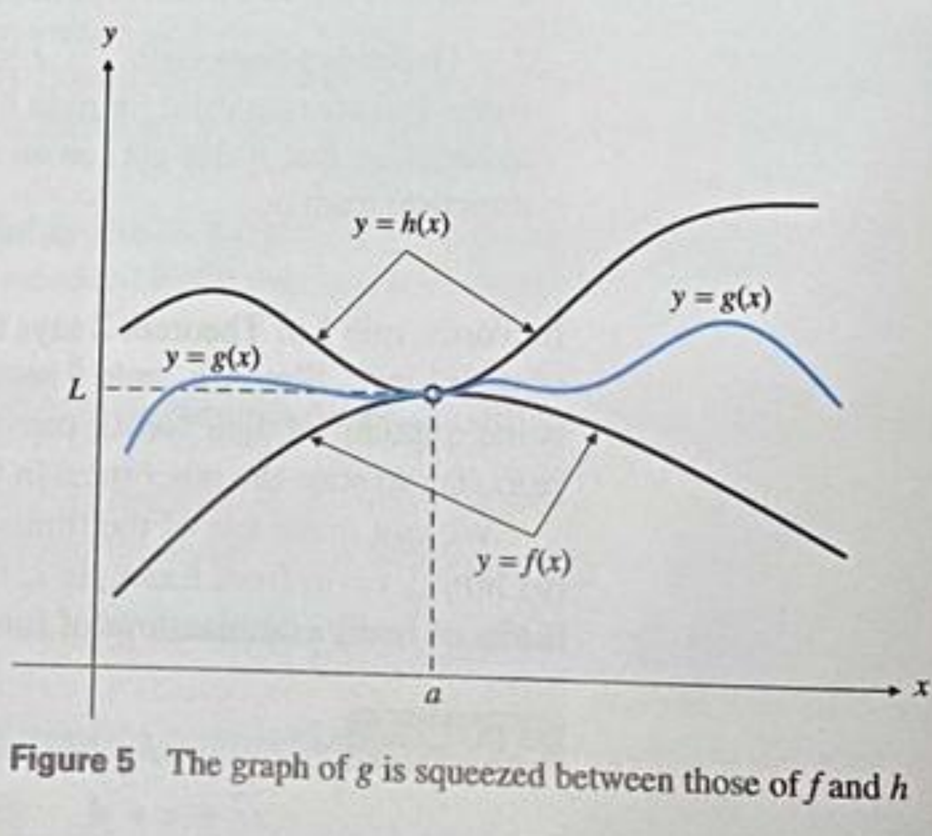


Figure 5 The graph of g is squeezed between those of f and h

Theorem 4: The Squeeze Theorem

Suppose that $f(x) \le g(x) \le h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim_{x\to a} g(x) = L$ also. Similar statements hold for left and right limits.

HOMEWORK 5

Given that $3 - x^2 \le u(x) \le 3 + x^2$ for all $x \ne 0$, find $\lim_{x\to 0} u(x)$.

EXAMPLE 6

Show that if $\lim_{x\to a} |f(x)| = 0$, then $\lim_{x\to a} f(x) = 0$.

SOLUTION Since $-|f(x)| \le f(x) \le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches a, so does $f(x)$ by the Squeeze Theorem.

1.1 Exercises

1. Find: (a) $\lim_{x\to -1} f(x)$, (b) $\lim_{x\to 0} f(x)$, and (c) $\lim_{x\to 1} f(x)$, for the function f whose graph is shown in **Figure 6**.

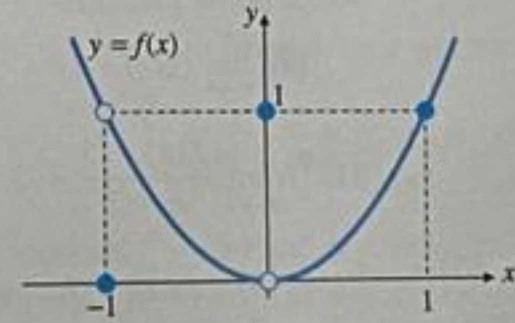


Figure 6

In Exercises 2–3, find the indicated one-sided limit of the function g whose graph is given in Figure 6.

2. $\lim_{x\to 1-} g(x)$

3. $\lim_{x\to 3+} g(x)$

In Exercises 4–18, evaluate the limit or explain why it does not exist.

4. $\lim_{x\to 4} (x^2 - 4x + 1)$

5. $\lim_{x\to 3} \dfrac{x+3}{x+6}$

6. $\lim_{x\to 1} \dfrac{x^2 - 1}{x + 1}$

7. $\lim_{x\to 3} \dfrac{x^2 - 6x + 9}{x^2 - 9}$

8. $\lim_{h\to 2} \dfrac{1}{4 - h^2}$

9. $\lim_{x\to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

10. $\lim \dfrac{(x-\pi)^2}{\ }$

11. $\lim_{x\to 0} \dfrac{|x-2|}{x-2}$

guess why?) Evaluate this limit for the functions f in Exercises 19–21.

19. $f(x) = x^2$ **20.** $f(x) = \dfrac{1}{x}$ **21.** $f(x) = \sqrt{x}$

In Exercises 22–27, find the indicated one-sided limit or explain why it does not exist.

22. $\lim_{x \to 2-} \sqrt{2-x}$ **23.** $\lim_{x \to -2-} \sqrt{2-x}$ **24.** $\lim_{x \to 0} \sqrt{x^3 - x}$

25. $\lim_{x \to 0+} \sqrt{x^3 - x}$ **26.** $\lim_{x \to a-} \dfrac{|x-a|}{x^2 - a^2}$ **27.** $\lim_{x \to 2-} \dfrac{x^2 - 4}{|x+2|}$

Exercises 28–29 refer to the function

$$f(x) = \begin{cases} x - 1 & \text{if } x \le -1 \\ x^2 + 1 & \text{if } -1 < x \le 0 \\ (x + \pi)^2 & \text{if } x > 0. \end{cases}$$

Find the indicated limits.

28. $\lim_{x \to -1-} f(x)$ **29.** $\lim_{x \to 0+} f(x)$

30. Suppose $\lim_{x \to 4} f(x) = 2$ and $\lim_{x \to 4} g(x) = -3$. Find:

(a) $\lim_{x \to 4} (g(x) + 3)$ (b) $\lim_{x \to 4} xf(x)$ (c) $\lim_{x \to 4} (g(x))^2$ (d) $\lim_{x \to 4} \dfrac{g(x)}{f(x) - 1}$.

31. If $\lim_{x \to 2} \dfrac{f(x) - 5}{x - 2} = 3$, find $\lim_{x \to 2} f(x)$.

Using the Squeeze Theorem

32. If $2 - x^2 \le g(x) \le 2 \cos x$ for all x, find $\lim_{x \to 0} g(x)$.

33. On what intervals is $x^{1/3} < x^3$? On what intervals is $x^{1/3} > x^3$? If the graph of $y = h(x)$ always lies between the graphs of $y = x^{1/3}$ and $y = x^3$, for what real numbers a can you determine the value of $\lim_{x \to a} h(x)$? Find the limit for each of these values of a.

7.2 Limits at Infinity and Infinite Limits

- Limits at Infinity
- Limits at Infinity for Rational Functions
- Infinite Limits

In this section we will extend the concept of limit to allow for two situations not covered by the definitions of limit and one-sided limit in the previous section:

(i) limits at infinity, where x becomes arbitrarily large, positive or negative;

(ii) infinite limits, which are not really limits at all but provide useful symbolism for describing the behavior of functions whose values become arbitrarily large, positive or negative.

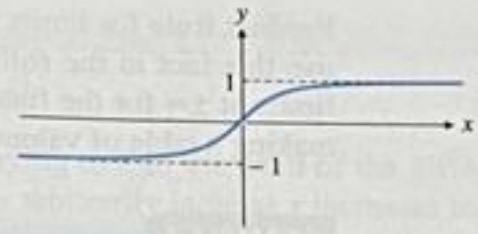


Figure 7 The graph of $x/\sqrt{x^2+1}$

Limits at Infinity Consider the function

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

whose graph is shown in **Figure 7** and for which some values (rounded to 7 decimal places) are given in Table 1. The values of $f(x)$ seem to approach 1 as x takes on larger and larger positive values, and -1 as x takes on negative values that get larger and larger in absolute value. We express this behavior by writing

$\lim_{x\to\infty} f(x) = 1$ "$f(x)$ approaches 1 as x approaches infinity."

$\lim_{x\to-\infty} f(x) = -1$ "$f(x)$ approaches -1 as x approaches negative infinity."

Table 1

x	$f(x) = x/\sqrt{x^2+1}$
−1,000	−0.9999995
−100	−0.9999500
−10	−0.9950372
−1	−0.7071068
0	0.0000000
1	0.7071068
10	0.9950372
100	0.9999500
1,000	0.9999995

The graph of f conveys this limiting behavior by approaching the horizontal lines $y = 1$ as x moves far to the right and $y = -1$ as x moves far to the left. These lines are called **horizontal asymptotes** of the graph. In general, if a curve approaches a straight line as it recedes very far away from the origin, that line is called an **asymptote** of the curve.

Definition 3 Limits at infinity and negative infinity (informal definition)

If the function f is defined on an interval (a, ∞) and if we can ensure that $f(x)$ is as close as we want to the number L by taking x large enough, then we say that $f(x)$ **approaches the limit L as x approaches infinity**, and we write

$$\lim_{x\to\infty} f(x) = L.$$

If f is defined on an interval $(-\infty, b)$ and if we can insure that $f(x)$ is as close as we want to the number M by taking x negative and large enough in absolute value, then we say that $f(x)$ **approaches the limit M as x approaches negative infinity**, and we write

$$\lim_{x\to-\infty} f(x) = M.$$

Recall that the symbol ∞, called **infinity**, does *not* represent a real number. We cannot use ∞ in arithmetic in the usual way, but we can use the phrase "approaches ∞" to mean "becomes arbitrarily large positive" and the phrase "approaches $-\infty$" to mean "becomes arbitrarily large negative."

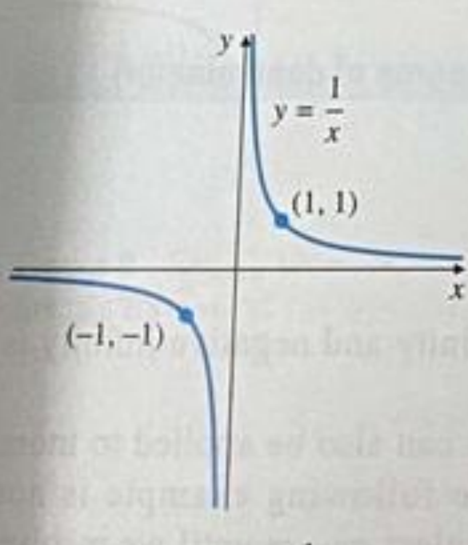


Figure 8 $\lim_{x\to\pm\infty} \frac{1}{x} = 0$

EXAMPLE 1

In **Figure 8**, we can see that $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$. The x-axis is a horizontal asymptote of the graph $y = 1/x$.

The theorems of Section 7.1 have suitable counterparts for limits at infinity or negative infinity. In particular, it follows from the example above and from the

Product Rule for limits that $\lim_{x\to\pm\infty} 1/x^n = 0$ for any positive integer n. We will use this fact in the following examples. Example 2 shows how to obtain the limits at $\pm\infty$ for the function $x/\sqrt{x^2+1}$ by algebraic means, without resorting to making a table of values or drawing a graph, as we did above.

EXAMPLE 2

Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

SOLUTION Rewrite the expression for $f(x)$ as follows:

$$f(x) = \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} = \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}$$

Remember $\sqrt{x^2} = |x|$.

$$= \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}$$

$$= \frac{\operatorname{sgn} x}{\sqrt{1+\frac{1}{x^2}}},$$

where $\operatorname{sgn} x = \dfrac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$

The factor $\sqrt{1+(1/x^2)}$ approaches 1 as x approaches ∞ or $-\infty$, so $f(x)$ must have the same limits as $x \to \pm\infty$ as does sgn (x). Therefore (see **Figure 7**),

$$\lim_{x\to\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -1.$$

Limits at Infinity for Rational Functions The only polynomials that have limits at $\pm\infty$ are constant ones, $P(x) = c$. The situation is more interesting for rational functions. Recall that a rational function is a quotient of two polynomials. The following examples show how to render such a function in a form where its limits at infinity and negative infinity (if they exist) are apparent. The way to do this is to *divide the numerator and denominator by the highest power of x appearing in the denominator*. The limits of a rational function at infinity and negative infinity either both fail to exist or both exist and are equal.

HOMEWORK 1 **(Numerator and denominator of the same degree)**

Evaluate $\lim_{x\to\pm\infty} \dfrac{2x^2-x+3}{3x^2+5}$.

HOMEWORK 2 **(Degree of numerator less than degree of denominator)**

Evaluate $\lim_{x\to\pm\infty} \dfrac{5x+2}{2x^3-1}$.

SUMMARY OF LIMITS AT $\pm\infty$ FOR RATIONAL FUNCTIONS

Let $P_m(x) = a_m x^m + \cdots + a_0$ and $Q_n(x) = b_n x^n + \cdots + b_0$ be polynomials of degree m and n, respectively, so that $a_m \neq 0$ and $b_n \neq 0$. Then

$$\lim_{x\to\pm\infty} \frac{P_m(x)}{Q_n(x)}$$

(a) equals zero if $m < n$,
(b) equals $\dfrac{a_m}{b_n}$ if $m = n$,
(c) does not exist if $m > n$.

The limiting behavior of rational functions at infinity and negative infinity is summarized at the left.

The technique used in the previous examples can also be applied to more general kinds of functions. The function in the following example is not rational, and the limit seems to produce a meaningless $\infty - \infty$ until we resolve matters by rationalizing the numerator.

EXAMPLE 3

Find $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$.

SOLUTION We are trying to find the limit of the difference of two functions, each of which becomes arbitrarily large as x increases to infinity. We rationalize the expression by multiplying the numerator and the denominator (which is 1) by the conjugate expression $\sqrt{x^2+x}+x$:

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x}\\ &= \lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x}\\ &= \lim_{x\to\infty}\frac{x}{x\sqrt{1+\frac{1}{x}}+x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}.\end{aligned}$$

(Here, $\sqrt{x^2}=x$ because $x>0$ as $x\to\infty$.)

Remark The limit $\lim_{x\to-\infty}(\sqrt{x^2+x}-x)$ is not nearly so subtle. Since $-x>0$ as $x\to-\infty$, we have $\sqrt{x^2+x}-x>\sqrt{x^2+x}$, which grows arbitrarily large as $x\to-\infty$. The limit does not exist.

Infinite Limits A function whose values grow arbitrarily large can sometimes be said to have an infinite limit. Since infinity is not a number, infinite limits are not really limits at all, but they provide a way of describing the behavior of functions that grow arbitrarily large positive or negative. A few examples will make the terminology clear.

HOMEWORK 3 **(A two-sided infinite limit)**

Describe the behavior of the function $f(x)=1/x^2$ near $x=0$.

EXAMPLE 4 **(One-sided infinite limits)**

Describe the behavior of the function $f(x)=1/x$ near $x=0$. (See **Figure 9**.)

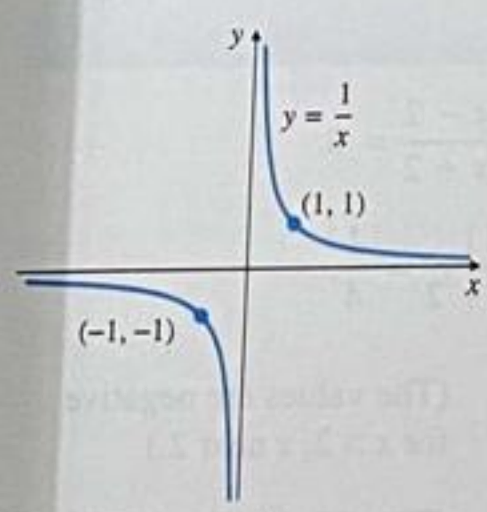


Figure 9 $\lim_{x\to0-}1/x=-\infty$, $\lim_{x\to0+}1/x=\infty$

SOLUTION As x approaches 0 from the right, the values of $f(x)$ become larger and larger positive numbers, and we say that f has right-hand limit infinity at $x=0$:

$$\lim_{x\to0+}f(x)=\infty.$$

Similarly, the values of $f(x)$ become larger and larger negative numbers as x approaches 0 from the left, so f has left-hand limit $-\infty$ at $x=0$:

$$\lim_{x\to0-}f(x)=-\infty.$$

These statements do not say that the one-sided limits *exist*; they do not exist because ∞ and $-\infty$ are not numbers. Since the one-sided limits are not equal even as infinite symbols, all we can say about the two-sided $\lim_{x\to0}f(x)$ is that it does not exist.

HOMEWORK 4 **(Polynomial behavior at infinity)**

(a) $\lim_{x\to\infty}(3x^3 - x^2 + 2) = \infty$

(b) $\lim_{x\to-\infty}(3x^3 - x^2 + 2) = -\infty$

(c) $\lim_{x\to\infty}(x^4 - 5x^3 - x) = \infty$

(d) $\lim_{x\to-\infty}(x^4 - 5x^3 - x) = \infty$

The highest-degree term of a polynomial dominates the other terms as $|x|$ grows large, so the limits of this term at ∞ and $-\infty$ determine the limits of the whole polynomial. For the polynomial in parts (a) and (b) we have

$$3x^3 - x^2 + 2 = 3x^3\left(1 - \frac{1}{3x} + \frac{2}{3x^3}\right).$$

The factor in the large parentheses approaches 1 as x approaches $\pm\infty$, so the behavior of the polynomial is just that of its highest-degree term $3x^3$.

We can now say a bit more about the limits at infinity and negative infinity of a rational function whose numerator has higher degree than the denominator. Earlier in this section we said that such a limit *does not exist*. This is true, but we can assign ∞ or $-\infty$ to such limits, as the following example shows.

EXAMPLE 5 **(Rational functions with numerator of higher degree)**

Evaluate $\lim_{x\to\infty} \dfrac{x^3 + 1}{x^2 + 1}$.

$= -\infty$

SOLUTION Divide the numerator and the denominator by x^2, the largest power of x in the denominator:

$$\lim_{x\to\infty} \frac{x^3 + 1}{x^2 + 1} = \lim_{x\to\infty} \frac{x + \dfrac{1}{x^2}}{1 + \dfrac{1}{x^2}} = \frac{\lim_{x\to\infty}\left(x + \dfrac{1}{x^2}\right)}{1} = \infty.$$

A polynomial $Q(x)$ of degree $n > 0$ can have at most n *zeros*; that is, there are at most n different real numbers r for which $Q(r) = 0$. If $Q(x)$ is the denominator of a rational function $R(x) = P(x)/Q(x)$, that function will be defined for all x except those finitely many zeros of Q. At each of those zeros, $R(x)$ may have limits, infinite limits, or one-sided infinite limits. Here are some examples.

HOMEWORK 5

(a) $\lim_{x\to 2} \dfrac{(x-2)^2}{x^2 - 4} = \lim_{x\to 2} \dfrac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \dfrac{x-2}{x+2} = 0.$

(b) $\lim_{x\to 2} \dfrac{x-2}{x^2 - 4} = \lim_{x\to 2} \dfrac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \dfrac{1}{x+2} = \dfrac{1}{4}.$

(c) $\lim_{x\to 2+} \dfrac{x-3}{x^2 - 4} = \lim_{x\to 2+} \dfrac{x-3}{(x-2)(x+2)} = -\infty.$ (The values are negative for $x > 2$, x near 2.)

(d) $\lim_{x\to 2-} \dfrac{x-3}{x^2 - 4} = \lim_{x\to 2-} \dfrac{x-3}{(x-2)(x+2)} = \infty.$ (The values are positive for $x < 2$, x near 2.)

(e) $\lim_{x\to 2} \dfrac{x-3}{x^2 - 4} = \lim_{x\to 2} \dfrac{x-3}{(x-2)(x+2)}$ does not exist.

(f) $\lim_{x\to 2} \dfrac{2-x}{(x-2)^3} = \lim \dfrac{-(x-2)}{(x-2)^3} = \lim \dfrac{-1}{\ldots} = -\infty$

7.3 Continuity

الاتصال

- Continuity at a Point
- Continuity on an Interval
- There Are Lots of Continuous Functions
- Continuous Extensions and Removable Discontinuities
- Continuous Functions on Closed, Finite Intervals
- Finding Maxima and Minima Graphically
- Finding Roots of Equations

When a car is driven along a highway, its distance from its starting point depends on time in a *continuous* way, changing by small amounts over short intervals of time. But not all quantities change in this way. When the car is parked in a parking lot where the rate is quoted as "\$2.00 per hour or portion," the parking charges remain at \$2.00 for the first hour and then suddenly jump to \$4.00 as soon as the first hour has passed. The function relating parking charges to parking time will be called *discontinuous* at each hour. In this section we will define continuity and show how to tell whether a function is continuous. We will also examine some important properties possessed by continuous functions.

Continuity at a Point Most functions that we encounter have domains that are intervals, or unions of separate intervals. A point P in the domain of such a function is called an **interior point** of the domain if it belongs to some *open* interval contained in the domain. If it is not an interior point, then P is called an **endpoint** of the domain. For example, the domain of the function $f(x) = \sqrt{4 - x^2}$ is the closed interval $[-2, 2]$, which consists of interior points in the interval $(-2, 2)$, a left endpoint -2, and a right endpoint 2. The domain of the function $g(x) = 1/x$ is the union of open intervals $(-\infty, 0) \cup (0, \infty)$ and consists entirely of interior points. Note that although 0 is an endpoint of each of those intervals, it does not belong to the domain of g and so is not an endpoint of that domain.

int {2, -2}
end -2, 2
نقاط داخلية
end 0 لا تعتبر الـ

الاتصال عند نقطة

Definition 4 Continuity at an interior point

We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

If either $\lim_{x\to c} f(x)$ fails to exist or it exists but is not equal to $f(c)$, then we will say that f is **discontinuous** at c.

In graphical terms, f is continuous at an interior point c of its domain if its graph has no break in it at the point $(c, f(c))$; in other words, if you can draw the graph through that point without lifting your pen from the paper. Consider **Figure 10**. In (a), f is continuous at c. In (b), f is discontinuous at c because $\lim_{x\to c} f(x) \neq f(c)$. In (c), f is discontinuous at c because $\lim_{x\to c} f(x)$ does not exist. In both (b) and (c) the graph of f has a break at $x = c$.

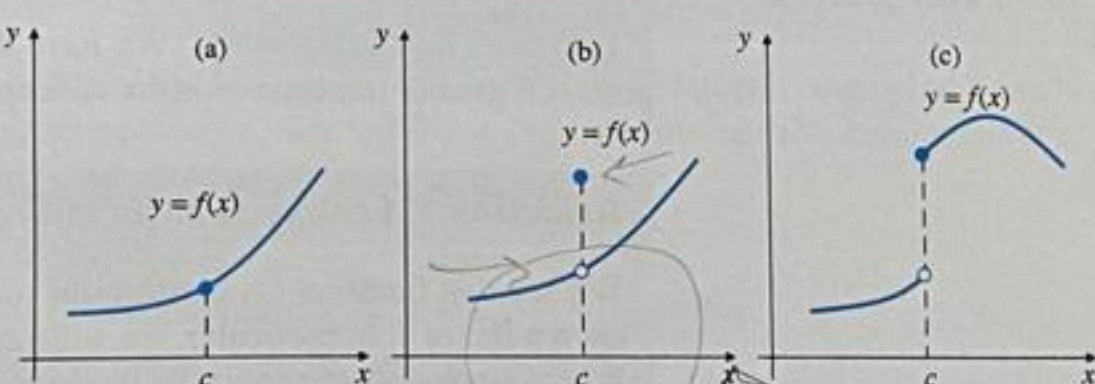


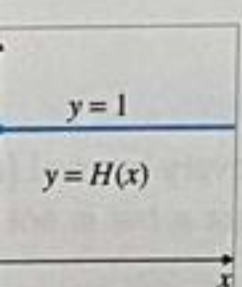
Figure 10
(a) f is continuous at c
(b) $\lim_{x\to c} f(x) \neq f(c)$
(c) $\lim_{x\to c} f(x)$ does not exist

Although a function cannot have a limit at an endpoint of its domain, it can still have a one-sided limit there. We extend the definition of continuity to provide for such situations.

Definition 5 Right and left continuity

We say that f is **right continuous** at c if $\lim_{x\to c+} f(x) = f(c)$.

We say that f is **left continuous** at c if $\lim_{x\to c-} f(x) = f(c)$.



Heaviside

EXAMPLE 1

The Heaviside function $H(x)$, whose graph is shown in **Figure 11**, is continuous at every number x except 0. It is right continuous at 0 but is not left continuous or continuous there.

The relationship between continuity and one-sided continuity is summarized in the following theorem.

Theorem 5

Function f is continuous at c if and only if it is both right continuous and left continuous at c.

Definition 6 Continuity at an endpoint

We say that f is continuous at a left endpoint c of its domain if it is right continuous there.
We say that f is continuous at a right endpoint c of its domain if it is left continuous there.

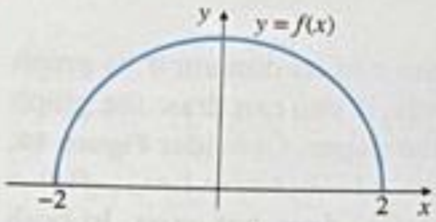


Figure 12 $f(x)=\sqrt{4-x^2}$ is continuous at every point of its domain

HOMEWORK 1

The function $f(x)=\sqrt{4-x^2}$ has domain $[-2, 2]$. It is continuous at the right endpoint 2 because it is left continuous there, that is, because $\lim_{x\to 2-} f(x)=0=f(2)$. It is continuous at the left endpoint -2 because it is right continuous there: $\lim_{x\to -2+} f(x)=0=f(-2)$. Of course, f is also continuous at every interior point of its domain. If $-2<c<2$, then $\lim_{x\to c} f(x)=\sqrt{4-c^2}=f(c)$. (See **Figure 12**.)

Continuity on an Interval We have defined the concept of continuity at a point. Of greater importance is the concept of continuity on an interval.

Definition 7 Continuity on an interval

We say that function f is **continuous on the interval** I if it is continuous at each point of I. In particular, we will say that f is a **continuous function** if f is continuous at every point of its domain.

EXAMPLE 2

The function $f(x)=\sqrt{x}$ is a continuous function. Its domain is $[0, \infty)$. It is continuous at the left endpoint 0 because it is right continuous there. Also, f is continuous at every number $c>0$ since $\lim_{x\to c}\sqrt{x}=\sqrt{c}$.

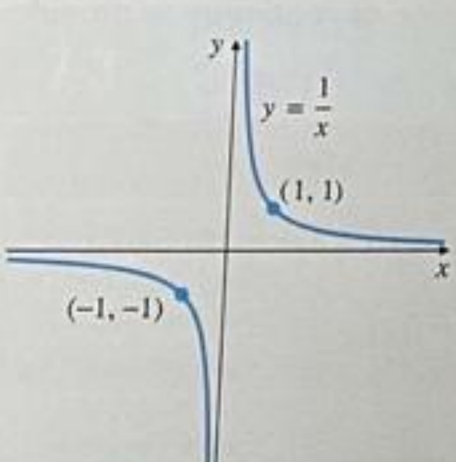


Figure 13 $1/x$ is continuous on its domain

HOMEWORK 2

The function $g(x)=1/x$ is also a continuous function. This may seem wrong to you at first glance because its graph is broken at $x=0$. (See **Figure 13**.) However, the number 0 is not in the domain of g, so we will prefer to say that g is undefined rather than discontinuous there. (Some authors would say that g is discontinuous at $x=0$.) If we were to define $g(0)$ to be some number, say 0, then we would say that $g(x)$ is discontinuous at 0. There is no way of defining $g(0)$ so that g becomes continuous at 0.

EXAMPLE 3

The greatest integer function $[x]$ is continuous on every interval $[n, n+1)$, where n is an integer. It is right continuous at each integer n but is not left continuous there, so it is discontinuous at the integers.

$$\lim_{x\to n+}[x]=n=[n], \qquad \lim_{x\to n-}[x]=n-1\neq n=[n].$$

There Are Lots of Continuous Functions The following functions are continuous wherever they are defined:

(a) all polynomials;

(b) all rational functions;

(c) all rational powers $x^{m/n} = \sqrt[n]{x^m}$;

(d) the sine, cosine, tangent, secant, cosecant, and cotangent functions; and

(e) the absolute value function $|x|$.

Theorem 3 of Section 7.1 assures us that every polynomial is continuous everywhere on the real line, and every rational function is continuous everywhere on its domain (which consists of all real numbers except the finitely many where its denominator is zero). If m and n are integers and $n \neq 0$, the rational power function $x^{m/n}$ is defined for all positive numbers x, and also for all negative numbers x if n is odd. The domain includes 0 if and only if $m/n \geq 0$.

The following theorems show that if we combine continuous functions in various ways, the results will be continuous.

Theorem 6: Combining continuous functions

If the functions f and g are both defined on an interval containing c and both are continuous at c, then the following functions are also continuous at c:

1. the sum $f + g$ and the difference $f - g$;
2. the product fg;
3. the constant multiple kf, where k is any number;
4. the quotient f/g (provided $g(c) \neq 0$); and
5. the nth root $(f(x))^{1/n}$, provided $f(c) > 0$ if n is even.

The proof involves using the various limit rules in Theorem 2 of Section 7.1. For example,

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c),$$

so $f + g$ is continuous.

Theorem 7: Composites of continuous functions are continuous

If $f(g(x))$ is defined on an interval containing c, and if f is continuous at L and $\lim_{x\to c} g(x) = L$, then

$$\lim_{x\to c} f(g(x)) = f(L) = f(\lim_{x\to c} g(x)).$$

In particular, if g is continuous at c (so $L = g(c)$), then the composition $f \circ g$ is continuous at c:

$$\lim_{x\to c} f(g(x)) = f(g(c)).$$

HOMEWORK 3

The following functions are continuous everywhere on their respective domains:

(a) $3x^2 - 2x$ (b) $\dfrac{x-2}{x^2-4}$ (c) $|x^2 - 1|$

(d) $\sqrt{x}$ (e) $\sqrt{x^2 - 2x - 5}$ (f) $\dfrac{|x|}{\sqrt{|x+2|}}$.

Continuous Extensions and Removable Discontinuities As we have seen in Section 7.1, a rational function may have a limit even at a point where its denominator is zero. If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, we can define a new function $F(x)$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = c. \end{cases}$$

$F(x)$ is continuous at $x = c$. It is called the **continuous extension** of $f(x)$ to $x = c$. For rational functions f, continuous extensions are usually found by canceling common factors.

EXAMPLE 4

Show that $f(x) = \dfrac{x^2 - x}{x^2 - 1}$ has a continuous extension to $x = 1$, and find that extension.

SOLUTION Although $f(1)$ is not defined, if $x \neq 1$ we have

$$f(x) = \frac{x^2 - x}{x^2 - 1} = \frac{x(x-1)}{(x+1)(x-1)} = \frac{x}{x+1}.$$

The function

$$F(x) = \frac{x}{x+1}$$

is equal to $f(x)$ for $x \neq 1$ but is also continuous at $x = 1$, having there the value $1/2$. The graph of f is shown in **Figure 14**. The continuous extension of $f(x)$ to $x = 1$ is $F(x)$. It has the same graph as $f(x)$ except with no hole at $(1, 1/2)$.

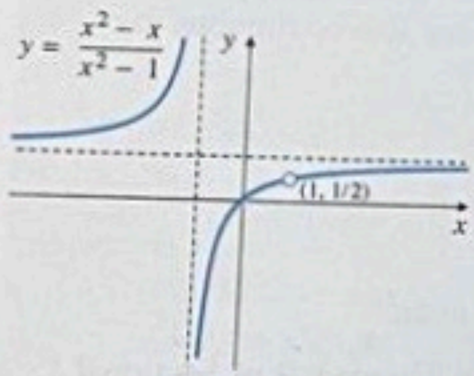


Figure 14 This function has a continuous extension to $x = 1$

If a function f is undefined or discontinuous at a point a but can be (re)defined at that *single point* so that it becomes continuous there, then we say that f has a **removable discontinuity** at a. The function f in the above example has a removable discontinuity at $x = 1$. To remove it, define $f(1) = 1/2$.

HOMEWORK 4

The function $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ has a removable discontinuity at $x = 2$. To remove it, redefine $g(2) = 2$. (See **Figure 15**.)

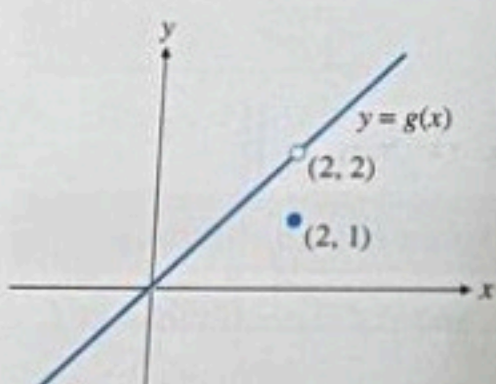


Figure 15 g has a removable discontinuity at 2

Continuous Functions on Closed, Finite Intervals Continuous functions that are defined on *closed, finite intervals* have special properties that make them particularly useful in mathematics and its applications. We will discuss two of these properties here. Although they may appear obvious, these properties are much more subtle than the results about limits stated earlier in this chapter; their proofs require a careful study of the implications of the completeness property of the real numbers.

The first of the properties states that a function $f(x)$ that is continuous on a closed, finite interval $[a, b]$ must have an **absolute maximum value** and an **absolute minimum value**. This means that the values of $f(x)$ at all points of the interval lie between the values of $f(x)$ at two particular points in the interval; the graph of f has a highest point and a lowest point.

Theorem 8: The Max-Min Theorem

If $f(x)$ is continuous on the closed, finite interval $[a, b]$, then there exist numbers p and q in $[a, b]$ such that for all x in $[a, b]$,

$$f(p) \le f(x) \le f(q).$$

Thus f has the absolute minimum value $m = f(p)$, taken on at the point p, and the absolute maximum value $M = f(q)$, taken on at the point q.

Many important problems in mathematics and its applications come down to having to find maximum and minimum values of functions. Calculus provides some very useful tools for solving such problems. Observe, however, that the theorem above merely asserts that minimum and maximum values *exist*; it doesn't tell us how to find them. For now, we can solve some simple maximum and minimum value problems involving quadratic functions by completing the square without using any calculus.

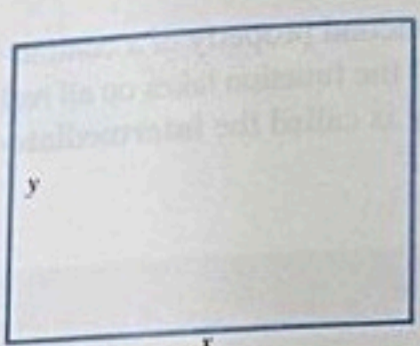


Figure 16 Rectangular field: perimeter $= 2x + 2y$, area $= xy$

EXAMPLE 5

What is the largest possible area of a rectangular field that can be enclosed by 200 m of fencing?

SOLUTION If the sides of the field are x m and y m (**Figure 16**), then its perimeter is $P = 2x + 2y$ m, and its area is $A = xy$ m^2. We are given that $P = 200$, so $x + y = 100$, and $y = 100 - x$. Neither side can be negative, so x must belong to the closed interval $[0, 100]$. The area of the field can be expressed as a function of x by substituting $100 - x$ for y:

$$A = x(100 - x) = 100x - x^2.$$

We want to find the maximum value of the quadratic function $A(x) = 100x - x^2$ on the interval $[0, 100]$. Theorem 8 assures us that such a maximum exists.

To find the maximum, we complete the square of the function $A(x)$. Note that $x^2 - 100x$ are the first two terms of the square $(x - 50)^2 = x^2 - 100x + 2{,}500$. Thus,

$$A(x) = 2{,}500 - (x - 50)^2.$$

Observe that $A(50) = 2{,}500$ and $A(x) < 2{,}500$ if $x \ne 50$, because we are subtracting a positive number $(x - 50)^2$ from 2,500 in this case. Therefore, the maximum value of $A(x)$ is 2,500. The largest field has area 2,500 m^2 and is actually a square with dimensions $x = y = 50$ m.

Theorem 8 implies that a function that is continuous on a closed, finite interval is **bounded**. This means that it cannot take on arbitrarily large positive or negative values; there must exist a number K such that

$$|f(x)| \le K; \qquad \text{that is,} \qquad -K \le f(x) \le K.$$

In fact, for K we can use the larger of the numbers $|f(p)|$ and $|f(q)|$ in the theorem.

The conclusions of Theorem 8 may fail if the function f is not continuous or if the interval is not closed. See **Figures 17–20** for examples of how such failure can occur.

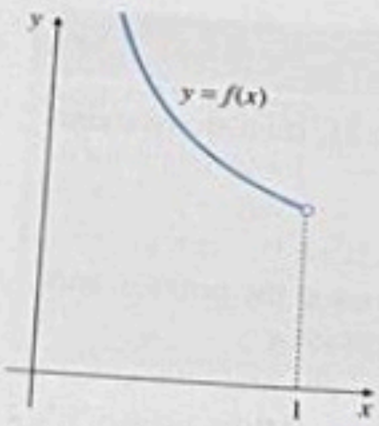


Figure 17 $f(x) = 1/x$ is continuous on the open interval $(0, 1)$. It is not bounded and has neither a maximum nor a minimum value

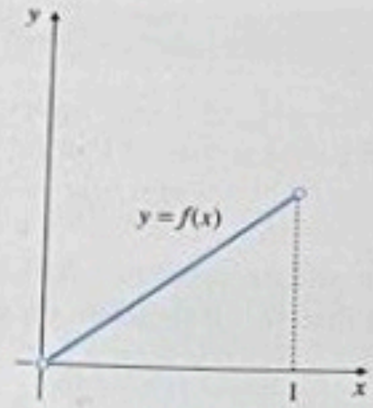


Figure 18 $f(x) = x$ is continuous on the open interval $(0, 1)$. It is bounded but has neither a maximum nor a minimum value

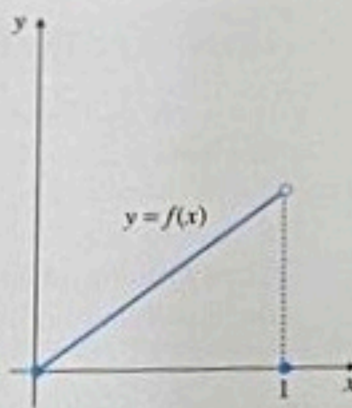


Figure 19 This function is defined on the closed interval $[0, 1]$ but is discontinuous at the endpoint $x = 1$. It has a minimum value but no maximum value

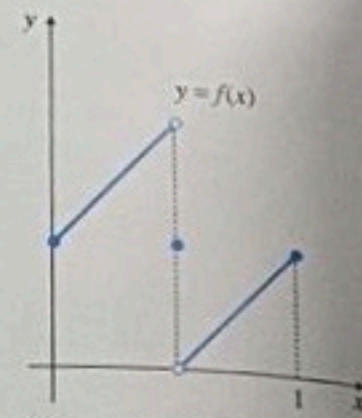


Figure 20 This function is discontinuous at an interior point of its domain, the closed interval $[0, 1]$. It is bounded but has neither maximum nor minimum values

Finding Maxima and Minima Graphically The second property of a continuous function defined on a closed, finite interval is that the function takes on all real values between any two of its values. This property is called the **intermediate-value property**.

Theorem 9: The Intermediate-Value Theorem

If $f(x)$ is continuous on the interval $[a, b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = s$.

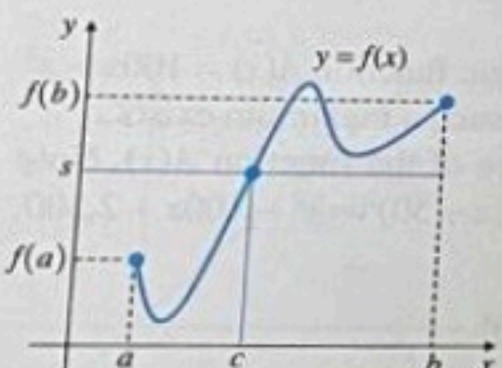


Figure 21 The continuous function f takes on the value s at some point c between a and b

In particular, a continuous function defined on a closed interval takes on all values between its minimum value m and its maximum value M, so its range is also a closed interval, $[m, M]$.

Figure 21 shows a typical situation. The points $(a, f(a))$ and $(b, f(b))$ are on opposite sides of the horizontal line $y = s$. Being unbroken, the graph $y = f(x)$ must cross this line in order to go from one point to the other. In the figure, it crosses the line only once, at $x = c$. If the line $y = s$ were somewhat higher, there might have been three crossings and three possible values for c.

Theorem 9 is the reason why the graph of a function that is continuous on an interval I cannot have any breaks. It must be **connected**, a single, unbroken curve with no jumps.

HOMEWORK 5

Determine the intervals on which $f(x) = x^3 - 4x$ is positive and negative.

Finding Roots of Equations Among the many useful tools that calculus will provide are ones that enable us to calculate solutions to equations of the form $f(x) = 0$ to any desired degree of accuracy. Such a solution is called a **root** of the equation, or a **zero** of the function f. Using these tools usually requires previous knowledge that the equation has a solution in some interval. The Intermediate-Value Theorem can provide this information.

EXAMPLE 6

Show that the equation $x^3 - x - 1 = 0$ has a solution in the interval $[1, 2]$.

SOLUTION The function $f(x) = x^3 - x - 1$ is a polynomial and is therefore continuous everywhere. Now $f(1) = -1$ and $f(2) = 5$. Since 0 lies between -1 and 5, the Intermediate-Value Theorem assures us that there must be a number c in $[1, 2]$ such that $f(c) = 0$.

One method for finding a zero of a function that is continuous and changes sign on an interval involves bisecting the interval many times, each time determining which half of the previous interval must contain the root, because the function has opposite signs at the two ends of that half. This method is slow. For example, if the original interval has length 1, it will take 11 bisections to cut down to an interval of length less than 0.0005 (because $2^{11} > 2{,}000 = 1/(0.0005)$), and thus to ensure that we have found the root correct to 3 decimal places. But this method requires no graphics hardware and is easily implemented with a calculator, preferably one into which the formula for the function can be programmed.

HOMEWORK 6 **(The Bisection Method)**

Solve the equation $x^3 - x - 1 = 0$ of Example 6 correct to 3 decimal places by successive bisections.

Remark The Max-Min Theorem and the Intermediate-Value Theorem are examples of what mathematicians call **existence theorems**. Such theorems assert that something exists without telling you how to find it. Students sometimes complain that mathematicians worry too much about proving that a problem has a solution and not enough about how to find that solution. They argue: "If I can calculate a solution to a problem, then surely I do not need to worry about whether a solution exists." This is, however, false logic. Suppose we pose the problem: "Find the largest positive integer." Of course this problem has no solution; there is no largest positive integer because we can add 1 to any integer and get a larger integer. Suppose, however, that we forget this and try to calculate a solution. We could proceed as follows:

Let N be the largest positive integer.
Since 1 is a positive integer, we must have $N \geq 1$.
Since N^2 is a positive integer, it cannot exceed the largest positive integer.
Therefore, $N^2 \leq N$ and so $N^2 - N \leq 0$.
Thus, $N(N-1) \leq 0$ and we must have $N - 1 \leq 0$.
Therefore, $N \leq 1$. Since also $N \geq 1$, we have $N = 1$.
Therefore, 1 is the largest positive integer.

The only error we have made here is in the assumption (in the first line) that the problem has a solution. It is partly to avoid logical pitfalls like this that mathematicians prove existence theorems.