Complex numbers

A complex number z can be represented as a sum of real and imaginary part $z = x + yi$, where x and y are real numbers and $i = \sqrt{-1}\left(i^2 = -1\right)$.

- The complex number $x + yi$ can be represented by the order pair (x, y), and plotted in a plane (called the Argand plane) as shown in figure 1. In the Argand plane the horizontal axis is called the real axis and the vertical axis called the imaginary axis.

The **real part** of the complex number $x + yi$ is the real number x and the **imaginary part** is the real number y. Thus, the real part of $5 - 7i$ is 5 and the imaginary part is -7.

- **$z = x + yi$ is called a Cartesian complex number**
- Two complex numbers $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ are **equal** if $x_1 = x_2$ and $y_1 = y_2$.
- Complex numbers are **used** to solve polynomials (for example $x^2 = -1$ is a polynomial of 2 degree so it should have 2 roots, $x = \pm i$), used to solve differential equations (ODE and PDE). Also used for analyzing oscillation and waves (phase component).

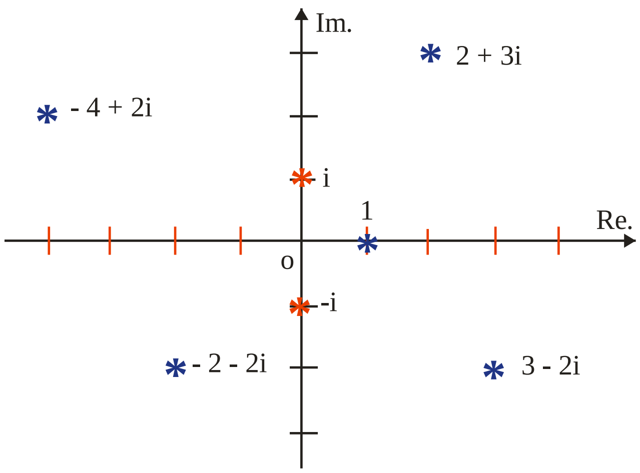


Figure 1. Complex numbers as points in the Argand plane.

- The **sum and difference** of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts. (It is same as we do with the real numbers)

If $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$, then $z_1 \pm z_2 = \left(x_1 \pm x_2\right) + i\left(y_1 \pm y_2\right)$

Example **If** $z_1 = 1 - i$ **and** $z_2 = 4 + 7i$**,** $z_1 + z_2 = (1 - i) + (4 + 7i) = 5 + 6i$

Multiplication of complex numbers: Multiplication of complex numbers is achieved by assuming all quantities involved are real and then using $i^2 = -1$ simplify by separating real and imaginary parts. Multiplication is the most interesting operation in complex numbers. For $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$

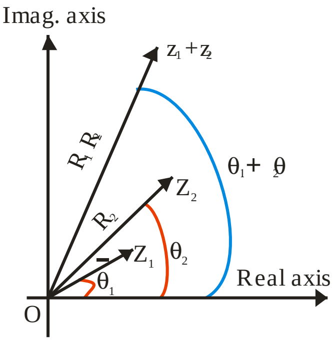
$$z_1z_2 = \left(x_1 + i\,y_1\right)\left(x_2 + i\,y_2\right) = x_1x_2 + x_1\left(i\,y_2\right) + \left(i\,y_1\right)x_2 + i^2y_1y_2 = x_1x_2 + i\left(x_1y_2 + y_1x_2\right) - y_1y_2$$

$$\Rightarrow z_1z_2 = \left(x_1x_2 - y_1y_2\right) + i\left(x_1y_2 + x_2y_1\right)$$

- It is much simpler and easier to multiply and divide the complex numbers in polar coordinates system.

- $z_1z_2 = R_1R_2e^{i\theta_1}e^{i\theta_2} = R_1R_2e^{i(\theta_1+\theta_2)}$ $\qquad \dfrac{z_1}{z_2} = \dfrac{R_1e^{i\theta_1}}{R_2e^{i\theta_2}} = \dfrac{R_1}{R_2}e^{i(\theta_1-\theta_2)}$

- In complex plane: A nice geometrical interpretation of complex number multiplication is shown in the following figure. Simple multiply the magnitudes R_1R_2 and add the angles $\theta_1+\theta_2$.

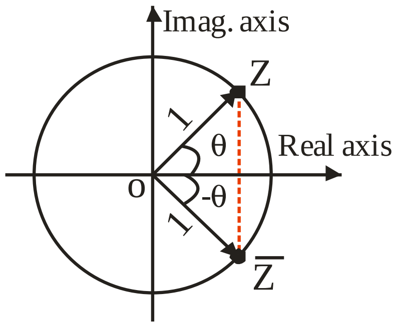


Example If $z_1 = 3+2i$ and $z_2 = 4-5i$ **,**

$z_1z_2 = (3+2i)(4-5i) = 12-15i+8i-10i^2 = 12-7i+10 = 22-7i$

Complex conjugate: The **complex conjugate** of $z = x+yi$ is $\bar{z} = x-yi$. It is very important and is the mirror image of the number in the real axis.

- $z+\bar{z} = x+yi+x-yi = 2x$, is a real number and
- $z\bar{z} = (x+iy)(x-iy) = x^2 - ixy + ixy + y^2 \Rightarrow z\bar{z} = x^2+y^2$ is a real number and is equal to the length of z.



Division of complex numbers: For the quotient (division) of two complex numbers, to get rid of complex term from the denominator we multiply the numerator and denominator by the **complex conjugate** of the denominator.

For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $\dfrac{z_1}{z_2} = \dfrac{x_1+iy_1}{x_2+iy_2}$, Multiplying and dividing by the complex conjugate of z_2, $\dfrac{z_1}{z_2} = \dfrac{x_1+iy_1}{x_2+iy_2}\times\dfrac{x_2-iy_2}{x_2-iy_2} = \dfrac{z_1\bar{z}_2}{|z_2|^2}$

Example Express the number $\dfrac{-1+3i}{2+5i}$ in the form $x+yi$.

Solution. We multiply numerator and denominator by the complex conjugate of the denominator that is, $2-5i$.

$$\frac{-1+3i}{2+5i} = \frac{-1+3i}{2+5i}\times\frac{2-5i}{2-5i} = \frac{-1\times(2-5i)+3i(2-5i)}{2\times(2-5i)+5i(2-5i)} = \frac{-2+5i+6i-15i^2}{4-10i+10i-25i^2} = \frac{-2+11i-15(-1)}{4-25(-1)} = \frac{13+11i}{29} = \frac{13}{29}+\frac{11i}{29}$$

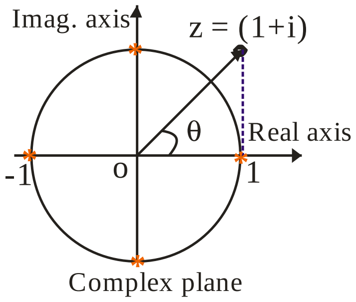
Example. Find the roots of the equation: $x^2+1=0$

$x^2=-1$, No real solution. Invent $i=\sqrt{-1}\left(i^2=-1\right)$.

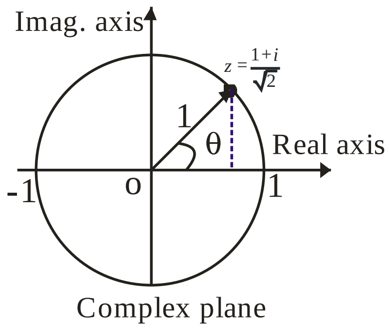
$x^2 = i^2 \Rightarrow x^2 - i^2 = 0 \Rightarrow (x+i)(x-i) = 0 \Rightarrow x+i=0 \ and \ x-i=0 \Rightarrow x=-i \ and \ x=i$

Example. Find the roots of the equation: $x^4 = 1$, four degree equation have four solutions

$x^4 - 1 = 0 \Rightarrow (x^2 - 1)(x^2 + 1) = 0 \Rightarrow (x+1)(x-1)(x+i)(x-i) = 0 \Rightarrow x = -1,\ x = +1\ \ x = -i\ \ x = i$.



- The complex number $z = \dfrac{1+i}{\sqrt{2}}$ is shown in the figure find



(i) $\bar{z} = ?\ \ z = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}} \Rightarrow \bar{z} = \dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$

(ii) $z^2 = ?\ \ z^2 = \left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right) \Rightarrow z^2 = \dfrac{1}{2} + \dfrac{i}{2} + \dfrac{i}{2} - \dfrac{1}{2} \Rightarrow z^2 = i$

(iii) $z + \bar{z} = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} \Rightarrow z + \bar{z} = \sqrt{2}\ \ (a\ real\ number)$

(iv) $z\bar{z} = \left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}\right) = \dfrac{1}{2} - \dfrac{i}{2} + \dfrac{i}{2} + \dfrac{1}{2} \Rightarrow z\bar{z} = 1$

(v) Find 8th solution of the equation $z^8 = 1$, plot these solutions on the complex plane.

Problem If $z = -1 + i2$

Find (i) $\bar{z}$ (ii) $z\bar{z}$ (iii) $z + \bar{z}$ (iv) Plot each result in the complex plane

Example. Find the roots of the equation: $x^2 + x + 1 = 0$

Using quadratic formula, $x = \dfrac{-1 \pm \sqrt{1^2 - 4}}{2} \Rightarrow x = \dfrac{-1 \pm \sqrt{-3}}{2} \Rightarrow x = \dfrac{-1 \pm i\sqrt{3}}{2}$

The roots are: $\left(\dfrac{-1 + i\sqrt{3}}{2}, \dfrac{-1 - i\sqrt{3}}{2}\right)$**.** The solution of the example are complex conjugate of each other.

Example. Evaluate $(a)\, i^3,\ (b)\, i^4,\ (c)\, i^{23},\ (d)\, \dfrac{-4}{i^9}$

$$i^3 = i^2 \times i = (-1) \times i = -i, \text{ since } i^2 = -1$$

$$i^4 = i^2 \times i^2 = (-1)\times(-1) = 1$$

$$i^{23} = i \times i^{22} = i \times (i^2)^{11} = i \times (-1)^{11} = i \times (-1) = -i$$

$$i^9 = i \times i^8 = i \times (i^2)^4 = i \times (-1)^4 = i \times 1 = i$$

Hence, $\dfrac{-4}{i^9} = \dfrac{-4}{i} = \dfrac{-4}{i} \times \dfrac{i}{i} = \dfrac{-4i}{i^2} = \dfrac{-4i}{-1} = 4i$

Problem. Evaluate $(a) i^8, \ (b) -\frac{1}{i^7}, \ (c) \frac{4}{2i^{13}}$

Answer: $(a) 1, \ (b) -i, \ (c) -2i$

Problem. Evaluate in $a+ib$ form, given: $z_1 = 1+2i, \ z_2 = 4-3i, \ z_3 = -2+3i \ \ and \ \ z_4 = -5-i.$

$(1) \frac{z_1 z_3}{z_1 + z_3} \quad (2) z_2 + \frac{z_1}{z_4} + z_3 \qquad answer\left((1) \frac{3}{26} + \frac{41}{26} i \quad (2) \frac{45}{26} - \frac{9}{26} i \right)$

Problem. Show that: $\frac{-25}{2}\left(\frac{1+2i}{3+4i} - \frac{2-5i}{-i} \right) = 57 + 24i$

Complex equations

If two complex numbers are equal, then their real parts are equal and their imaginary parts are equal. Hence if $a+ib = c+id, \quad then \ \ a = c \ \ and \ \ b = d$

Example. Solve the complex equation $(x-2yi)+(y-3xi) = 2+3i$

$(x-2yi)+(y-3xi) = 2+3i \Rightarrow (x+y)+(-2y-3x)i = 2+3i$, Equating real and imaginary parts gives

$\Rightarrow x + y = 2 \qquad (1)$

$-2y - 3x = 3 \qquad (2)$

Solving equation (1) and (2), Multiplying equation (1) with 2 and then add both equations

$2x + 2y = 4 \qquad (1)$

$-2y - 3x = 3 \qquad (2)$

$\Rightarrow x = -7$, Substituting x in equation (2) gives $y = 9$.

Problem. Solve the complex equation

$(x-2yi)-(y-xi) = 2+i \qquad answer\,(x = 3, \ y = 1)$

Hyperbolic functions

Pretty useful function especially in complex numbers.

$$\sinh(z) = \frac{e^z - e^{-z}}{2}, \quad \cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}} \Rightarrow \tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} \times \frac{e^z}{e^z} = \frac{e^{2z} - 1}{e^{2z} + 1}$$

z is complex number.

Trignometric functions

Pretty useful function especially in complex numbers.

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

- Sin and cos is between +1 and -1, but it is not true for complex z .

$|\sin(10i)| = \left| \frac{e^{-10} - e^{10}}{2i} \right| \geq 10000$ a huge number

- Hyperbolic, trignometir and exponancial functions are interrelated.

Identities:

- $i\sin(z) = \sinh(iz)$ $\qquad\qquad$ $\cos(z) = \cosh(iz)$
- $\sin(iz) = i\sinh(z)$ $\qquad\qquad$ $\cos(iz) = \cosh(z)$
- $e^{z} = \cosh(z) + \sinh(z)$ $\qquad\qquad$ $e^{z} = \cos(iz) - i\sin(iz)$

The Polar form of a complex number

Let a complex number Z be $x + yi$ as shown in the Argand diagram of figure 2. Let distance OZ be r and the angle OZ makes with the positive real axis be θ.

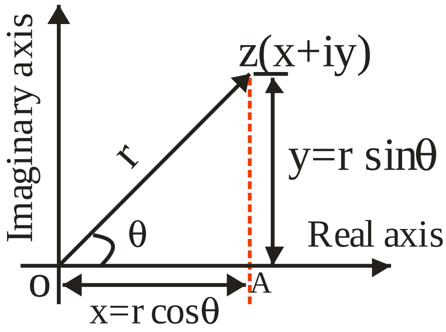


Figure 2.

From trigonometry, $x = r\cos\theta$ *and* $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Where r is called the modulus (or magnitude) of Z and is written as mod Z or $|Z|$. r is determined using Pythagoras' theorem on triangle OAZ and θ is called the argument (or amplitude) of Z and is written as arg Z. Whenever changing from Cartesian form to polar form, or vice-versa, a sketch is invaluable for determining the quadrant in which the complex number occurs.

Hence $z = x + yi = r\cos\theta + ir\sin\theta \Rightarrow Z = r(\cos\theta + i\sin\theta)$**,** where r is the distance and θ is direction.

$$(x+yi)^2 = r^2(\cos\theta + i\sin\theta)^2 = r^2\left(\cos^2\theta - \sin^2\theta + i(2\sin\theta\cos\theta)\right) = r^2(\cos 2\theta + i\sin 2\theta)$$

- $x + yi = r(\cos\theta + i\sin\theta)$
  $(x+yi)^2 = r^2(\cos 2\theta + i\sin 2\theta)$

In polar coordinates the numbers are multiplied and the angles are added

- Using Euler (Swiss Mathematician 1707 - 1783) formula (Euler identity) (One of the most beautiful formula in mathematics and is the most important formula in complex analysis (is the heart of complex numbers)) $\cos\theta + i\sin\theta = e^{i\theta}$ **,** $r^2(\cos\theta + i\sin\theta)^2 = r^2 e^{i2\theta}$

- $e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$
- $e^{i\pi} = \cos\pi + i\sin\pi = -1$
- $i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$ *is a real number*

$Z = r(\cos\theta + i\sin\theta)$ is usually abbreviated to $Z = r\angle\theta$ which is known as the **polar form** of a complex number.

Problem $\qquad$ For the the complex number $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ find (i) r (ii) $\theta(45^o)$ (iii) $e^{i\theta}$ (iv) z^2

Function of complex variable

$$z^n = (x+yi)^n \Rightarrow z^n = \left(r(\cos\theta + i\sin\theta)\right)^n = r^n(\cos\theta + i\sin\theta)^n \Rightarrow z^n = r^n\left(e^{i\theta}\right)^n \Rightarrow z^n = r^n e^{in\theta}$$

$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$ is **De Moivre's** formula.