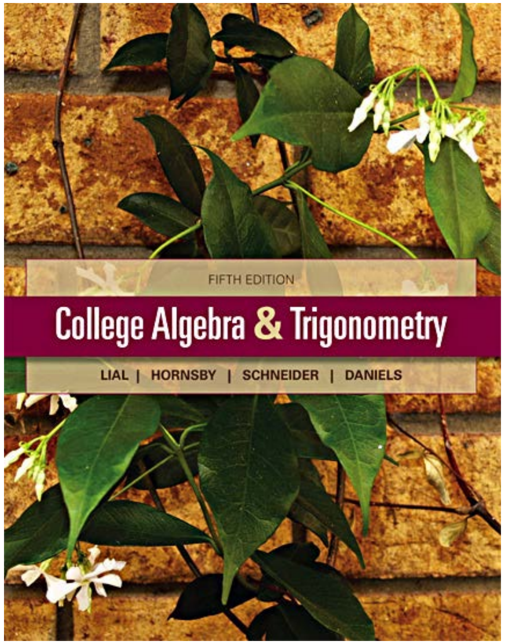
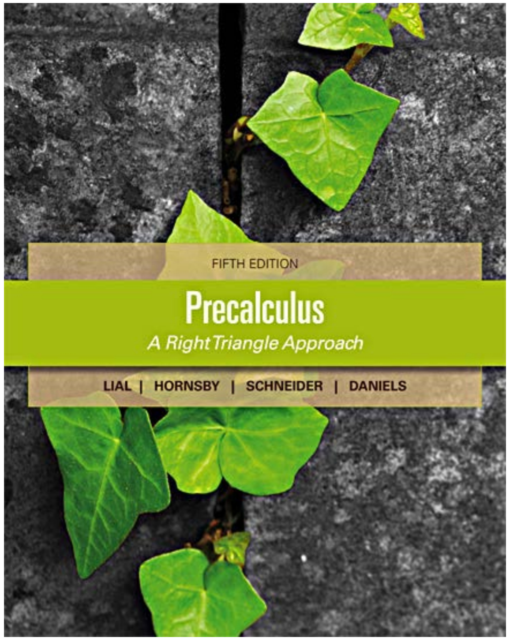
5

Inverse, Exponential, and Logarithmic Functions





5.1 Inverse Functions

- Objectives:
- One-to-One Functions
- Inverse Functions
- Equations of Inverses
- An Application of Inverse Functions to Cryptography ملغي

Pre-Questions

1) What the different between the two functions

F={(1,2),(3,2),(4,5)},

G={(1,2),(3,4),(5,6)}?

2) Is every function has an inverse function?

3) What the type of function that has inverse?

One-to-One Functions

Suppose we define the function

$$F = \{(-2, 2), (-1, 1), (0, 0), (1, 3), (2, 5)\}.$$

We can form another set of ordered pairs from *F* by interchanging the *x*- and *y*-values of each pair in *F*. We call this set *G*, so

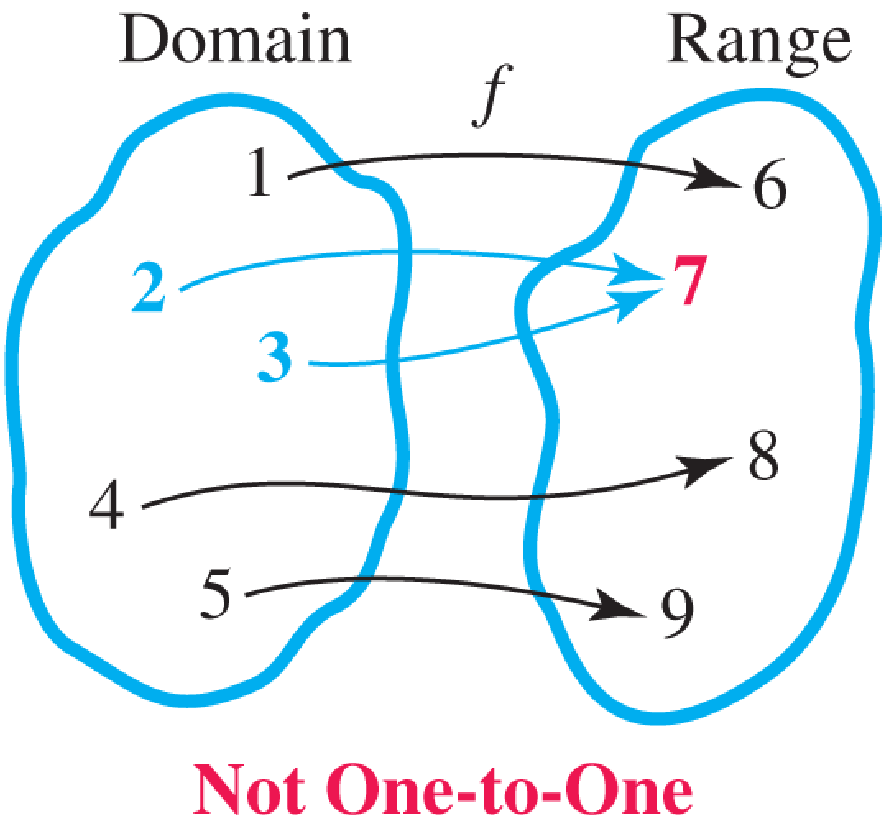
$$G = \{(2, -2), (1, -1), (0, 0), (3, 1), (5, 2)\}.$$

One-to-One Functions

To show that these two sets are related, *G* is called the *inverse* of *F*. For a function f to have an inverse, f must be a *one-to-one function.*

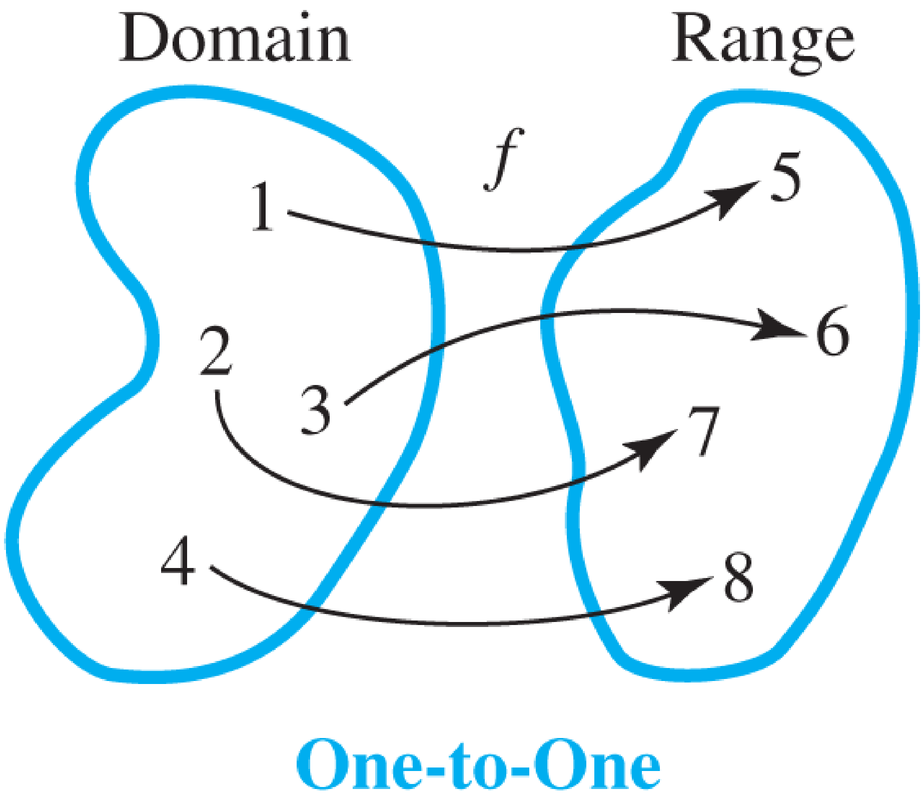
In a one-to-one function, each x-value corresponds to only one y-value, and each y-value corresponds to only one x-value.

One-to-One Functions



This function is not one-to-one because the *y*-value 7 corresponds to *two x*-values, 2 and 3. That is, the ordered pairs (2, 7) and (3, 7) both belong to the function.

One-to-One Functions



This function is one-to-one.

Quiz 1

Which function is **not** one-to-one ?

A. $\{(0,1),(1,2),(2,3),(3,4)\}$

B. $\{(0,1),(1,0),(2,0),(3,2)\}$

C. $\{(0,0),(1,1),(2,2),(3,3)\}$

D. $\{(0,1),(1,0),(2,3),(3,2)\}$

One-to-One Function

A function f is a **one-to-one function** if, for elements *a* and *b* in the domain of f,

$$a \neq b \quad \text{implies} \quad f(a) \neq f(b).$$

One-to-One Functions

Using the concept of the *contrapositive* from the study of logic, the last line in the preceding box is equivalent to

$$f(a) = f(b) \quad \text{implies} \quad a = b.$$

We use this statement to decide whether a function f is one-to-one in the next example.

Not One-to-One Function

A function f is not **one-to-one function** if, for elements *a* and *b* in the domain of f, $a \neq b$ implies $f(a) = f(b)$.

Example 1 DECIDING WHETHER FUNCTIONS ARE ONE-TO-ONE

Decide whether each function is one-to-one.

(a) $f(x) = -4x + 12$

Solution We must show that $f(a) = f(b)$ leads to the result $a = b$.

$$f(a) = f(b)$$

$$-4a + 12 = -4b + 12 \qquad f(x) = -4x + 12$$

$$-4a = -4b \qquad \text{Subtract 12.}$$

$$a = b \qquad \text{Divide by } -4.$$

By the definition, $f(x) = -4x + 12$ is one-to-one.

Example 1 DECIDING WHETHER FUNCTIONS ARE ONE-TO-ONE

Decide whether each function is one-to-one.

(b) $f(x) = \sqrt{25 - x^2}$

Solution If we choose $a = 3$ and $b = -3$, then $3 \neq -3$, but

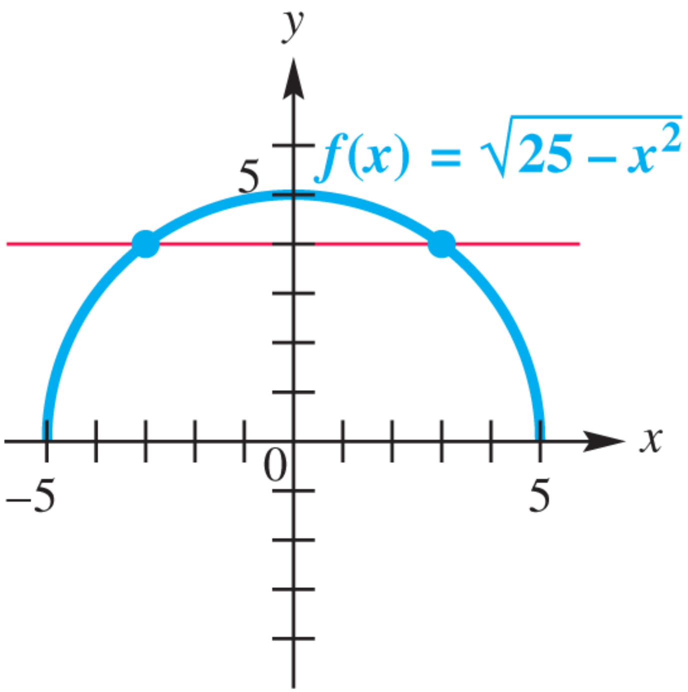
$$f(3) = \sqrt{25 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4$$

and $$f(-3) = \sqrt{25 - (-3)^2} = \sqrt{25 - 9} = 4.$$

Here, even though $3 \neq -3$, $f(3) = f(-3) = 4$. By the definition, f is *not* a one-to-one function.

Horizontal Line Test

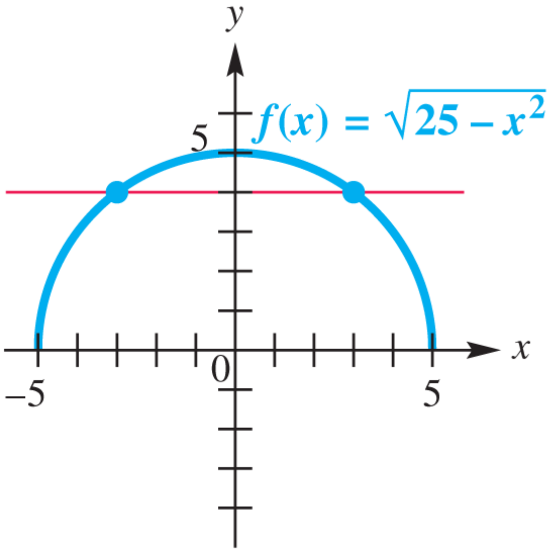
As illustrated in **Example 1(b),** a way to show that a function is *not* one-to-one is to produce a pair of different domain elements that lead to the same function value. There is also a useful graphical test, the **horizontal line test,** that tells whether or not a function is one-to-one.



Horizontal Line Test

A function is one-to-one if every horizontal line intersects the graph of the function at most once.

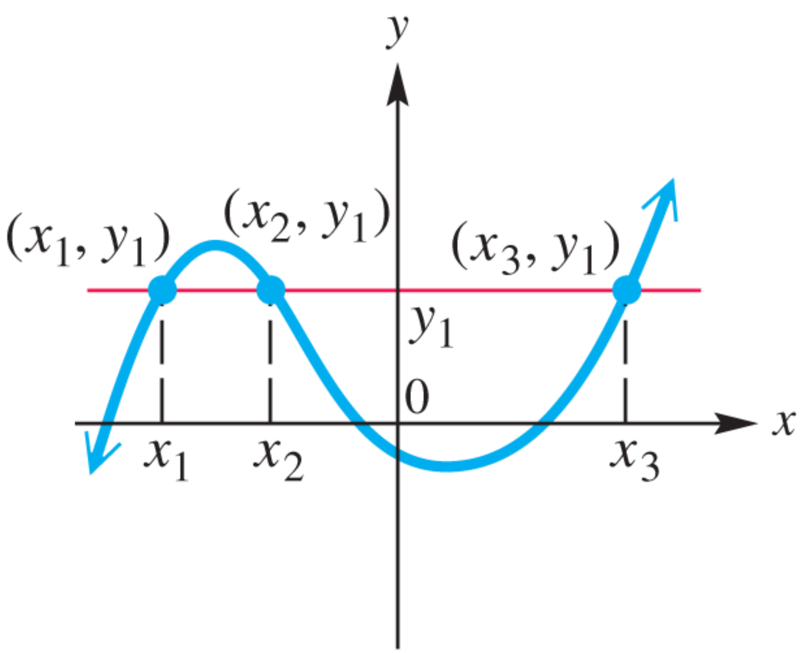
Note In **Example 1(b),** the graph of the function is a semicircle, as shown in the figure. Because there is at least one horizontal line that intersects the graph in more than one point, this function is not one-to-one.



Homework 1 USING THE HORIZONTAL LINE TEST

Determine whether each graph is the graph of a one-to-one function.

(a)



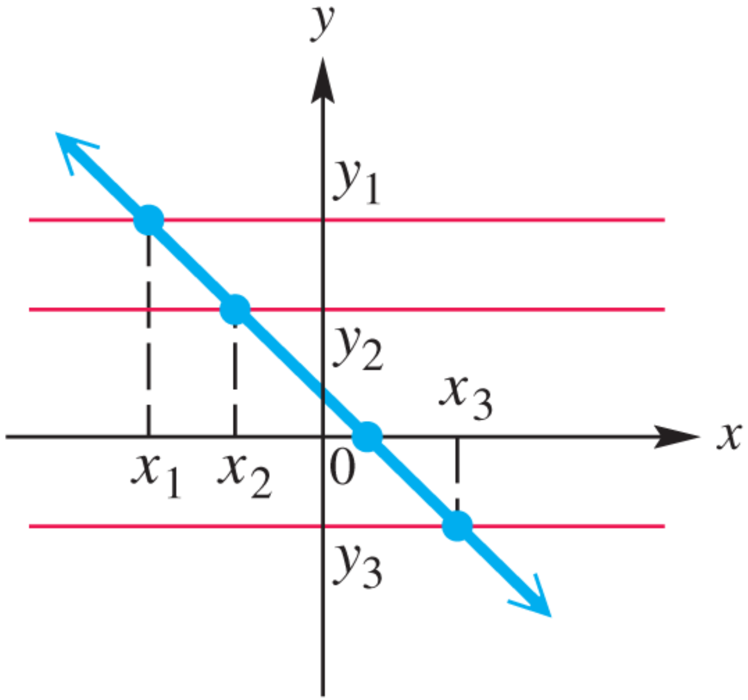
Solution

Each point where the horizontal line intersects the graph has the same value of *y* but a different value of *x*. Since more than one (here three) different values of *x* lead to the same value of *y*, the function is not one-to-one.

Homework 1 USING THE HORIZONTAL LINE TEST

Determine whether each graph is the graph of a one-to-one function.

(b)

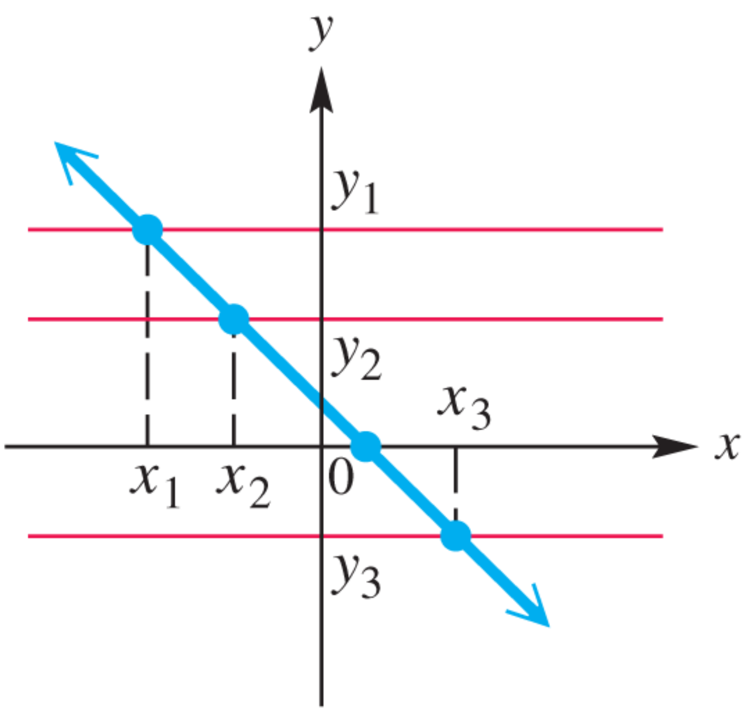


Solution

Since every horizontal line will intersect the graph at exactly one point, this function is one-to-one.

One-to-One Functions

Notice that the function graphed in **Example 2(b)** decreases on its entire domain. ***In general, a function that is either increasing or decreasing on its entire domain, such as $f(x) = -x$, $g(x) = x^3$, and $h(x) = \sqrt{x}$, must be one-to-one.***



Tests to Determine Whether a Function is One-to-One

1. Show that $f(a) = f(b)$ implies $a = b$. This means that f is one-to-one. **(Example 1(a))**

2. In a one-to-one function every *y*-value corresponds to no more than one *x*-value. To show that a function is not one-to-one, find at least two *x*-values that produce the same *y*-value. **(Example 1(b))**

Tests to Determine Whether a Function is One-to-One

3. Sketch the graph and use the horizontal line test. **(Example 2)**

4. If the function either increases or decreases on its entire domain, then it is one-to-one. A sketch is helpful here, too. **(Example 2(b))**

Inverse Functions

Consider the functions

$$f(x) = 8x + 5 \quad \text{and} \quad g(x) = \frac{1}{8}x - \frac{5}{8}.$$

Let us choose an arbitrary element from the domain of *f*, say 10. Evaluate *f*(10).

$$f(10) = 8 \cdot 10 + 5 = 85$$

Inverse Functions

$$f(x) = 8x + 5 \quad \text{and} \quad g(x) = \frac{1}{8}x - \frac{5}{8}.$$

Now, we evaluate $g(85)$.

$$g(85) = \frac{1}{8}(85) - \frac{5}{8}$$ Let $x = 85$.

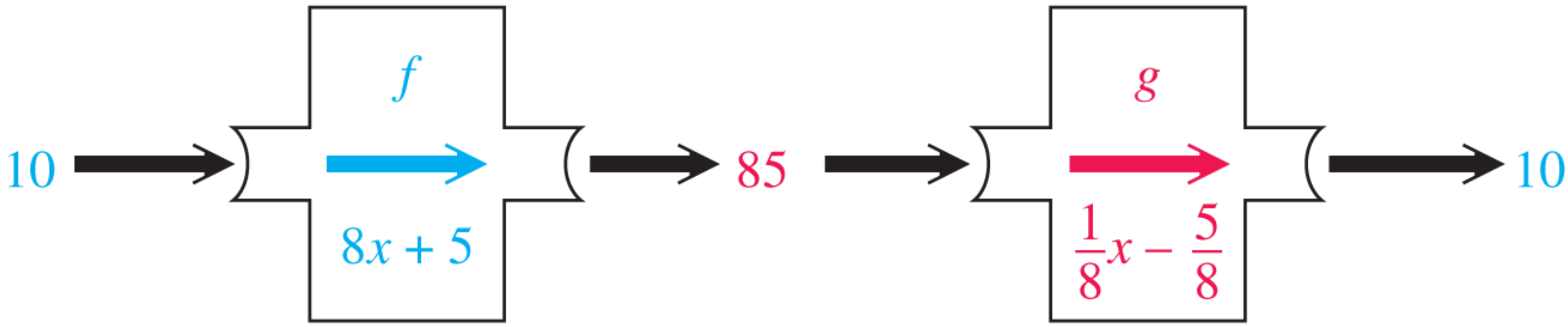
$$= \frac{85}{8} - \frac{5}{8}$$ Multiply.

$$= \frac{80}{8}$$ Subtract.

$$g(85) = 10$$ Divide.

Inverse Functions

Starting with 10, we “applied” function f and then “applied” function g to the result, which returned the number 10.



Inverse Functions

As further examples, check that

$$f(3) = 29 \quad \text{and} \quad g(29) = 3,$$

$$f(-5) = -35 \quad \text{and} \quad g(-35) = -5,$$

$$g(2) = -\frac{3}{8} \quad \text{and} \quad f\left(-\frac{3}{8}\right) = 2.$$

In particular, for this pair of functions,

$$f(g(2)) = 2 \quad \text{and} \quad g(f(2)) = 2.$$

Inverse Functions

In fact, for *any* value of *x*,

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x.$$

Using the notation for composition introduced in **Section 2.4,** these two equations can be written as follows.

$$(f \circ g)(x) = x \quad \text{and} \quad (g \circ f)(x) = x.$$

Because the compositions of f and g yield the *identity* function, they are *inverses* of each other.

Inverse Function

Let f be a one-to-one function. Then g is the **inverse function** of f if

$$(f \circ g)(x) = x$$ for every x in the domain of g,

and

$$(g \circ f)(x) = x$$ for every x in the domain of f.

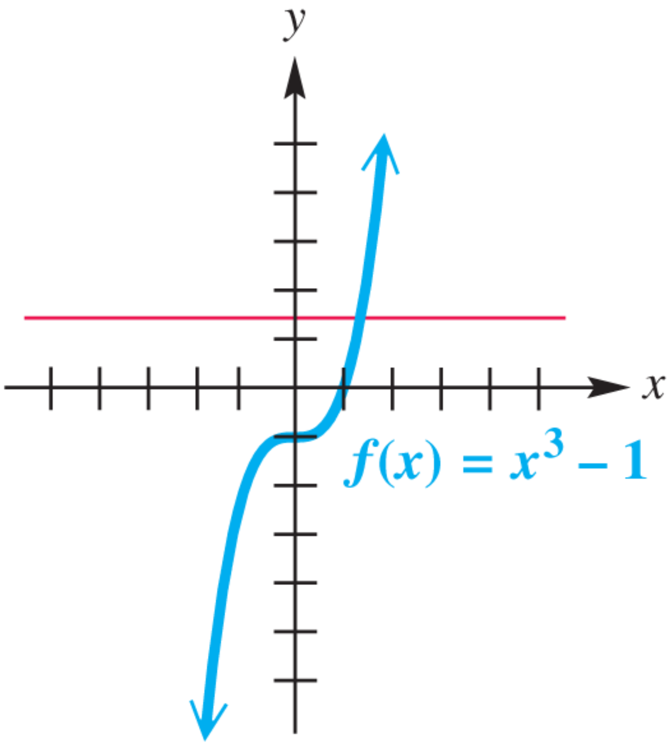
Inverse Function

The condition that f is one-to-one in the definition of inverse function is essential. Otherwise, g will not define a function.

Example 2 DECIDING WHETHER TWO FUNCTIONS ARE INVERSES

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively.
Is g the inverse function of f?

Solution The horizontal line test applied to the graph indicates that f is one-to-one, so the function does have an inverse. Since it is one-to-one, we now find $(f \circ g)(x)$ and $(g \circ f)(x)$.



Example 2 DECIDING WHETHER TWO FUNCTIONS ARE INVERSES

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively.
Is g the inverse function of f?

Solution

$$(f \circ g)(x) = f(g(x)) = \left(\sqrt[3]{x+1}\right)^3 - 1$$

$$= x + 1 - 1$$

$$= x$$

Example 2 DECIDING WHETHER TWO FUNCTIONS ARE INVERSES

Let functions f and g be defined by $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$, respectively.

Is g the inverse function of f?

Solution

$$(g \circ f)(x) = g(f(x)) = \sqrt[3]{(x^3 - 1) + 1}$$

$$= \sqrt[3]{x^3}$$

$$= x$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, function g is the inverse of function f.

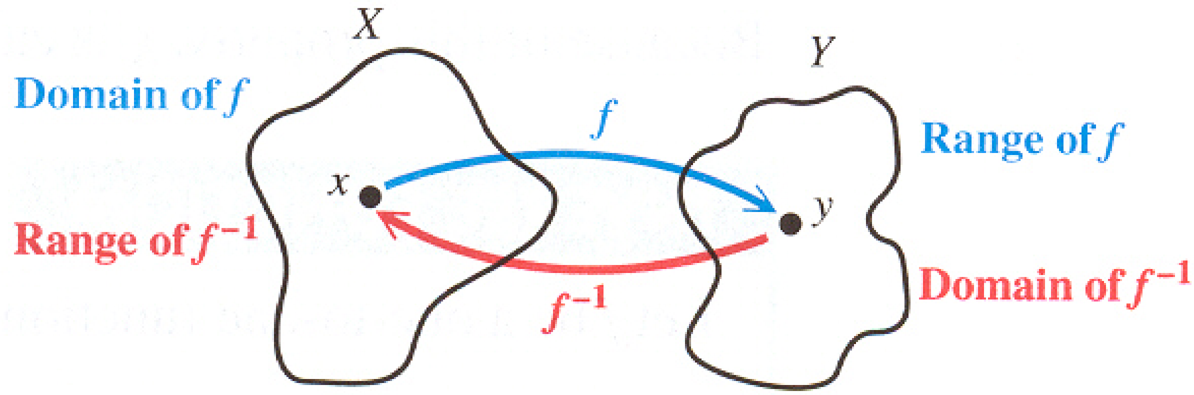
Special Notation

A special notation is used for inverse functions: If g is the inverse of a function f, then g is written as $\boldsymbol{f^{-1}}$ (read **"f-inverse"**). For $f(x) = x^3 - 1$, $f^{-1}(x) = \sqrt[3]{x+1}$

Caution ***Do not confuse the –1 in f^{-1} with a negative exponent.*** The symbol $f^{-1}(x)$ does not represent $\frac{1}{f(x)}$;it represents the inverse function of f.

Inverse Function

By the definition of inverse function, the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1} .



Homework 2 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of each function that is one-to-one.

(a) $F = \{(-2,1), (-1,0), (0,1), (1,2), (2,2)\}$

Solution Each *x*-value in *F* corresponds to just one *y*-value. However, the *y*-value 2 corresponds to two *x*-values, 1 and 2. Also, the *y*-value 1 corresponds to both -2 and 0. Because at least one *y*-value corresponds to more than one *x*-value, *F* is not one-to-one and does not have an inverse.

FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of each function that is one-to-one.

(b) $G = \{(3,1),(0,2),(2,3),(4,0)\}$

Solution Every *x*-value in *G* corresponds to only one *y*-value, and every *y*-value corresponds to only one *x*-value, so *G* is a one-to-one function. The inverse function is found by interchanging the *x*- and *y*-values in each ordered pair.

$$G^{-1} = \{(1,3),(2,0),(3,2),(0,4)\}$$

Notice how the domain and range of *G* becomes the range and domain, respectively, of G^{-1}.

Homework 2 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of each function that is one-to-one.

(c) The table shows the number of days in Illinois that were unhealthy for sensitive groups for selected years using the Air Quality Index (AQI). Let *f* be the function defined in the table, with the years forming the domain and the number of unhealthy days forming the range.

Year	Number of Unhealthy Days
2004	7
2005	32
2006	8
2007	24
2008	14
2009	13

Source: Illinois Environmental Protection Agency.

Homework 2 FINDING THE INVERSES OF ONE-TO-ONE FUNCTIONS

Find the inverse of each function that is one-to-one.

Year	Number of Unhealthy Days
2004	7
2005	32
2006	8
2007	24
2008	14
2009	13

Solution Each *x*-value in f corresponds to only one *y*-value and each *y*-value corresponds to only one *x*-value, so f is a one-to-one function. The inverse function is found by interchanging the *x*- and *y*-values in the table. The domain and range of f become the range and domain of f^{-1}.

$$f^{-1}(x) = \{(7,2004), (32,2005), (8,2006), (24,2007), (14,2008), (13,2009)\}$$

Equations of Inverses

The inverse of a one-to-one function is found by interchanging the *x*- and *y*-values of each of its ordered pairs. The equation of the inverse of a function defined by $y = f(x)$ is found in the same way.

Finding the Equation of the Inverse of $y = f(x)$

For a one-to-one function f defined by an equation $y = f(x)$, find the defining equation of the inverse as follows. (If necessary, replace $f(x)$ with y first. Any restrictions on x and y should be considered.)

Step 1 Interchange x and y.

Step 2 Solve for y.

Step 3 Replace y with $f^{-1}(x)$.

Example 3 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(a) $f(x) = 2x + 5$

Solution The graph of $y = 2x + 5$ is a non-horizontal line, so by the horizontal line test, f is a one-to-one function. To find the equation of the inverse, follow the steps in the preceding box, first replacing $f(x)$ with y.

Example 3 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

Solution

$$y = 2x + 5$$

Let $y = f(x)$.

$$x = 2y + 5$$

Interchange x and y.

$$x - 5 = 2y$$

Solve for y.

$$y = \frac{x-5}{2}$$

$$f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}$$

Replace y with $f^{-1}(x)$.

Example 3 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

Solution

In the function defined by $y = 2x + 5$, the value of *y* is found by starting with a value of *x*, multiplying by 2, and adding 5. The form $f^{-1}(x) = \frac{x-5}{2}$ for the equation of the inverse has us *subtract* 5 and then *divide* by 2. This shows how an inverse is used to “undo” what a function does to the variable *x*.

Example 3 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(b) $y = x^2 + 2$

Solution The equation has a parabola opening up as its graph, so some horizontal lines will intersect the graph at two points. For example, both $x = 3$ and $x = -3$ correspond to $y = 11$. Because of the presence of the x^2-term, there are many pairs of x-values that correspond to the same y-value. This means that the function defined by $y = x^2 + 2$ is not one-to-one and does not have an inverse.

Example 3 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(b) $y = x^2 + 2$

Solution The steps for finding the equation of an inverse lead to the following.

$$y = x^2 + 2$$

$$x = y^2 + 2 \quad \text{Interchange } x \text{ and } y.$$

$$x - 2 = y^2 \quad \text{Solve for } y.$$

$$\pm\sqrt{x-2} = y \quad \text{Square root property}$$

Remember *both* roots.

Example 3 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(b) $y = x^2 + 2$

Solution

$$\pm\sqrt{x-2} = y$$

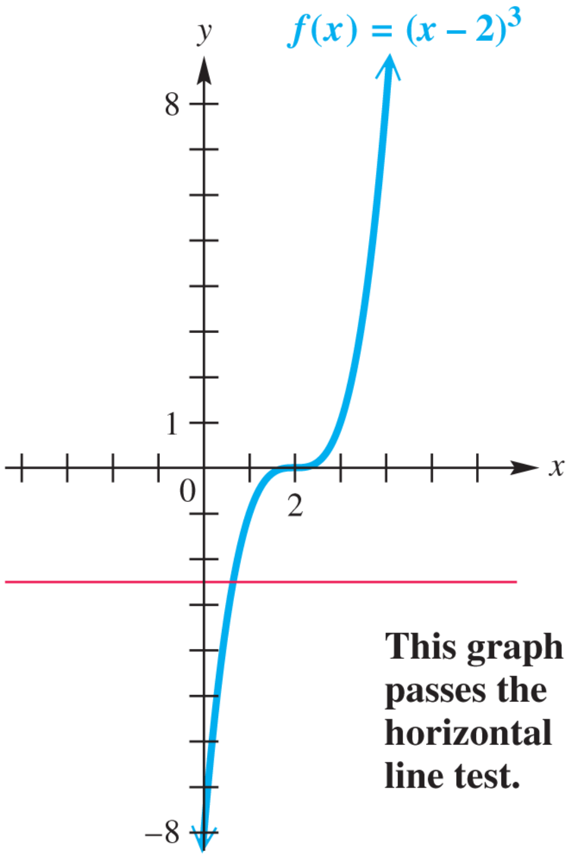
The last step shows that there are two *y*-values for each choice of *x* greater than 2, so the given function is not one-to-one and cannot have an inverse.

Example 3 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(c) $f(x) = (x-2)^3$

Solution The figure shows that the horizontal line test assures us that this horizontal translation of the graph of the cubing function is one-to-one.



Example 3 FINDING EQUATIONS OF INVERSES

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

Solution

$$f(x) = (x-2)^3$$

$$y = (x-2)^3$$ Replace $f(x)$ with y.

$$x = (y-2)^3$$ Interchange x and y.

$$\sqrt[3]{x} = \sqrt[3]{(y-2)^3}$$ Take the cube root on each side.

$$\sqrt[3]{x} = y-2$$

$$\sqrt[3]{x} + 2 = y$$ Solve for y by adding 2.

$$f^{-1}(x) = \sqrt[3]{x} + 2$$ Replace y with $f^{-1}(x)$. Rewrite.

Homework 3 FINDING THE EQUATION OF THE INVERSE OF A RATIONAL FUNCTION

The rational function $f(x) = \dfrac{2x+3}{x-4}$, $x \neq 4$, is a one-to-one function. Find its inverse.

Solution

$$f(x) = \frac{2x+3}{x-4}, \; x \neq 4$$

$$y = \frac{2x+3}{x-4}$$

Replace $f(x)$ with y.

$$x = \frac{2y+3}{y-4}, \; y \neq 4$$

Interchange x and y.

Homework 3 FINDING THE EQUATION OF THE INVERSE OF A RATIONAL FUNCTION

The rational function $f(x) = \dfrac{2x+3}{x-4}$, $x \neq 4$, is a one-to-one function. Find its inverse.

Solution

$$x(y-4) = 2y+3$$

Solve for *y*.

$$xy - 4x = 2y + 3$$

$$xy - 2y = 4x + 3$$

$$y(x-2) = 4x + 3$$

$$y = \frac{4x+3}{x-2}, \ x \neq 2$$

Replace *y* with $f^{-1}(x)$.

FINDING THE EQUATION OF THE INVERSE OF A RATIONAL FUNCTION

The rational function $f(x) = \dfrac{2x+3}{x-4}$, $x \neq 4$, is a one-to-one function. Find its inverse.

Solution

$$y = \frac{4x+3}{x-2}, \; x \neq 2$$

In the final line, we give the condition $x \neq 2$. (Note that 2 was not in the *range* of f, so it is Not in the domain of f^{-1}.)

$$f^{-1}(x) = \frac{4x+3}{x-2}, \; x \neq 2$$

Replace y with $f^{-1}(x)$.

Inverse Function

One way to graph the inverse of a function f whose equation is known follows.

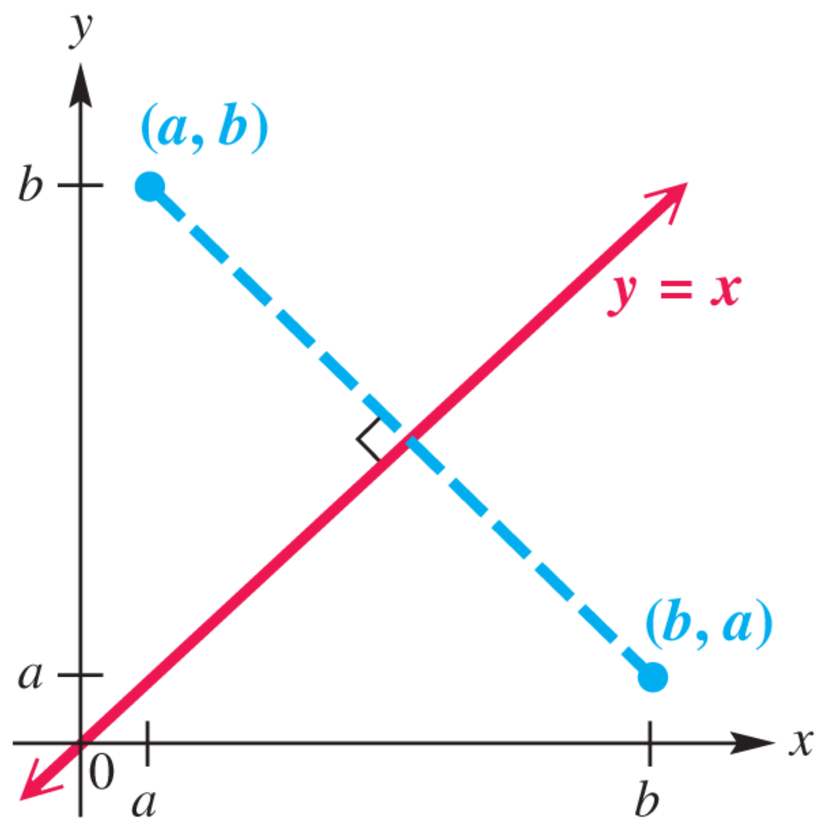
Step 1 Find some ordered pairs that are on the graph of f.

Step 2 Interchange x and y to get ordered pairs that are on the graph of f^{-1}.

Step 3 Plot those points, and sketch the graph of f^{-1} through them.

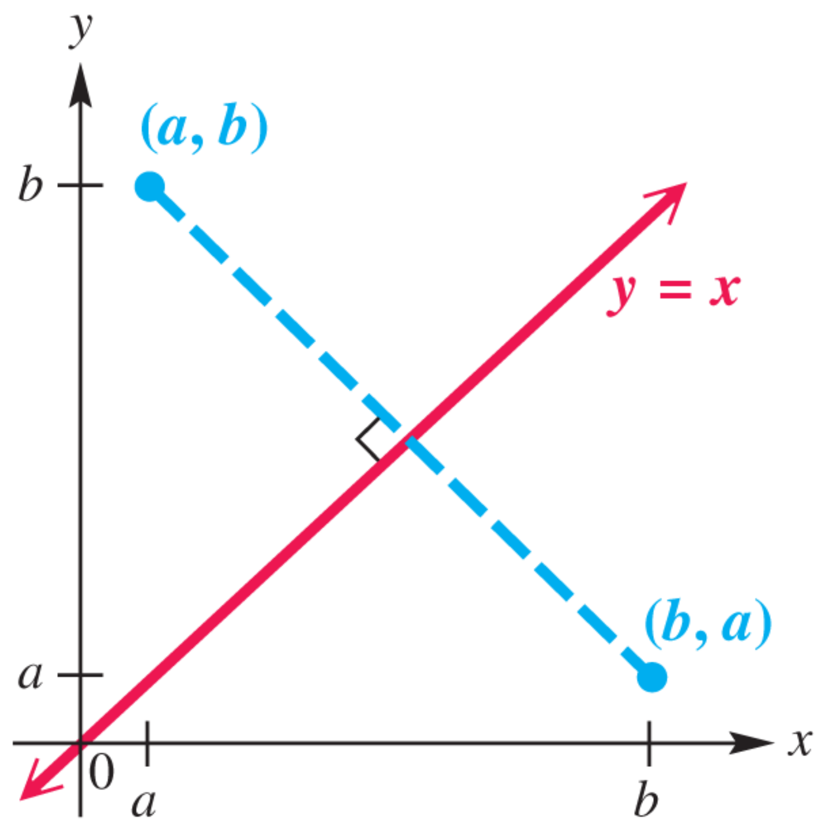
Inverse Function

Another way is to select points on the graph of f and use symmetry to find corresponding points on the graph of f^{-1}.



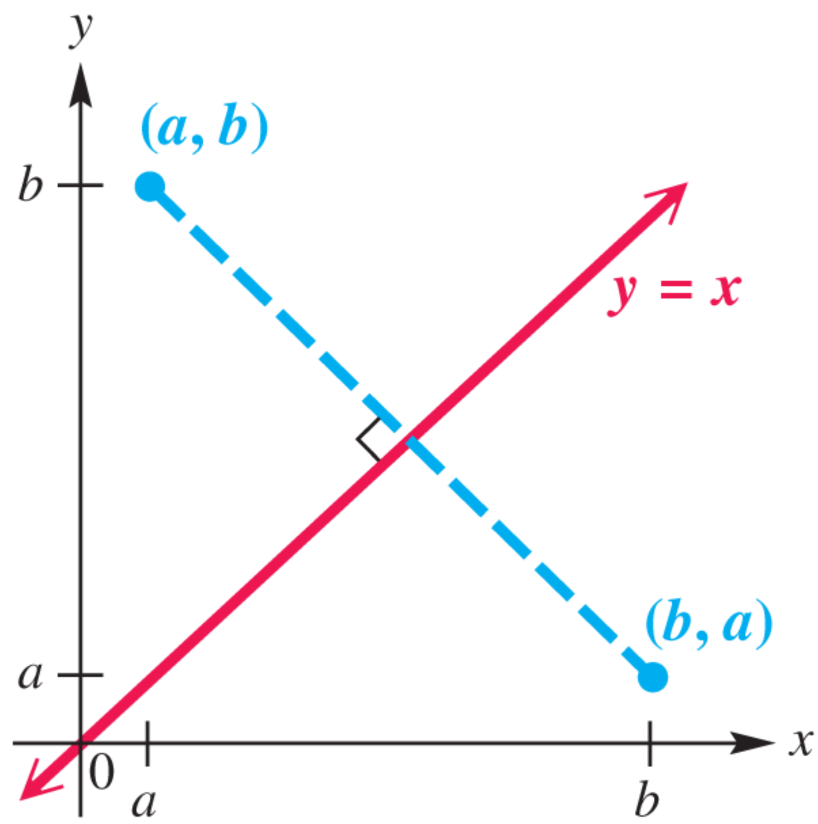
Inverse Function

For example, suppose the point (a, b) shown here is on the graph of a one-to-one function f.



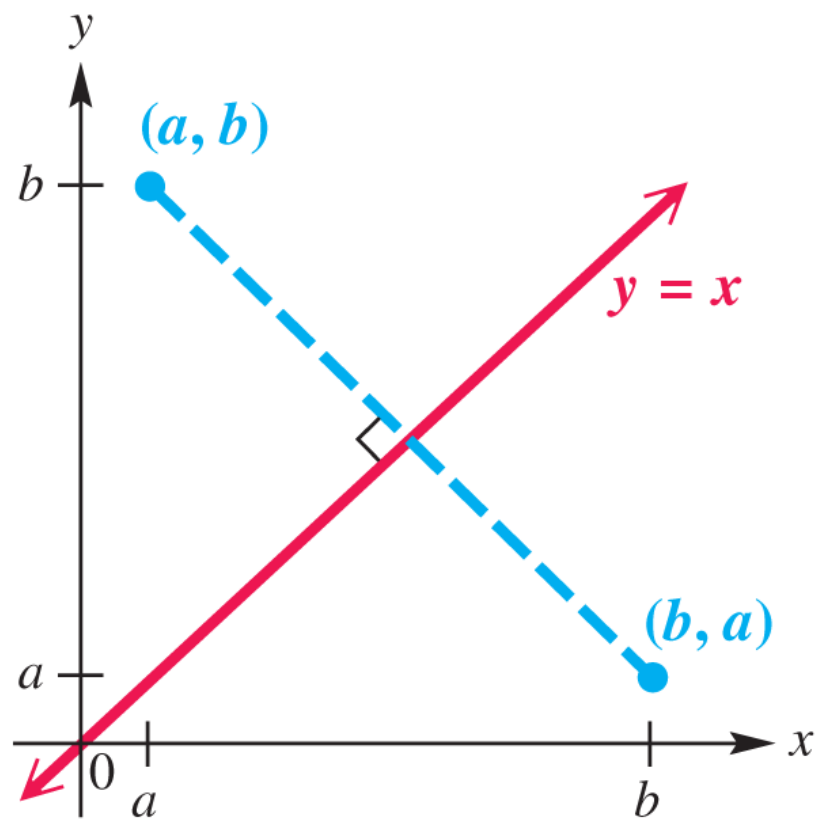
Inverse Function

Then the point (b, a) is on the graph of f^{-1}. The line segment connecting (a, b) and (b, a) is perpendicular to, and cut in half by, the line $y = x$. The points (a, b) and (b, a) are "mirror images" of each other with respect to $y = x$.



Inverse Function

Thus, we can find the graph of f^{-1} from the graph of f by locating the mirror image of each point in f with respect to the line $y = x$.

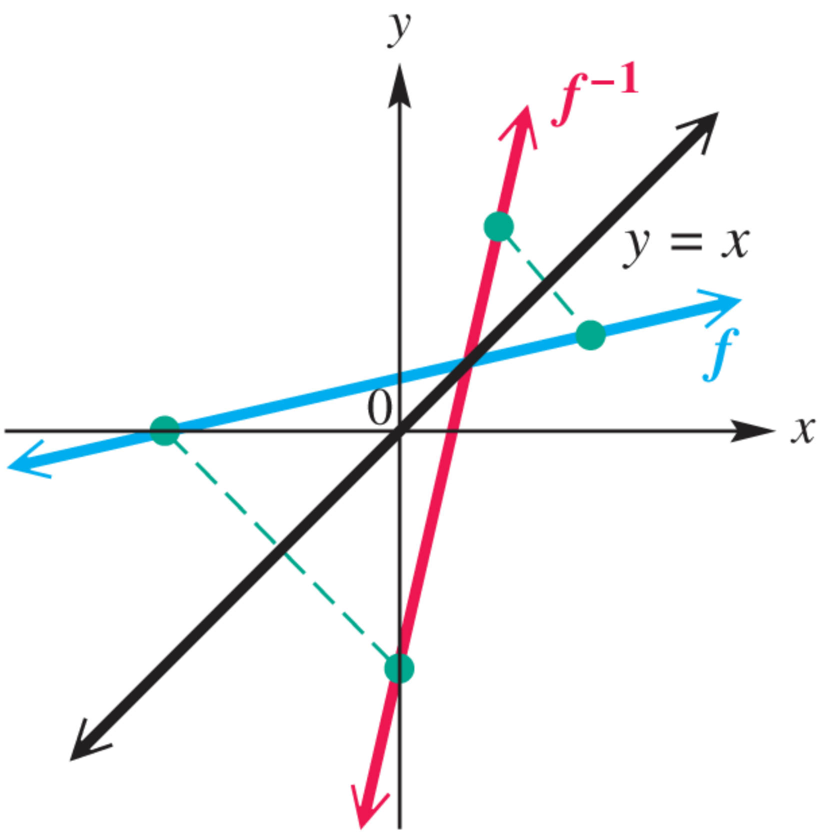


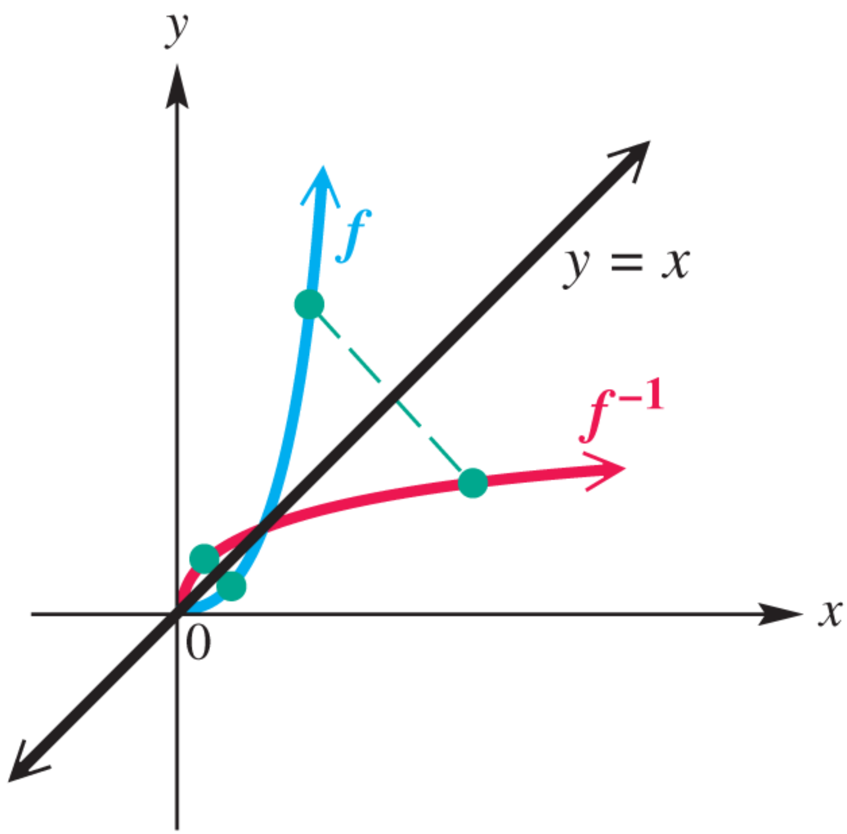
Example 4 GRAPHING f^{-1} GIVEN THE GRAPH OF f

In each set of axes, the graph of a one-to-one function f is shown in blue. Graph f^{-1} in red.

Solution On the next slide, the graphs of two functions f shown in blue are given with their inverses shown in red. In each case, the graph of f^{-1} is a reflection of the graph of f with respect to the line $y = x$.

Solution





Homework 4 FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN

Let $f(x) = \sqrt{x+5},\ x \geq -5.$ Find $f^{-1}(x).$

Solution First, notice that the domain of f is restricted to the interval $[-5, \infty)$. Function f is one-to-one because it is increasing on its entire domain and, thus, has an inverse function. Now we find the equation of the inverse.

FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN

Solution

$$f(x) = \sqrt{x+5}, \qquad x \geq -5$$

$$y = \sqrt{x+5}, \qquad x \geq -5$$ Replace $f(x)$ with y.

$$x = \sqrt{y+5}, \qquad y \geq -5$$ Interchange x and y.

$$x^2 = \left(\sqrt{y+5}\right)^2$$ Square each side.

$$x^2 = y + 5$$

$$y = x^2 - 5$$ Solve for y.

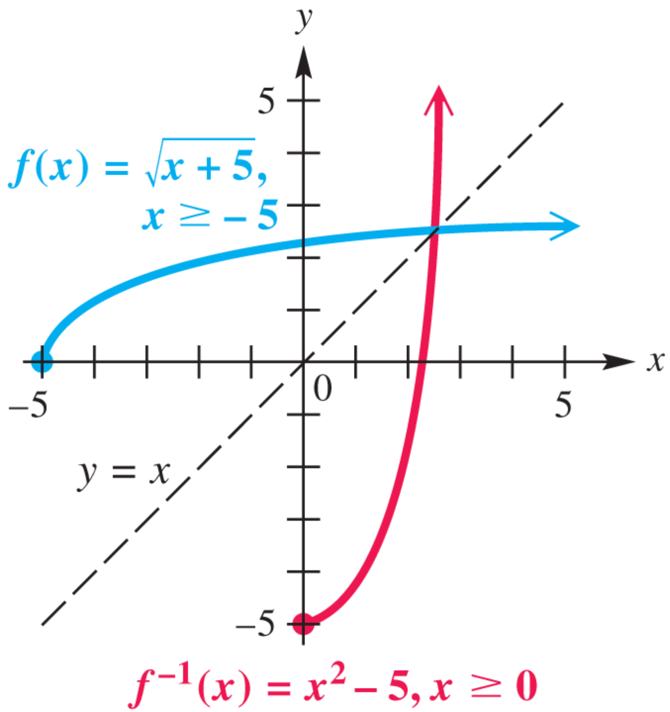
FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN

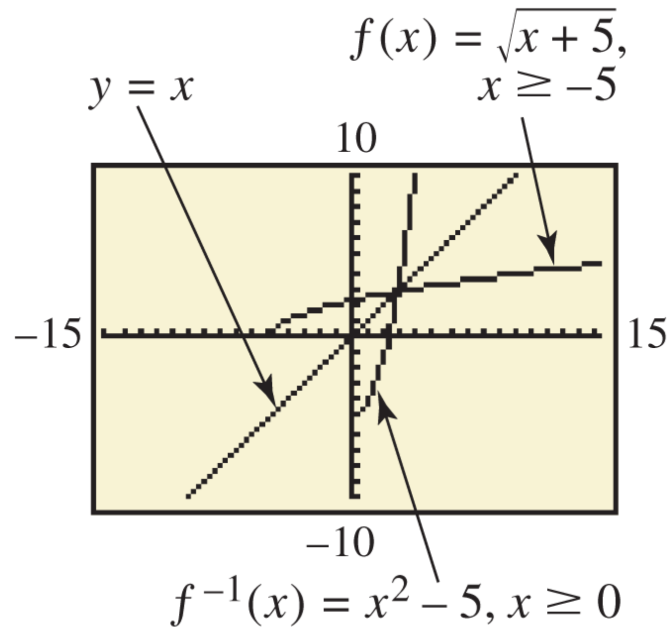
Solution However, we cannot define $f^{-1}(x)$ as $x^2 - 5$. The domain of f is $[-5, \infty)$, and its range is $[0, \infty)$. The range of f is the domain of f^{-1}, so f^{-1} must be defined as

$$f^{-1}(x) = x^2 - 5, \quad x \geq 0.$$

Homework 4 FINDING THE INVERSE OF A FUNCTION WITH A RESTRICTED DOMAIN

As a check, the range of f^{-1}, $[-5, \infty)$, is the domain of f. Graphs of f and f^{-1} are shown. The line $y = x$ is included on the graphs to show that the graphs are mirror images with respect to this line.





Important Facts About Inverses

1. If f is one-to-one, then f^{-1} exists.
2. The domain of f is the range of f^{-1},and the range of f is the domain of f^{-1}.
3. If the point (a, b) lies on the graph of f, then (b, a) lies on the graph of f^{-1}. The graphs of f and f^{-1} are reflections of each other across the line $y = x$.
4. To find the equation for f^{-1}, replace $f(x)$ with y, interchange x and y, and solve for y. This gives $f^{-1}(x)$.

An Application of Inverse Functions to Cryptography ملغي

A one-to-one function and its inverse can be used to make information secure. The function is used to encode a message, and its inverse is used to decode the coded message. In practice, complicated functions are used. We illustrate the process with a simple function in **Example 9.**

Example 9 USING FUNCTIONS TO ENCODE AND DECODE A MESSAGE

Use the one-to-one function $f(x) = 3x + 1$ and the following numerical values assigned to each letter of the alphabet to encode and decode the message BE MY FACEBOOK FRIEND.

A	1	**H**	8	**O**	15	**V**	22
B	2	**I**	9	**P**	16	**W**	23
C	3	**J**	10	**Q**	17	**X**	24
D	4	**K**	11	**R**	18	**Y**	25
E	5	**L**	12	**S**	19	**Z**	26
F	6	**M**	13	**T**	20		
G	7	**N**	14	**U**	21		

Example 9 USING FUNCTIONS TO ENCODE AND DECODE A MESSAGE

Use the one-to-one function $f(x) = 3x + 1$ and the following numerical values assigned to each letter of the alphabet to encode and decode the message BE MY FACEBOOK FRIEND.

Solution The message BE MY FACEBOOK FRIEND would be encoded as

7 16 40 76 19 4 10 16 7
46 46 34 19 55 28 16 43 13

because

B corresponds to 2 and $f(2) = 3(2) + 1 = 7$,

E corresponds to 5 and $f(5) = 3(5) + 1 = 16$,

and so on.

Example 9 USING FUNCTIONS TO ENCODE AND DECODE A MESSAGE

Solution The message BE MY FACEBOOK FRIEND would be encoded as

7 16 40 76 19 4 10 16 7
46 46 34 19 55 28 16 43 13

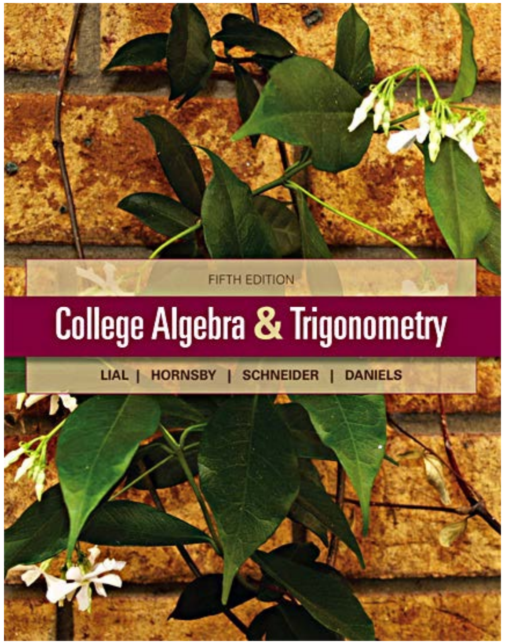
Using the inverse $f^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$ to decode yields

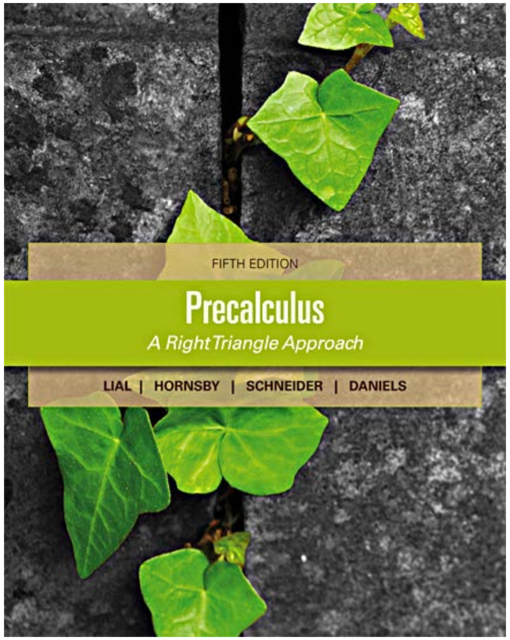
$f^{-1}(7) = \frac{1}{3}(7) - \frac{1}{3} = 2$, which corresponds to B,

$f^{-1}(16) = \frac{1}{3}(16) - \frac{1}{3} = 5$, corresponds to E, and so on.

5

Inverse, Exponential, and Logarithmic Functions





5.2 Exponential Functions

- Exponents and Properties
- Exponential Functions
- Exponential Equations
- Compound Interestملغي
- The Number *e* and Continuous Compounding
- Exponential Models ملغي

Exponents and Properties

Recall the definition of $a^{m/n}$: if a is a real number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number, then

For example,
$$a^{m/n} = \left(\sqrt[n]{a}\right)^m.$$

$$16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8,$$

$$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3},$$

and
$$64^{-1/2} = \frac{1}{64^{1/2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

Exponents and Properties

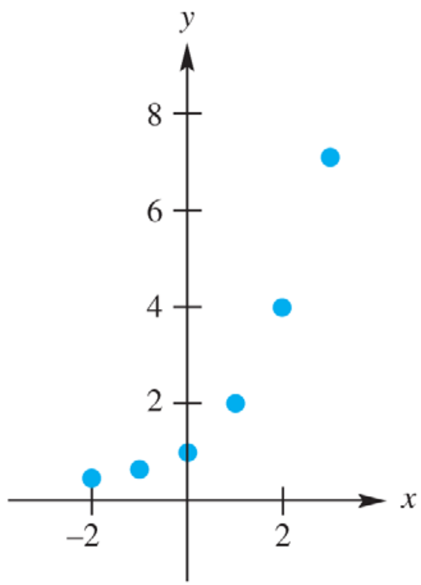
In this section we extend the definition of a^r to include all real (not just rational) values of the exponent *r*. For example, $2^{\sqrt{3}}$ might be evaluated by *approximating* the exponent $\sqrt{3}$ with the rational numbers 1.7, 1.73, 1.732, and so on.

Exponents and Properties

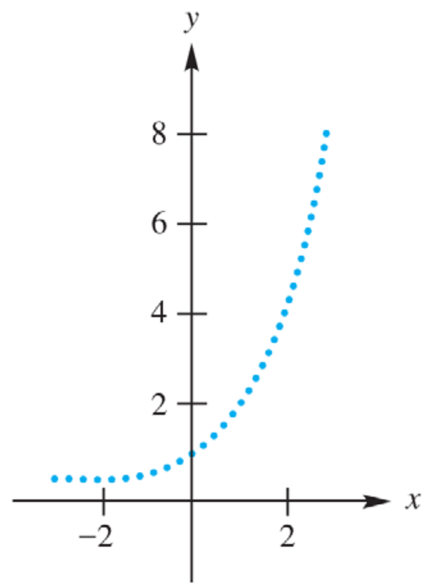
Since these decimals approach the value of $\sqrt{3}$ more and more closely, it seems reasonable that $2^{\sqrt{3}}$ should be approximated more and more closely by the numbers $2^{1.7}$, $2^{1.73}$, $2^{1.732}$, and so on. (Recall, for example, that $2^{1.7} = 2^{17/10} = \left(\sqrt[10]{2}\right)^{17}$.)

Exponents and Properties

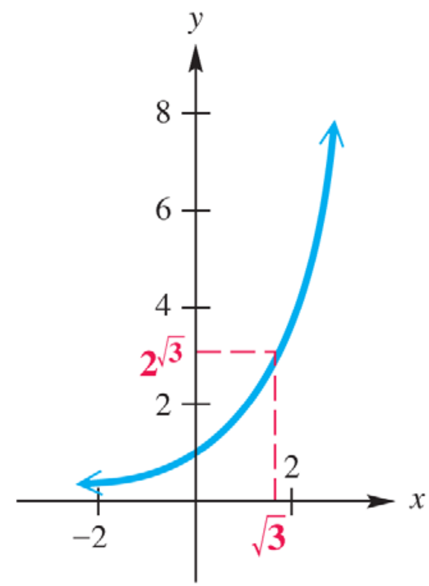
To show that this assumption is reasonable, see the graphs of the function $f(x) = 2^x$ with three different domains.



$f(x) = 2^x$;
integers as domain



$f(x) = 2^x$;
selected rational numbers as domain



$f(x) = 2^x$;
real numbers as domain

Exponents and Properties

Using this interpretation of real exponents, all rules and theorems for exponents are valid for all real number exponents, not just rational ones. In addition to the rules for exponents presented earlier, we use several new properties in this chapter.

Additional Properties of Exponents

For any real number $a > 0$, $a \neq 1$, the following statements are true.

(a) a^x is a unique real number for all real numbers x.

(b) $a^b = a^c$ if and only if $b = c$.

(c) If $a > 1$ and $m < n$, then $a^m < a^n$.

(d) If $0 < a < 1$ and $m < n$, then $a^m > a^n$.

Properties of Exponents

Properties (a) and (b) require $a > 0$ so that a^x is always defined. For example, $(-6)^x$ is not a real number if $x = \frac{1}{2}$. This means that a^x will always be positive, since a must be positive. In property (a), a cannot equal 1 because $1^x = 1$ for every real number value of x, so each value of x leads to the same real number, 1. For property (b) to hold, a must not equal 1 since, for example, $1^4 = 1^5$, even though $4 \neq 5$.

Properties of Exponents

Properties (c) and (d) say that ***when a > 1, increasing the exponent on "a" leads to a greater number, but when* 0 < *a* < 1, *increasing the exponent on "a" leads to a lesser number.***

Example 1 EVALUATING AN EXPONENTIAL EXPRESSION

If $f(x) = 2^x$, find each of the following.

(a) $f(-1)$

Solution

$$f(-1) = 2^{-1} = \frac{1}{2}$$ Replace x with -1.

Example 1 EVALUATING AN EXPONENTIAL EXPRESSION

If $f(x) = 2^x$, find each of the following.

(b) $f(3)$

Solution

$$f(3) = 2^3 = 8$$

Example 1 EVALUATING AN EXPONENTIAL EXPRESSION

If $f(x) = 2^x$, find each of the following.

(c) $f\left(\dfrac{5}{2}\right)$

Solution

$$f\left(\frac{5}{2}\right) = 2^{5/2} = (2^5)^{1/2}$$

$$= 32^{1/2} = \sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$$

Example 1 EVALUATING AN EXPONENTIAL EXPRESSION

If $f(x) = 2^x$, find each of the following.

(d) $f(4.92)$

Solution

$$f(4.92) = 2^{4.92} \approx 30.2738447$$

Use a calculator.

Exponential Function

If $a > 0$ and $a \neq 1$, then

$$f(x) = a^x$$

defines the **exponential function with base *a*.**

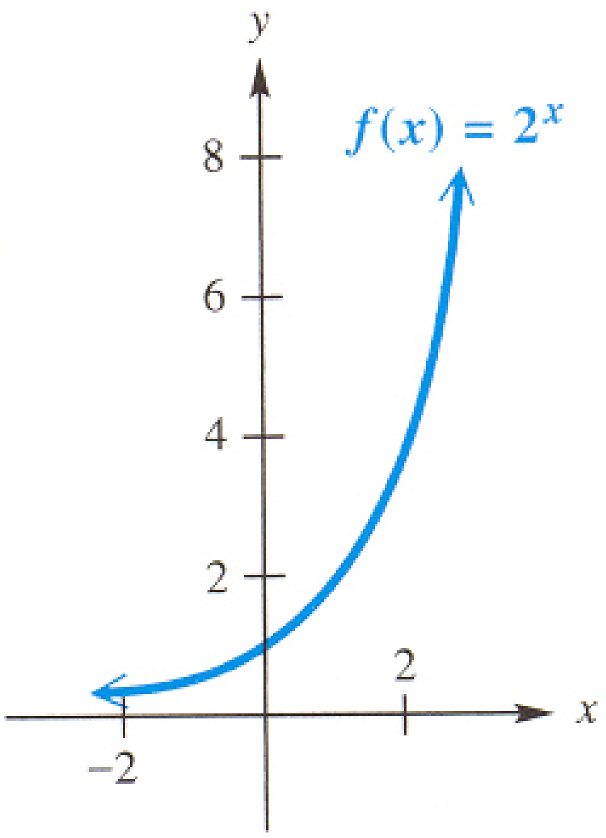
Motion Problems

Note ***We do not allow 1 as the base for an exponential function.*** If $a = 1$, the function becomes the constant function defined by $f(x) = 1$, which is not an exponential function.

Exponential Functions

Slide 7 showed the graph of $f(x) = 2^x$ with three different domains. We repeat the final graph (with real numbers as domain) here.

- The y-intercept is $y = 2^0 = 1$.
- Since $2^x > 0$ for all x and $2^x \to 0$ as $x \to -\infty$, the x-axis is a horizontal asymptote.
- As the graph suggests, the domain of the function is $(-\infty, \infty)$ and the range is $(0, \infty)$.
- The function is increasing on its entire domain, and is one-to-one.



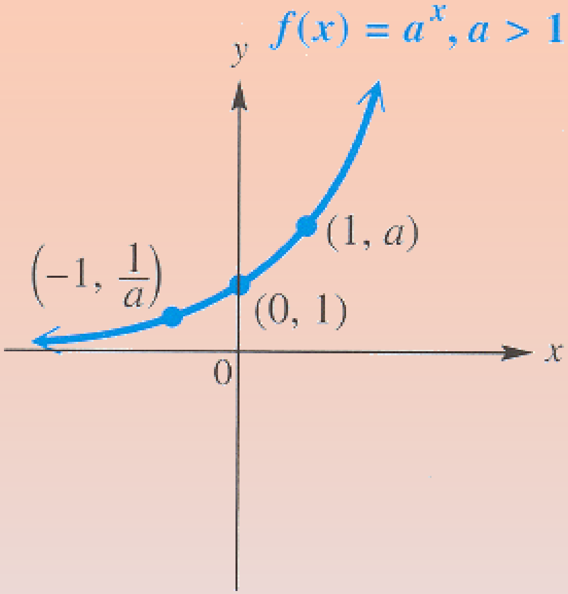
EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: $(-\infty, \infty)$ ثابت لا يتغير

Range: $(0, \infty)$ قابل للتغير

For $f(x) = 2^x$:

x	$f(x)$
–2	¼
–1	½
0	1
1	2
2	4
3	8



- $f(x) = a^x$, for $a > 1$, is increasing and continuous on its entire domain, $(-\infty, \infty)$.

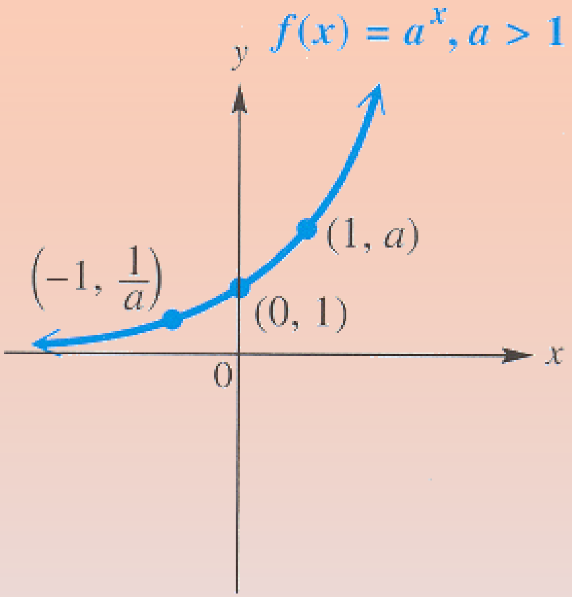
EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

For $f(x) = 2^x$:

x	$f(x)$
–2	¼
–1	½
0	1
1	2
2	4
3	8



- The x-axis is a horizontal asymptote as $x \to -\infty$.

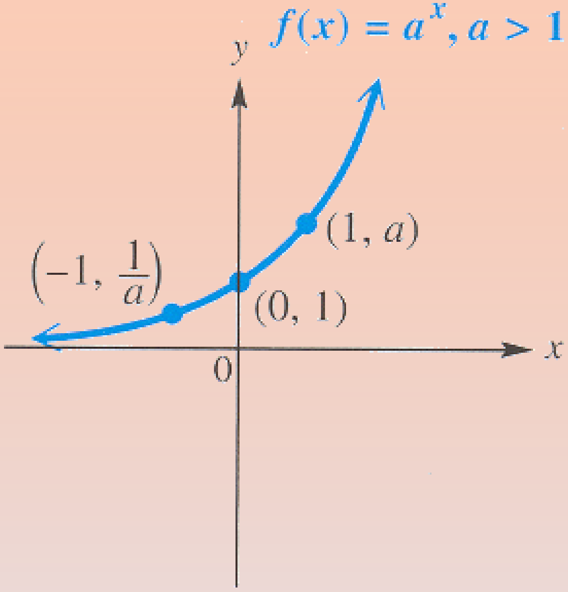
EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

For $f(x) = 2^x$:

x	$f(x)$
–2	¼
–1	½
0	1
1	2
2	4
3	8



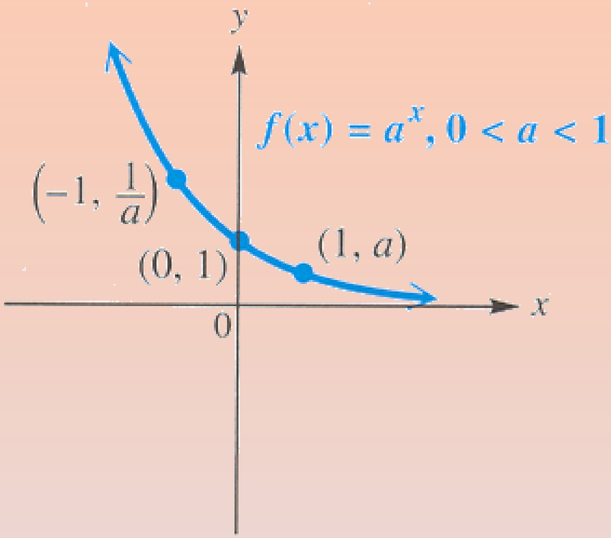
- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, $(0,1)$, and $(1,a)$.

EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

For $f(x) = (½)^x$:

x	$f(x)$
–3	8
–2	4
–1	2
0	1
1	½
2	¼



- $f(x) = a^x$, for $0 < a < 1$, is decreasing and continuous on its entire domain, $(-\infty, \infty)$.

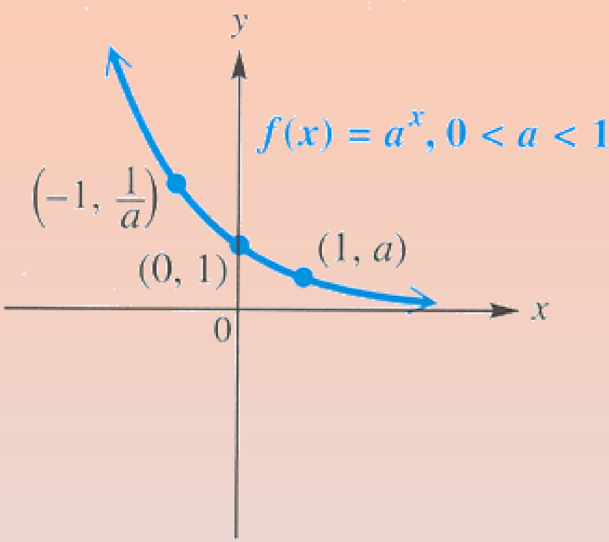
EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

For $f(x) = (½)^x$:

x	$f(x)$
–3	8
–2	4
–1	2
0	1
1	½
2	¼



- The x-axis is a horizontal asymptote as $x \to \infty$.

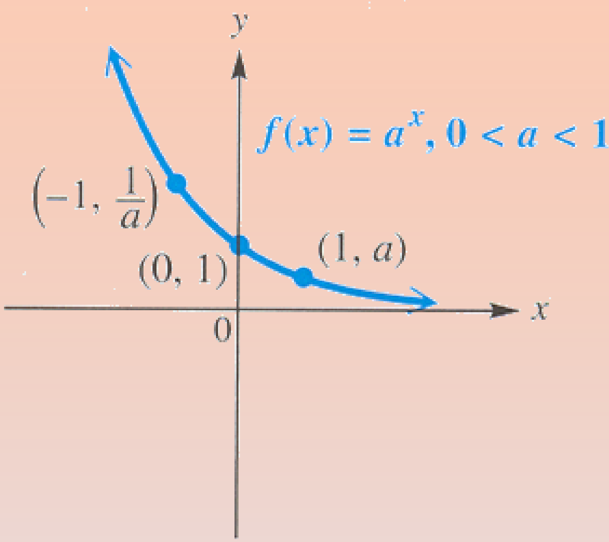
EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

For $f(x) = (½)^x$:

x	$f(x)$
–3	8
–2	4
–1	2
0	1
1	½
2	¼



- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, $(0,1)$, and $(1,a)$.

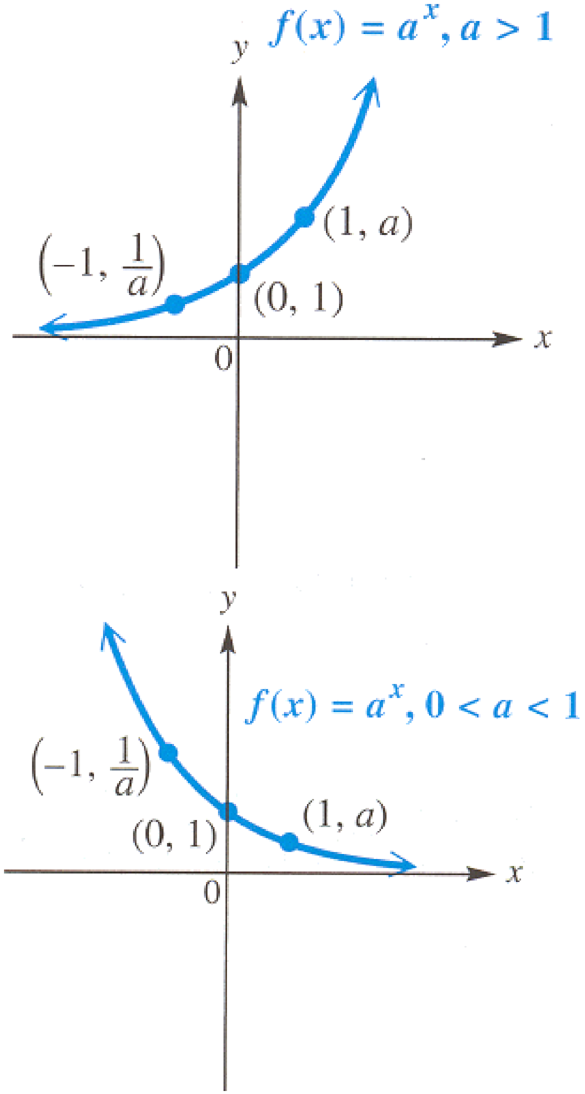
Exponential Function

From **Section 2.7,** the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis. Thus, we have the following.

If $f(x) = 2^x$, then

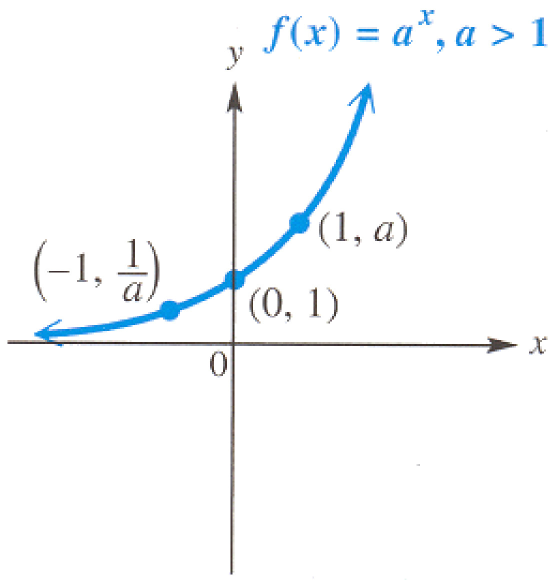
$f(-x) = 2^{-x} = (2^{-1})^x = 2^{-1 \cdot x} = (½)^x$.

This is supported by the graphs shown.

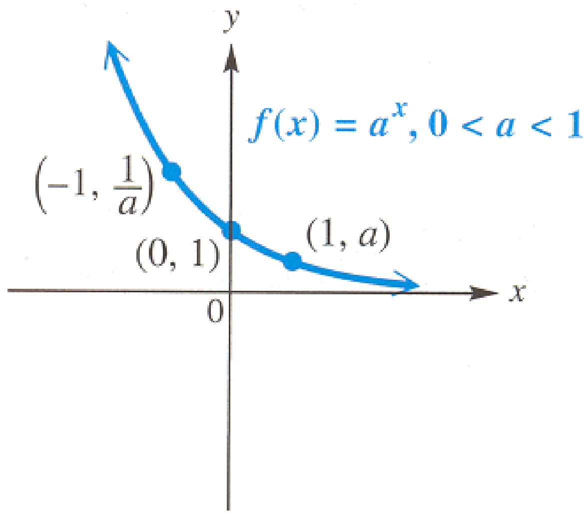


Exponential Function

The graph of $f(x) = 2^x$ is typical of graphs of $f(x) = a^x$ where $a > 1$. For larger values of a, the graphs rise more steeply, but the general shape is similar.

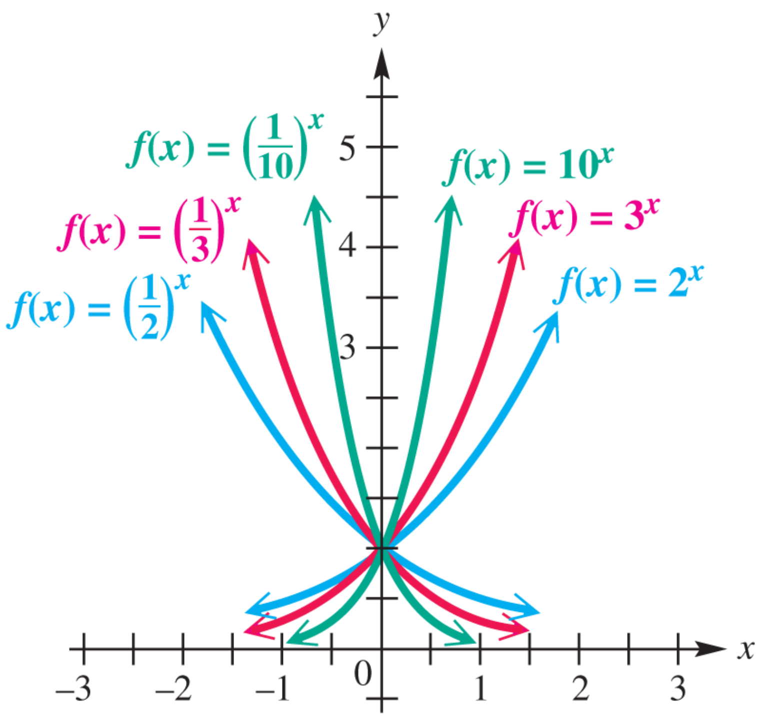


When $0 < a < 1$, the graph decreases in a manner similar to the graph of $f(x) = (1/2)^x$.



Exponential Function

The graphs of several typical exponential functions illustrate these facts.



$f(x) = a^x$

Domain: $(-\infty, \infty)$; Range: $(0, \infty)$

- When $a > 1$, the function is increasing.
- When $0 < a < 1$, the function is decreasing.
- In every case, the x-axis is a horizontal asymptote.

Characteristics of the Graph of $f(x) = a^x$

1. The points $\left(-1, \frac{1}{a}\right)$, $(0,1)$, and $(1,a)$ are on the graph.

2. If $a > 1$, then f is an increasing function. If $0 < a < 1$, then f is a decreasing function.

3. The x-axis is a horizontal asymptote.

4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

Homework 1 GRAPHING AN EXPONENTIAL FUNCTION

Graph $f(x) = 5^x$. Give the domain and range.

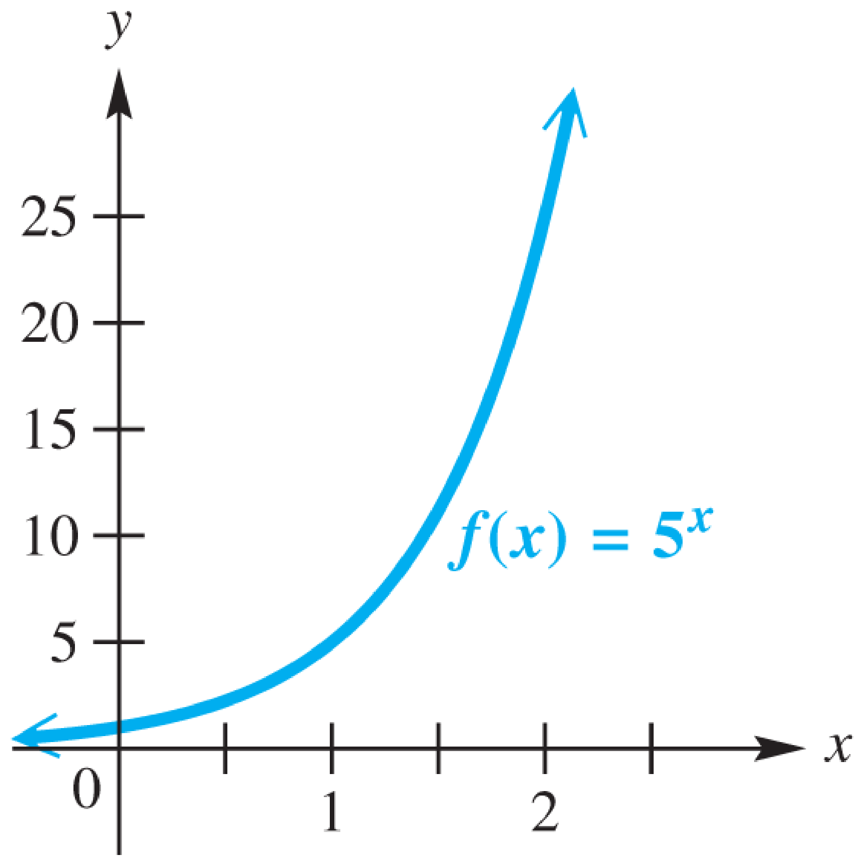
Solution The *y*-intercept is 1, and the *x*-axis is a horizontal asymptote. Plot a few ordered pairs, and draw a smooth curve through them. Like the function $f(x) = 2^x$, this function also has domain $(-\infty, \infty)$ and range $(0, \infty)$ and is one-to-one. The function is increasing on its entire domain.

GRAPHING AN EXPONENTIAL FUNCTION

Graph $f(x) = 5^x$. Give the domain and range.

Solution

x	$f(x)$
–1	0.2
0	1
0.5	≈ 2.2
1	5
1.5	≈ 11.2
2	25



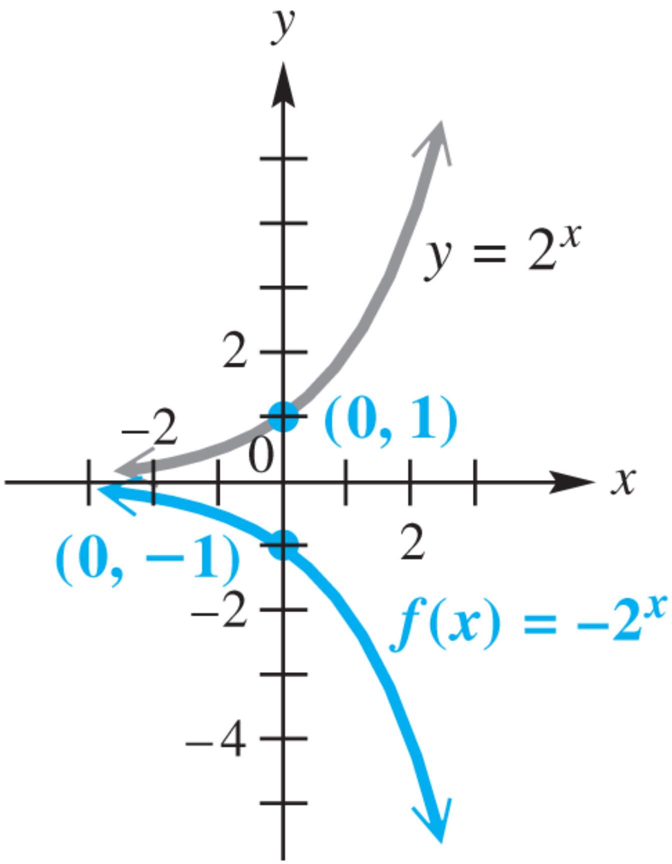
Example 2 GRAPHING REFLECTIONS AND TRANSLATIONS

Graph each function. Show the graph of $y = 2^x$ for comparison. Give the domain and range.

(a) $f(x) = -2^x$

Solution

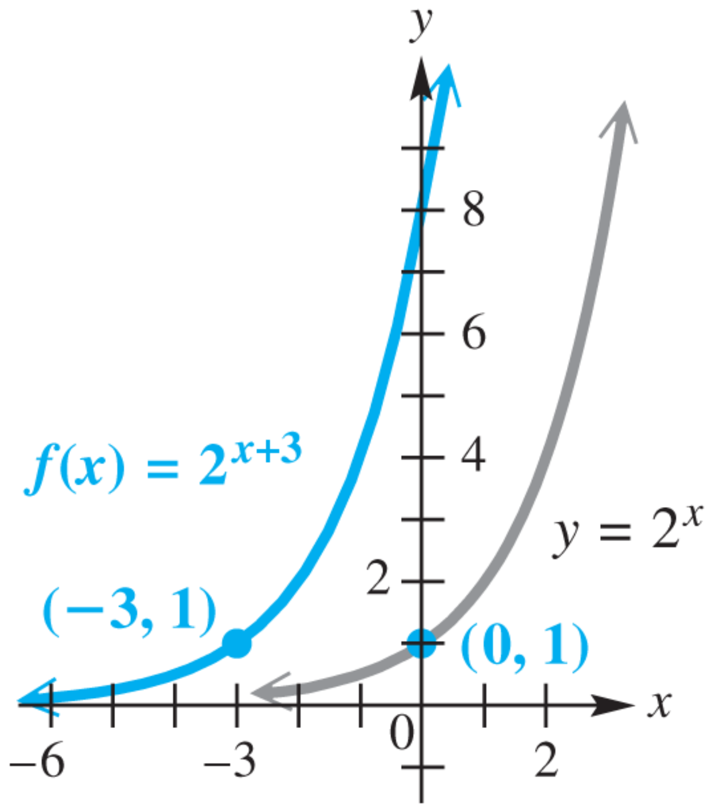
The graph of $f(x) = -2^x$ is that of $f(x) = 2^x$ reflected across the x-axis. The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0)$.



Example 2 GRAPHING REFLECTIONS AND TRANSLATIONS

Graph each function. Show the graph of $y = 2^x$ for comparison. Give the domain and range.

(b) $f(x) = 2^{x+3}$



Solution

The graph of $f(x) = 2^{x+3}$ is the graph of $f(x) = 2^x$ translated 3 units to the left. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

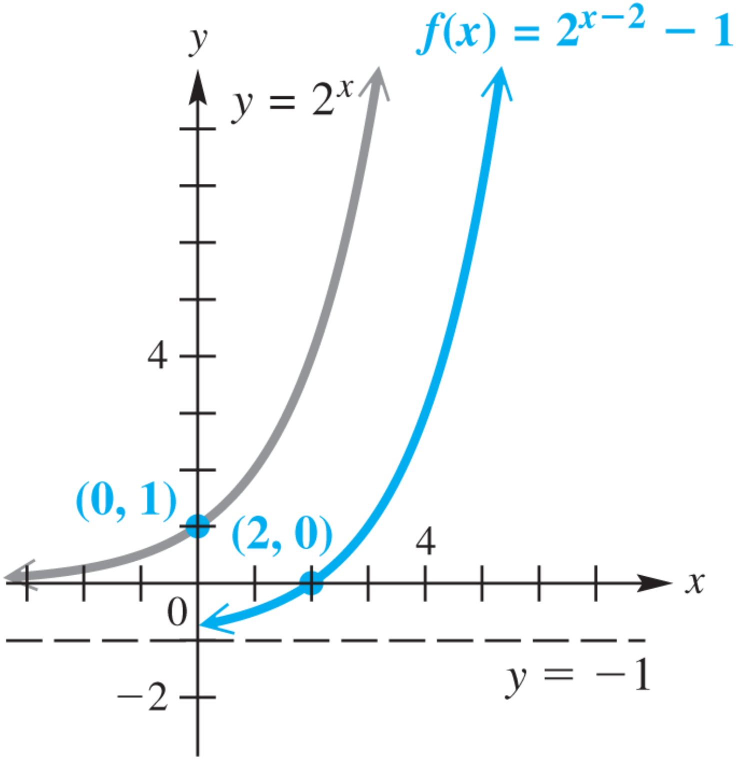
Example 2 GRAPHING REFLECTIONS AND TRANSLATIONS

Graph each function. Show the graph of $y = 2^x$ for comparison. Give the domain and range.

(c) $f(x) = 2^{x-2} - 1$

Solution

The graph of $f(x) = 2^{x-2} - 1$ is the graph of $f(x) = 2^x$ translated 2 units to the right and 1 unit down. The domain is $(-\infty,\infty)$, and the range is $(-1,\infty)$.



Characteristics of the Graph of $f(x) = a^{x-h} + k$, a>1

1. If $a > 1$, then f is an increasing function. If $0 < a < 1$, then f is a decreasing function.

2. The $y=k$ is a horizontal asymptote.

3. The domain is $(-\infty, \infty)$, and the range is (k, ∞).

Characteristics of the Graph of $f(x) = -a^{x-h}+k$,

1. If $a > 1$, then f is an decreasing function.
If $0 < a < 1$, then f is a increasing function.

2. The $y=k$ is a horizontal asymptote.

3. The domain is $(-\infty, \infty)$, and the range is $(-\infty,\mathbf{k})$.

Homework 2 SOLVING AN EXPONENTIAL EQUATION

Solve $\left(\frac{1}{3}\right)^x = 81.$

Solution ***Write each side of the equation using a common base.***

$$\left(\frac{1}{3}\right)^x = 81$$

$$\left(3^{-1}\right)^x = 81 \qquad \text{Definition of negative exponent.}$$

Homework 2 SOLVING AN EXPONENTIAL EQUATION

Solve $\left(\frac{1}{3}\right)^x = 81.$

Solution

$3^{-x} = 81$ — $(a^m)^n = a^{mn}$

$3^{-x} = 3^4$ — Write 81 as a power of 3.

$-x = 4$ — Set exponents equal.

$x = -4$ — Multiply by –1.

The solution set of the original equation is {–4}.

Example 3 SOLVING AN EXPONENTIAL EQUATION

Solve $2^{x+4} = 8^{x-6}$.

Solution Write each side of the equation using a common base.

$$2^{x+4} = 8^{x-6}$$

$$2^{x+4} = (2^3)^{x-6}$$ Write 8 as a power of 2.

$$2^{x+4} = 2^{3x-18}$$ $(a^m)^n = a^{mn}$

$$x + 4 = 3x - 18$$ Set exponents equal .

$$-2x = -22$$ Subtract 3*x* and 4.

$$x = 11$$ Divide by −2.

Check by substituting 11 for *x* in the original equation. The solution set is {11}.

Homework 3 SOLVING AN EQUATION WITH A FRACTIONAL EXPONENT

Solve $x^{4/3} = 81.$

Solution Notice that the variable is in the base rather than in the exponent.

$$x^{4/3} = 81$$

$$\left(\sqrt[3]{x}\right)^4 = 81 \qquad \text{Radical notation for } a^{m/n}$$

$$\sqrt[3]{x} = \pm 3 \qquad \text{Take fourth roots on each side. Remember to use } \pm.$$

$$x = \pm 27 \qquad \text{Cube each side.}$$

Check *both* solutions in the original equation. Both check, so the solution set is $\{\pm 27\}$.

Homework 3 SOLVING AN EQUATION WITH A FRACTIONAL EXPONENT

Solve $x^{4/3} = 81.$

Solution *Alternative Method* There may be more than one way to solve an exponential equation, as shown here.

$$x^{4/3} = 81$$

$$\left(x^{4/3}\right)^3 = 81^3 \qquad \text{Cube each side.}$$

$$x^4 = \left(3^4\right)^3 \qquad \text{Write 81 as } 3^4.$$

$$x^4 = 3^{12} \qquad (a^m)^n = a^{mn}$$

$$x = \pm\sqrt[4]{3^{12}} \qquad \text{Take fourth roots on each side.}$$

$$x = \pm 3^3 \qquad \text{Simplify the radical.}$$

$$x = \pm 27 \qquad \text{Apply the exponent.}$$

The same solution set, {± 27}, results.

Compound Interest ملغي

Recall the formula for simple interest, $I = Prt$, where P is principal (amount deposited), r is annual rate of interest expressed as a decimal, and t is time in years that the principal earns interest. Suppose $t = 1$ yr. Then at the end of the year the amount has grown to

$$P + Pr = P(1+r),$$

the original principal plus interest. If this balance earns interest at the same interest rate for another year, the balance at the end of that year will be

$$[P(1+r)] + [P(1+r)]r = [P(1+r)](1+r) \quad \text{Factor.}$$

$$= P(1+r)^2.$$

Compound Interest

After the third year, this will grow to

$$[P(1+r)^2]+[P(1+r)^2]r=[P(1+r)^2](1+r) \quad \text{Factor.}$$

$$=P(1+r)^3.$$

Continuing in this way produces a formula for interest compounded annually.

$$A=P(1+r)^t$$

Compound Interest

If *P* dollars are deposited in an account paying an annual rate of interest *r* compounded (paid) *n* times per year, then after *t* years the account will contain *A* dollars, according to the following formula.

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$

Example 7 USING THE COMPOUND INTEREST FORMULA

Suppose \$1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

(a) Find the amount in the account after 10 yr with no withdrawals.

Solution

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$

Compound interest formula

$$A = 1000\left(1+\frac{0.04}{4}\right)^{10(4)}$$

Let P = 1000, r = 0.04, n = 4, and t = 10.

Example 7 USING THE COMPOUND INTEREST FORMULA

Suppose \$1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

(a) Find the amount in the account after 10 yr with no withdrawals.

Solution

$$A = 1000(1 + 0.01)^{40}$$ Simplify.

$$A = 1488.86$$ Round to the nearest cent.

Thus, \$1488.86 is in the account after 10 yr.

Example 7 USING THE COMPOUND INTEREST FORMULA

Suppose $1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

(b) How much interest is earned over the 10-yr period?

Solution The interest earned for that period is

$$\$1488.86 - \$1000 = \$488.86.$$

Example 8 FINDING PRESENT VALUE

Becky Anderson must pay a lump sum of $6000 in 5 yr.

(a) What amount deposited today (present value) at 3.1% compounded annually will grow to $6000 in 5 yr?

Solution

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$

Compound interest formula

$$6000 = P\left(1+\frac{0.031}{1}\right)^{5(1)}$$

Let A = 6000, r = 0.031, n = 1, and t =5.

$$6000 = P(1.031)^{5}$$

Simplify.

$$P \approx 5150.60$$

Use a calculator.

Example 8 FINDING PRESENT VALUE

Becky Anderson must pay a lump sum of $6000 in 5 yr.

(a) What amount deposited today (present value) at 3.1% compounded annually will grow to $6000 in 5 yr?

Solution

If Becky leaves $5150.60 for 5 yr in an account paying 3.1% compounded annually, she will have $6000 when she needs it. We say that $5150.60 is the present value of $6000 if interest of 3.1% is compounded annually for 5 yr.

Example 8 FINDING PRESENT VALUE

Becky Anderson must pay a lump sum of $6000 in 5 yr.

(b) If only $5000 is available to deposit now, what annual interest rate is necessary for the money to increase to $6000 in 5 yr?

Solution

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$

Compound interest formula

$$6000 = 5000(1+r)^5$$

Let $A = 6000$, $P = 5000$, $n = 1$, and $t = 5$.

Example 8 FINDING PRESENT VALUE

Becky Anderson must pay a lump sum of \$6000 in 5 yr.

(b) If only \$5000 is available to deposit now, what annual interest rate is necessary for the money to increase to \$6000 in 5 yr?

Solution

$$\frac{6}{5} = (1+r)^5$$

Divide by 5000.

$$\left(\frac{6}{5}\right)^{1/5} = 1+r$$

Take the fifth root on each side.

Example 8 FINDING PRESENT VALUE

Becky Anderson must pay a lump sum of \$6000 in 5 yr.

(b) If only \$5000 is available to deposit now, what annual interest rate is necessary for the money to increase to \$6000 in 5 yr?

Solution

$$\left(\frac{6}{5}\right)^{1/5} - 1 = r$$ Subtract 1.

$$r \approx 0.0371$$ Use a calculator.

An interest rate of 3.71% will produce enough interest to increase the \$5000 to \$6000 by the end of 5 yr.

Continuous Compounding

The more often interest is compounded within a given time period, the more interest will be earned. Surprisingly, however, there is a limit on the amount of interest, no matter how often it is compounded.

Continuous Compounding

Suppose that \$1 is invested at 100% interest per year, compounded n times per year. Then the interest rate (in decimal form) is 1.00 and the interest rate per period is $\frac{1}{n}$. According to the formula (with $P = 1$), the compound amount at the end of 1 yr will be

$$A = \left(1 + \frac{1}{n}\right)^n.$$

Continuous Compounding

A calculator gives the results shown for various values of *n*. The table suggests that as *n* increases, the value of $\left(1+\frac{1}{n}\right)^n$ gets closer and closer to some fixed number. This is indeed the case. This fixed number is called e. ***(Note that in mathematics, e is a real number and not a variable.)***

n	$\left(1+\frac{1}{n}\right)^n$ (rounded)
1	2
2	2.25
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
1,000,000	2.71828

Value of e

$$e \approx 2.718281828459045$$

Continuous Compounding

If P dollars are deposited at a rate of interest r compounded continuously for t years, the compound amount A in dollars on deposit is given by the following formula.

$$A = Pe^{rt}$$

Example 9 SOLVING A CONTINUOUS COMPOUNDING PROBLEM

Suppose $5000 is deposited in an account paying 3% interest compounded continuously for 5 yr. Find the total amount on deposit at the end of 5 yr.

Solution

$$A = Pe^{rt}$$ Continuous compounding formula

$$= 5000e^{0.03(5)}$$ Let $P = 5000$, $r = 0.03$, and $t = 5$.

$$= 5000e^{0.15}$$ Multiply exponents.

$$A \approx 5809.17 \quad \text{or} \quad \$5809.17$$ Use a calculator.

Check that daily compounding would have produced a compound amount about $0.03 less.

Example 10 COMPARING INTEREST EARNED AS COMPOUNDING IS MORE FREQUENT

In **Example 7,** we found that \$1000 invested at 4% compounded quarterly for 10 yr grew to \$1488.86. Compare this same investment compounded annually, semiannually, monthly, daily, and continuously.

Solution

Substitute 0.04 for *r*, 10 for *t*, and the appropriate number of compounding periods for *n* into

$$A = P\left(1+\frac{r}{n}\right)^{tn}$$

Compound interest formula

and also into

$$A = Pe^{rt}.$$

Continuous compounding formula

The results for amounts of \$1 and \$1000 are given in the table.

Example 10 COMPARING INTEREST EARNED AS COMPOUNDING IS MORE FREQUENT

Compounded	$1	$1000
Annually	$(1+0.04)^{10} \approx 1.48024$	$1480.24
Semiannually	$\left(1+\dfrac{0.04}{2}\right)^{10(2)} \approx 1.48595$	$1485.95
Quarterly	$\left(1+\dfrac{0.04}{4}\right)^{10(4)} \approx 1.48886$	$1488.86
Monthly	$\left(1+\dfrac{0.04}{12}\right)^{10(12)} \approx 1.49083$	$1490.83
Daily	$\left(1+\dfrac{0.04}{365}\right)^{10(365)} \approx 1.49179$	$1491.79
Continuously	$e^{10(0.04)} \approx 1.49182$	$1491.82

Example 10 COMPARING INTEREST EARNED AS COMPOUNDING IS MORE FREQUENT

Comparing the results, we notice the following.

- Compounding semiannually rather than annually increases the value of the account after 10 yr by $5.71.
- Quarterly compounding grows to $2.91 more than semiannual compounding after 10 yr.
- Daily compounding yields only $0.96 more than monthly compounding.
- Continuous compounding yields only $0.03 more than monthly compounding.

Each increase in compounding frequency earns less and less additional interest.

Exponential Models

The number e is important as the base of an exponential function in many practical applications. In situations involving growth or decay of a quantity, the amount or number present at time t often can be closely modeled by a function of the form

$$y = y_0 e^{kt},$$

where y_0 is the amount or number present at time $t = 0$ and k is a constant.

Example 11 USING DATA TO MODEL EXPONENTIAL GROWTH

Data from recent past years indicate that future amounts of carbon dioxide in the atmosphere may grow according to the table. Amounts are given in parts per million.

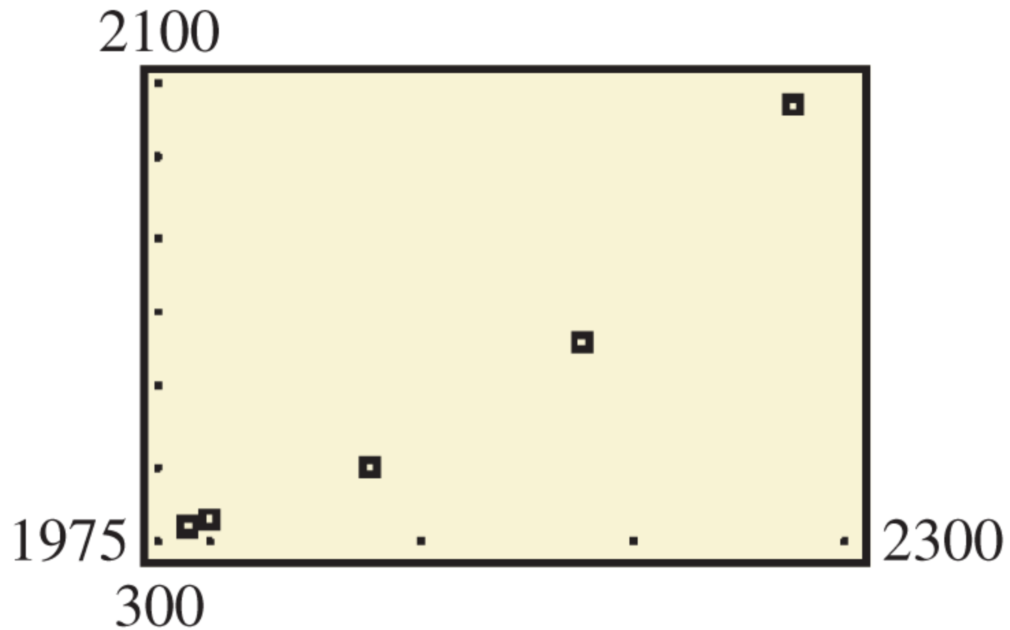
Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

Example 11 USING DATA TO MODEL EXPONENTIAL GROWTH

(a) Make a scatter diagram of the data. Do the carbon dioxide levels appear to grow exponentially?

Solution

The data appear to resemble the graph of an increasing exponential function.



Example 11 **USING DATA TO MODEL EXPONENTIAL GROWTH**

(b) One model for the data is the function

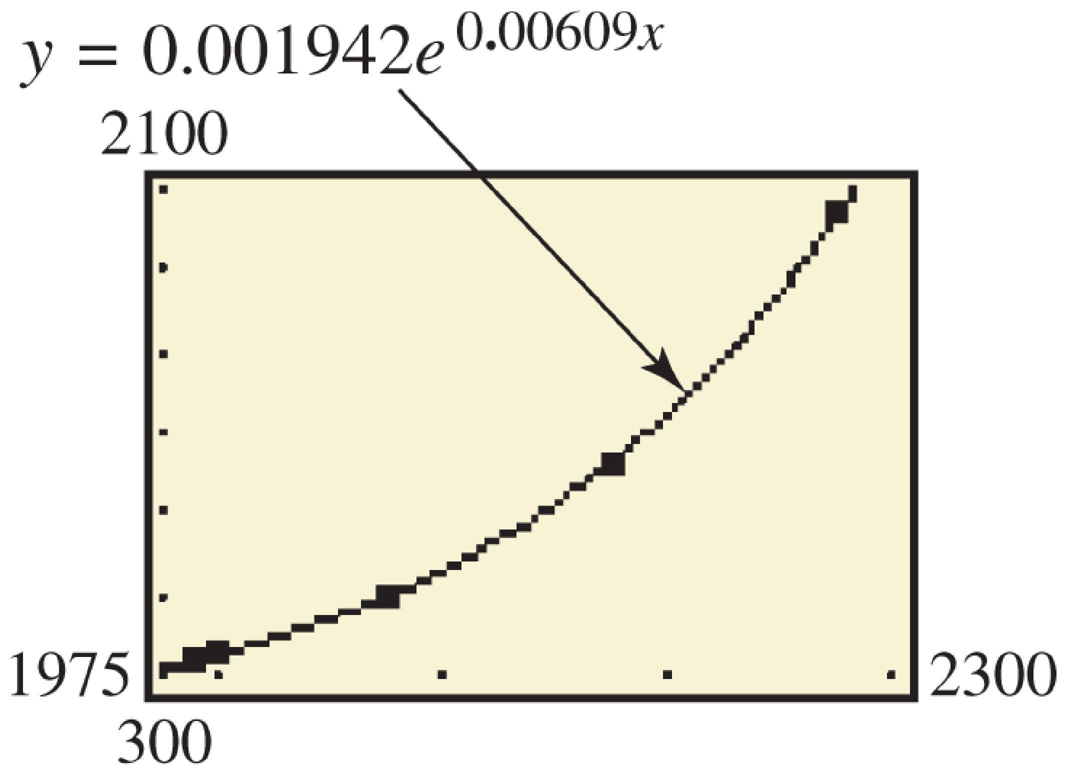
$$y = 0.001942e^{0.00609x},$$

where x is the year and $1990 \le x \le 2275$. Use a graph of this model to estimate when future levels of carbon dioxide will double and triple over the preindustrial level of 280 ppm.

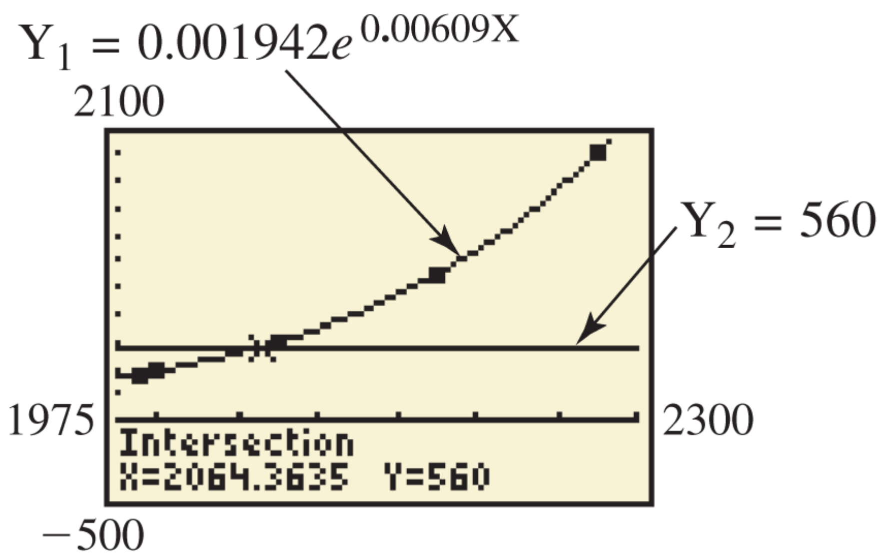
Example 11 **USING DATA TO MODEL EXPONENTIAL GROWTH**

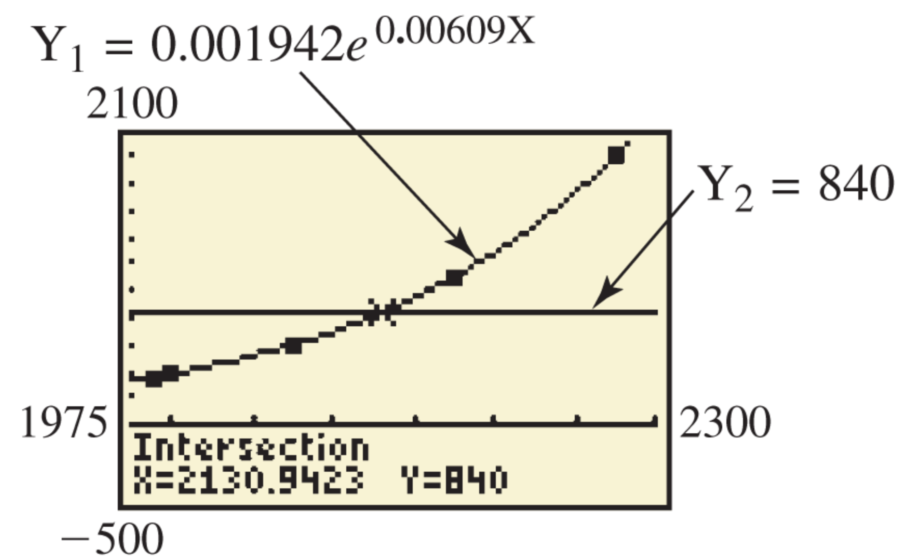
(b) Solution

A graph of $y = 0.001942e^{0.00609x}$ shows that it is very close to the data points.

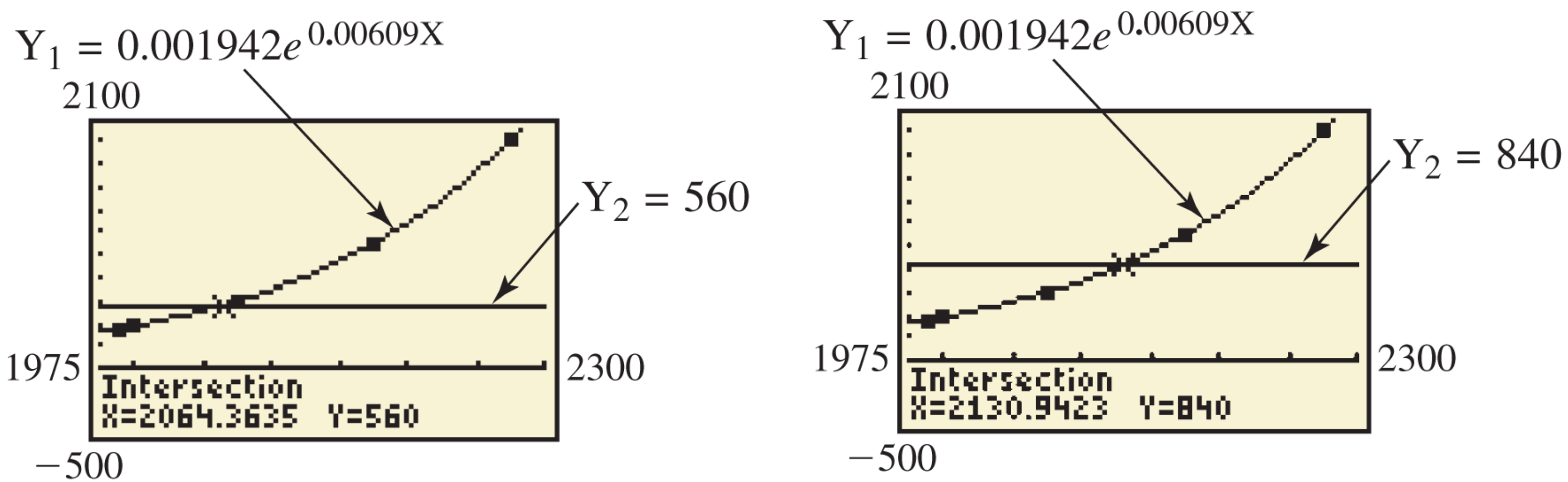


(b) We graph $y = 2\cdot280 = 560$ and $y = 3\cdot280 = 840$ on the same coordinate axes as the given function, and we use the calculator to find the intersection points.



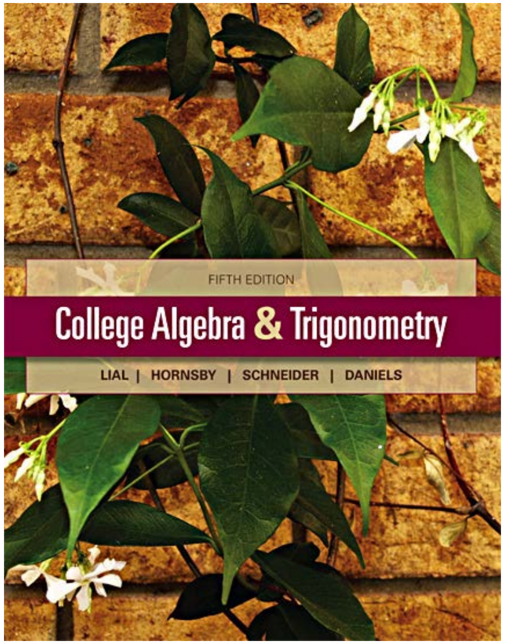


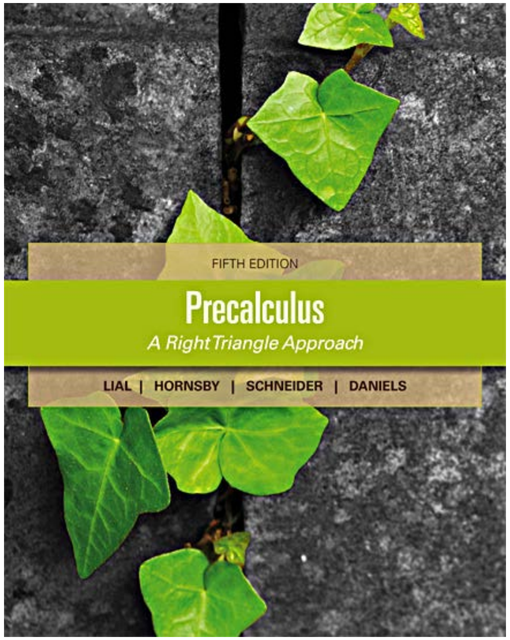
(b) The graph of the function intersects the horizontal lines at approximately 2064.4 and 2130.9. According to this model, carbon dioxide levels will have doubled by 2064 and tripled by 2131.



5

Inverse, Exponential, and Logarithmic Functions





5.3 Logarithmic Functions

- Logarithms
- Logarithmic Equations
- Logarithmic Functions
- Properties of Logarithms

Logarithms

The previous section dealt with exponential functions of the form $y = a^x$ for all positive values of a, where $a \neq 1$. The horizontal line test shows that exponential functions are one-to-one, and thus have inverse functions.

Logarithms

The equation defining the inverse of a function is found by interchanging x and y in the equation that defines the function. Starting with $y = a^x$ and interchanging x and y yields

$$x = a^y.$$

Logarithms

$$x = a^y$$

Here y is the exponent to which a must be raised in order to obtain x. We call this exponent a **logarithm,** symbolized by the abbreviation **"log."** The expression $\log_a x$ represents the logarithm in this discussion. The number a is called the **base** of the logarithm, and x is called the **argument** of the expression. It is read **"logarithm with base *a* of *x*,"** or **"logarithm of *x* with base *a*."**

Logarithm

For all real numbers *y* and all positive numbers *a* and *x*, where $a \neq 1$,

$$y = \log_a x \qquad \text{if and only if} \qquad x = a^y.$$

The expression* $\log_a x$ *represents the exponent to which the base a must be raised in order to obtain x.

Example 1 WRITING EQUIVALENT LOGARITHMIC AND EXPONENTIAL FORMS

The table shows several pairs of equivalent statements, written in both logarithmic and exponential forms.

Logarithmic Form	Exponential Form
$\log_2 8 = 3$	$2^3 = 8$
$\log_{1/2} 16 = -4$	$\left(\frac{1}{2}\right)^{-4} = 16$
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_3 \frac{1}{81} = -4$	$3^{-4} = \frac{1}{81}$
$\log_5 5 = 1$	$5^1 = 5$
$\log_{3/4} 1 = 0$	$\left(\frac{3}{4}\right)^0 = 1$

Example 1 WRITING EQUIVALENT LOGARITHMIC AND EXPONENTIAL FORMS

To remember the relationships among *a*, *x*, and *y* in the two equivalent forms $y = \log_a x$ and $x = a^y$, refer to these diagrams.

Exponent

Logarithmic form: $y = \log_a x$

Base

Exponent

Exponential form: $a^y = x$

Base

Homework 1 SOLVING LOGARITHMIC EQUATIONS

Solve each equation.

(a) $\log_x \frac{8}{27} = 3$

Solution $\log_x \frac{8}{27} = 3$

$x^3 = \frac{8}{27}$ Write in exponential form.

$x^3 = \left(\frac{2}{3}\right)^3$ $\frac{8}{27} = \left(\frac{2}{3}\right)^3$

$x = \frac{2}{3}$ Take cube roots

Homework 1 **SOLVING LOGARITHMIC EQUATIONS**

Check $\log_x \frac{8}{27} = 3$ Original equation

$\log_{2/3} \frac{8}{27} \stackrel{?}{=} 3$ Let x = 2/3.

$\left(\frac{2}{3}\right)^3 \stackrel{?}{=} \frac{8}{27}$ Write in exponential form.

$\frac{8}{27} = \frac{8}{27}$ True

The solution set is $\left\{\frac{2}{3}\right\}$.

Homework 1 SOLVING LOGARITHMIC EQUATIONS

Solve each equation.

(b) $\log_4 x = \frac{5}{2}$

Solution

$$\log_4 x = \frac{5}{2}$$

$$4^{5/2} = x$$ Write in exponential form.

$$(4^{1/2})^5 = x$$ $a^{mn} = (a^m)^n$

$$2^5 = x$$ $4^{1/2} = (2^2)^{1/2} = 2$

$$32 = x$$ Apply the exponent.

The solution set is {32}.

SOLVING LOGARITHMIC EQUATIONS

Solve each equation.

(c) $\log_{49} \sqrt[3]{7} = x$

Solution

$$49^x = \sqrt[3]{7}$$ Write in exponential form.

$$(7^2)^x = 7^{1/3}$$ Write with the same base.

$$7^{2x} = 7^{1/3}$$ Power rule for exponents.

$$2x = \frac{1}{3}$$ Set exponents equal.

$$x = \frac{1}{6}$$ Divide by 2.

The solution set is $\left\{\frac{1}{6}\right\}$.

Logarithmic Function

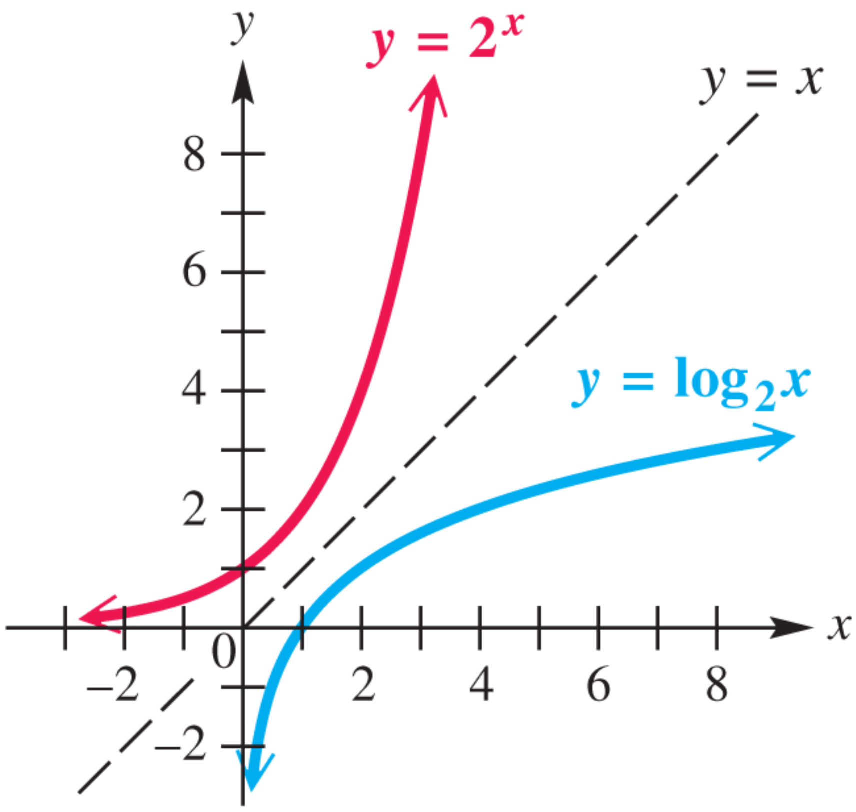
If $a > 0$, $a \neq 1$, and $x > 0$, then

$$f(x) = \log_a x$$

defines the **logarithmic function with base *a*.**

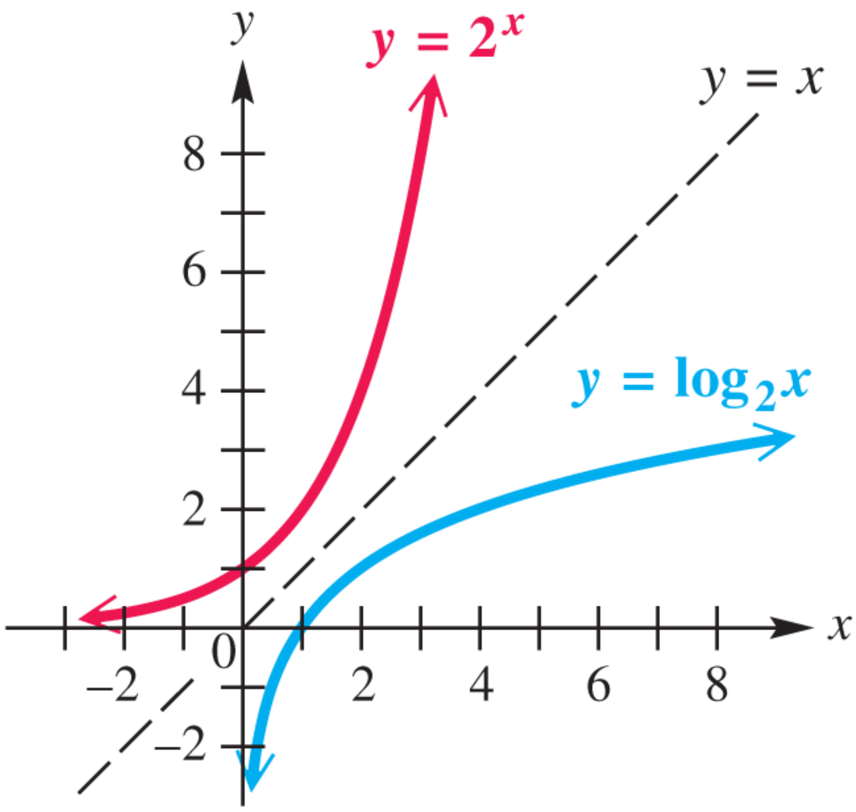
Logarithmic Functions

Exponential and logarithmic functions are inverses of each other. The graph of $y = 2^x$ is shown in red. The graph of its inverse is found by reflecting the graph of $y = 2^x$ across the line $y = x$.



Logarithmic Functions

The graph of the inverse function, defined by $y = \log_2 x$, shown in blue, has the y-axis as a vertical asymptote.



Logarithmic Functions

Since the domain of an exponential function is the set of all real numbers, the range of a logarithmic function also will be the set of all real numbers. In the same way, both the range of an exponential function and the domain of a logarithmic function are the set of all positive real numbers.

Thus, logarithms can be found for positive numbers only.

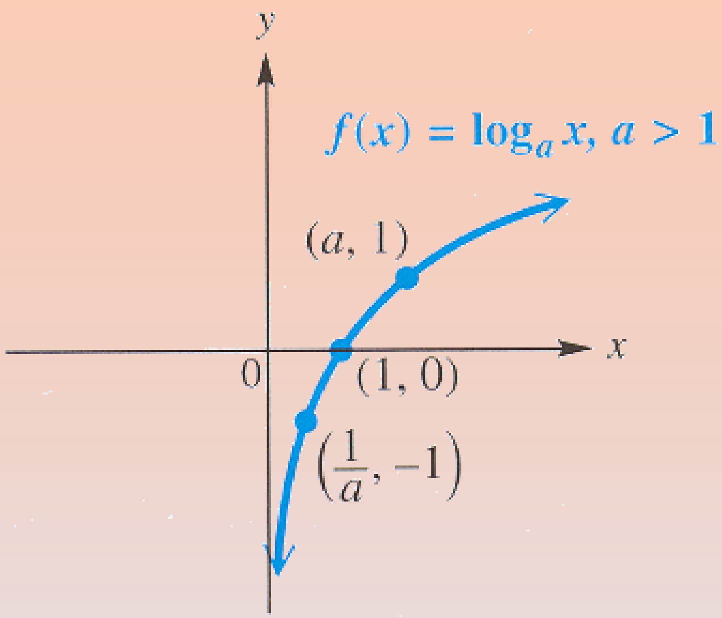
LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

x	$f(x)$
¼	−2
½	−1
1	0
2	1
4	2
8	3



- $f(x) = \log_a x$, for $a > 1$, is increasing and continuous on its entire domain, $(0, \infty)$.

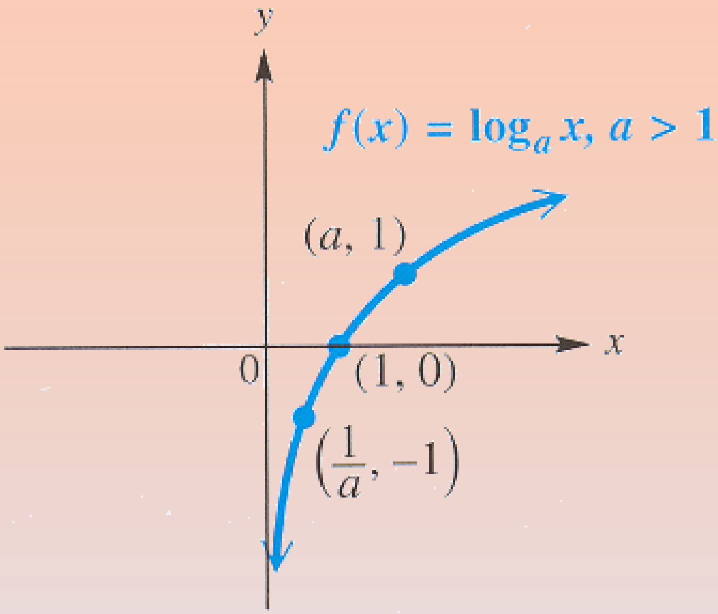
LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

x	$f(x)$
¼	−2
½	−1
1	0
2	1
4	2
8	3



- The y-axis is a vertical asymptote as $x \to 0$ from the right.

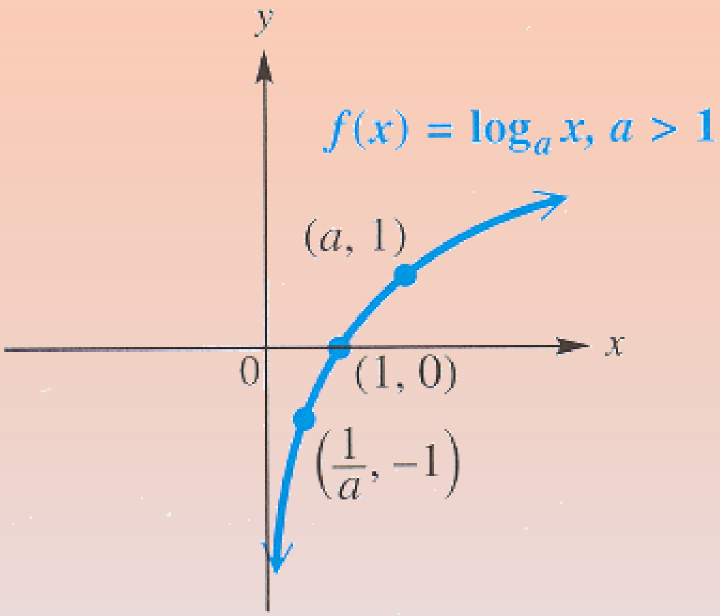
LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

x	$f(x)$
¼	−2
½	−1
1	0
2	1
4	2
8	3



- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

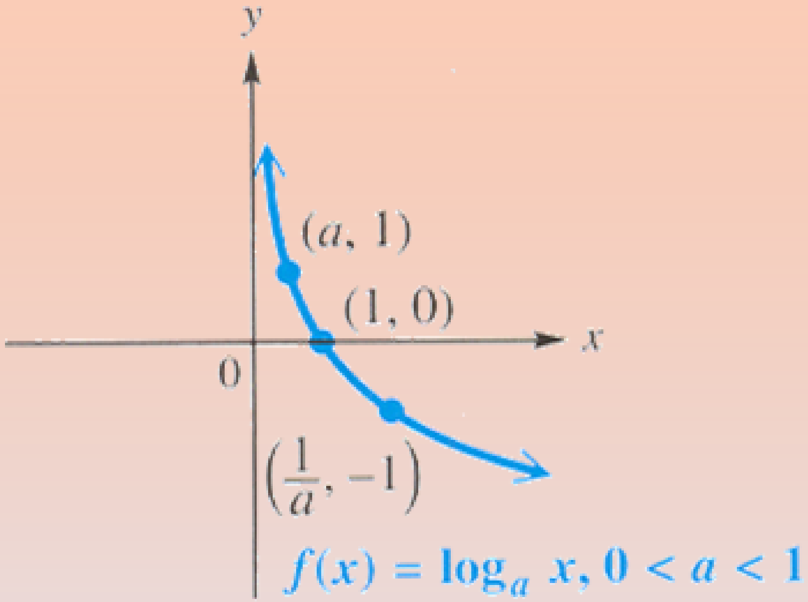
LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

For $f(x) = \log_{1/2} x$:

x	$f(x)$
¼	2
½	1
1	0
2	−1
4	−2
8	−3



- $f(x) = \log_a x$, for $0 < a < 1$, is decreasing and continuous on its entire domain, $(0, \infty)$.

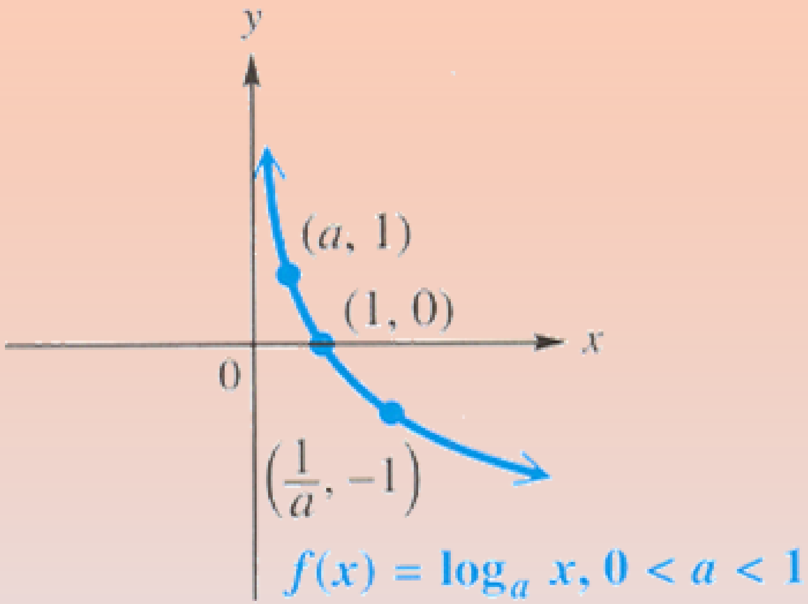
LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

For $f(x) = \log_{1/2} x$:

x	$f(x)$
¼	2
½	1
1	0
2	−1
4	−2
8	−3



- The y-axis is a vertical asymptote as $x \to 0$ from the right.

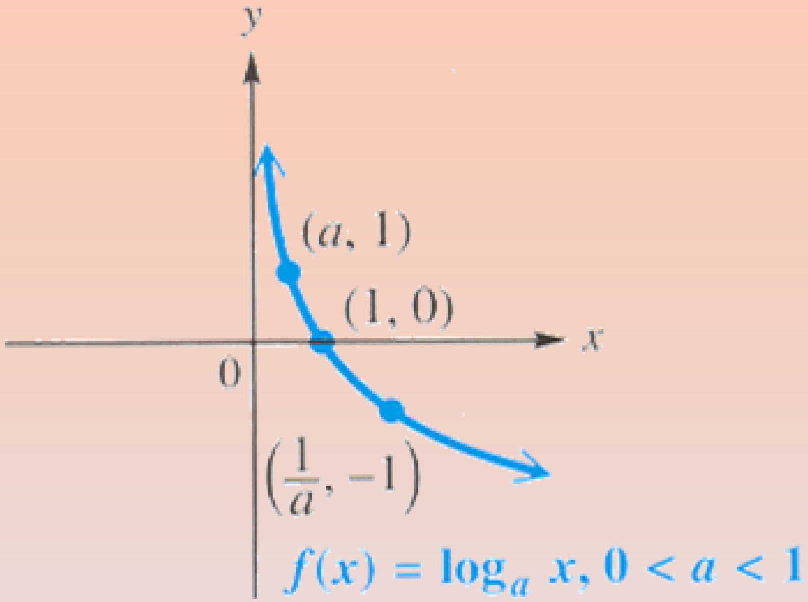
LOGARITHMIC FUNCTION $f(x) = \log_a x$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

For $f(x) = \log_{1/2} x$:

x	$f(x)$
¼	2
½	1
1	0
2	−1
4	−2
8	−3



- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1,0)$, and $(a,1)$.

Characteristics of the Graph of $f(x) = \log_a x$

1. The points $\left(\frac{1}{a}, -1\right)$, $(1,0)$, and $(a,1)$ are on the graph.
2. If $a > 1$, then f is an increasing function. If $0 < a < 1$, then f is a decreasing function.
3. The y-axis is a vertical asymptote.
4. The domain is $(0,\infty)$, and the range is $(-\infty, \infty)$.

Characteristics of the Graph of $f(x)=\log_a(x-h)+K$

1. The points $\left(\frac{1}{a}+h,-1+k\right)$, $(1+h,0+k)$, and $(a+h,1+$ are on the graph.

2. If $a > 1$, then f is an increasing function. If $0 < a < 1$, then f is a decreasing function.

3. The x=h is a vertical asymptote.

4. The domain is (h,∞), and the range is $(-\infty, \infty)$.

Example 2 GRAPHING LOGARITHMIC FUNCTIONS

Graph each function.

(a) $f(x) = \log_{1/2} x$

Solution

First graph $y = (½)^x$ which defines the inverse function of f, by plotting points. The graph of $f(x) = \log_{1/2} x$ is the reflection of the graph of $y = (½)^x$ across the line $y = x$. The ordered pairs for $y = \log_{1/2} x$ are found by interchanging the x- and y-values in the ordered pairs for $y = (½)^x$.

Example 2 GRAPHING LOGARITHMIC FUNCTIONS

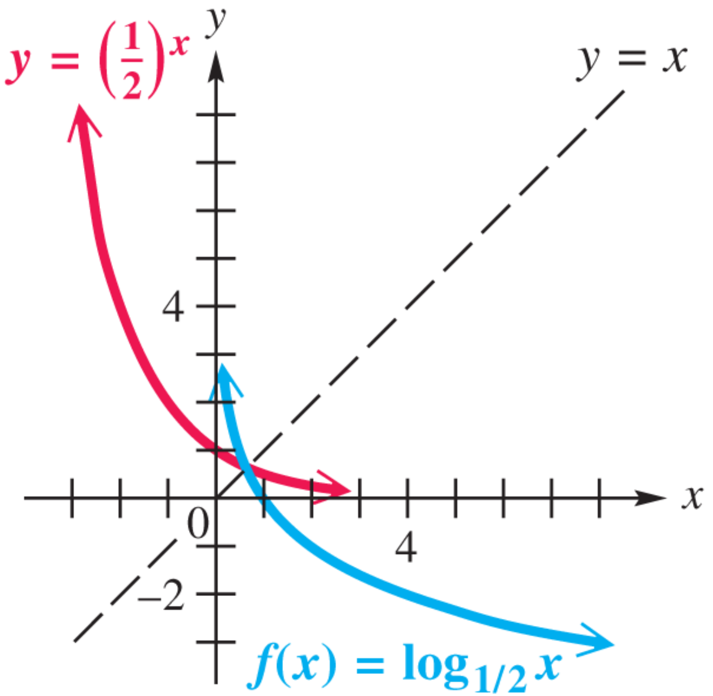
Graph each function.

(a) $f(x) = \log_{1/2} x$

Solution

x	$y = \left(\frac{1}{2}\right)^x$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
4	$\frac{1}{16}$

x	$f(x) = \log_{1/2} x$
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2
$\frac{1}{16}$	4



Example 2 GRAPHING LOGARITHMIC FUNCTIONS

Graph each function.

(b) $f(x) = \log_3 x$

Solution

Another way to graph a logarithmic function is to write $f(x) = y = \log_3 x$ in exponential form as $x = 3^y$, and then select y-values and calculate corresponding x-values.

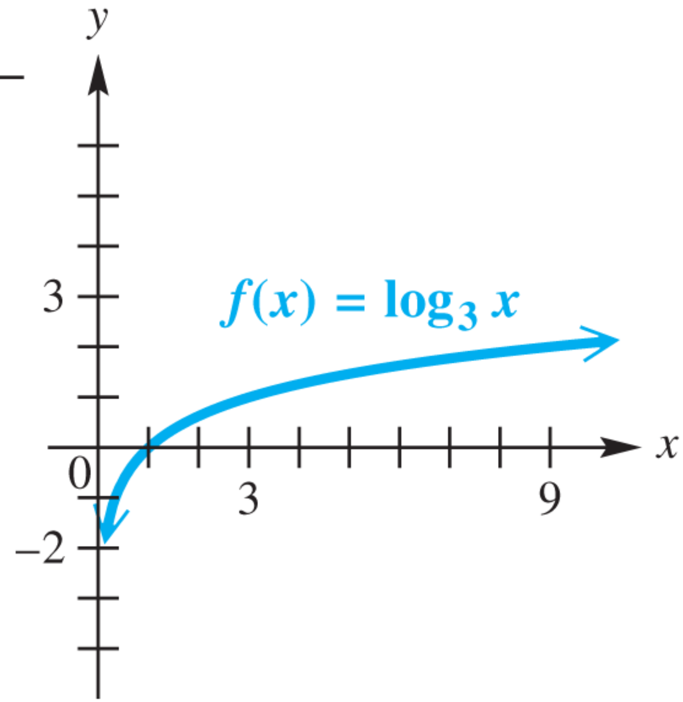
Graph each function.

(b) $f(x) = \log_3 x$

Solution

x	$f(x) = \log_3 x$
$\frac{1}{3}$	-1
1	0
3	1
9	2

Think: $x = 3^y$



Caution If you write a logarithmic function in exponential form to graph, as in **Example 3(b),** start *first* with *y*-values to calculate corresponding *x*-values. ***Be careful to write the values in the ordered pairs in the correct order.***

Homework 2

GRAPHING TRANSLATED LOGARITHMIC FUNCTIONS

Graph each function. Give the domain and range.

(a) $f(x) = \log_2(x-1)$

Solution

The graph of $f(x) = \log_2(x - 1)$ is the graph of $f(x) = \log_2 x$ translated 1 unit to the right. The vertical asymptote has equation $x = 1$. Since logarithms can be found only for positive numbers, we solve $x - 1 > 0$ to find the domain, $(1,\infty)$. To determine ordered pairs to plot, use the equivalent exponential form of the equation $y = \log_2 (x - 1)$.

GRAPHING TRANSLATED LOGARITHMIC FUNCTIONS

Graph each function. Give the domain and range.

(a) $f(x) = \log_2(x-1)$

Solution

$$y = \log_2(x-1)$$

$$x - 1 = 2^y$$

Write in exponential form.

$$x = 2^y + 1$$

Add 1.

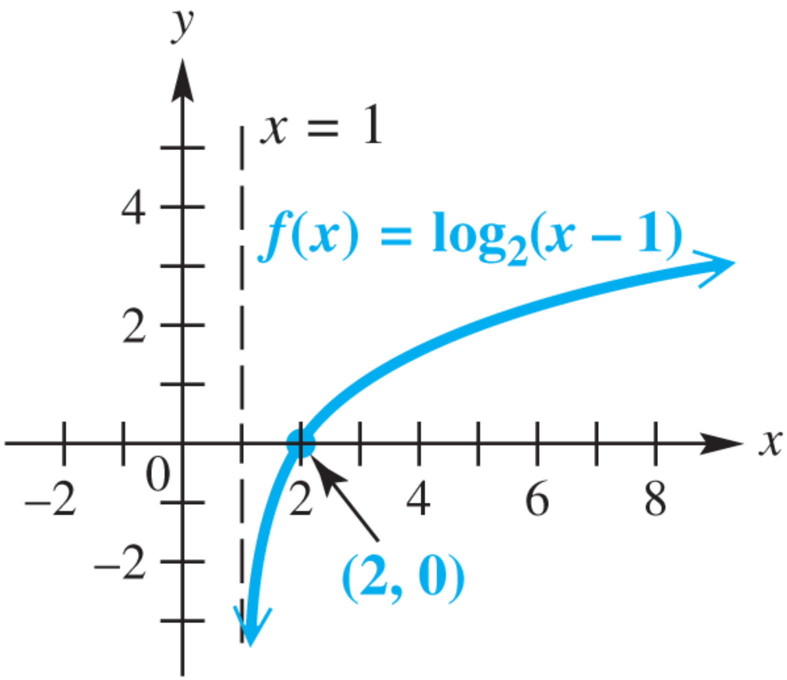
GRAPHING TRANSLATED LOGARITHMIC FUNCTIONS

Graph each function. Give the domain and range.

(a) $f(x) = \log_2(x - 1)$

Solution

We first choose values for y and then calculate each of the corresponding x-values. The range is $(-\infty, \infty)$.



GRAPHING TRANSLATED LOGARITHMIC FUNCTIONS

Graph each function. Give the domain and range.

(b) $f(x) = (\log_3 x) - 1$

Solution

The function $f(x) = (\log_3 x) - 1$ has the same graph as $g(x) = \log_3 x$ translated 1 unit down. We find ordered pairs to plot by writing the equation $y = (\log_3 x) - 1$ in exponential form.

GRAPHING TRANSLATED LOGARITHMIC FUNCTIONS

Graph each function. Give the domain and range.

(b) $f(x) = (\log_3 x) - 1$

Solution

$$y = (\log_3 x) - 1$$

$$y + 1 = \log_3 x$$

Add 1.

$$x = 3^{y+1}$$

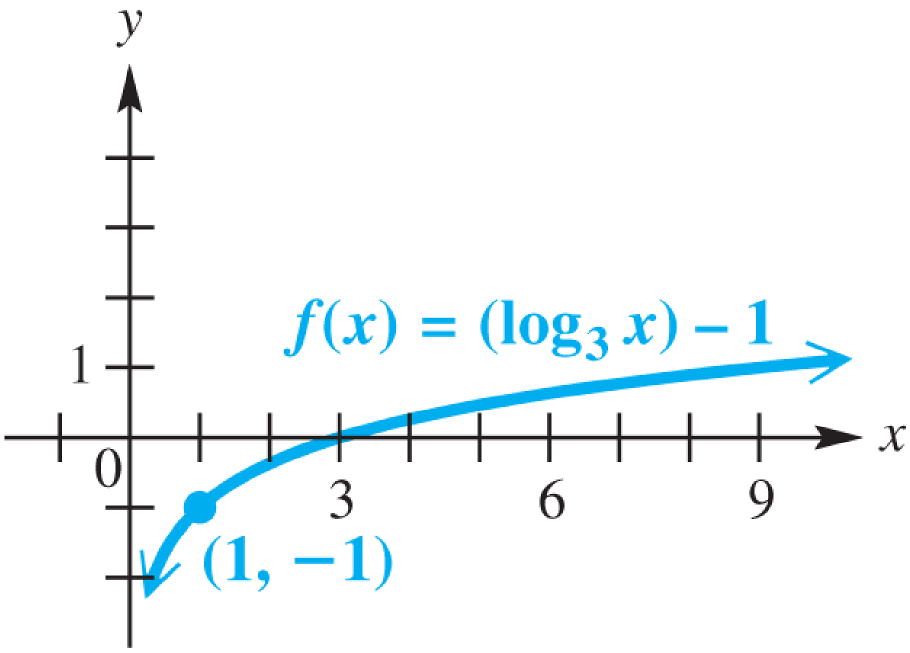
Write in exponential form.

Graph each function. Give the domain and range.

(b) $f(x) = (\log_3 x) - 1$

Solution

Again, choose *y*-values and calculate the corresponding *x*-values. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.



Graph each function. Give the domain and range.

(c) $f(x) = \log_4(x+2)+1$

Solution

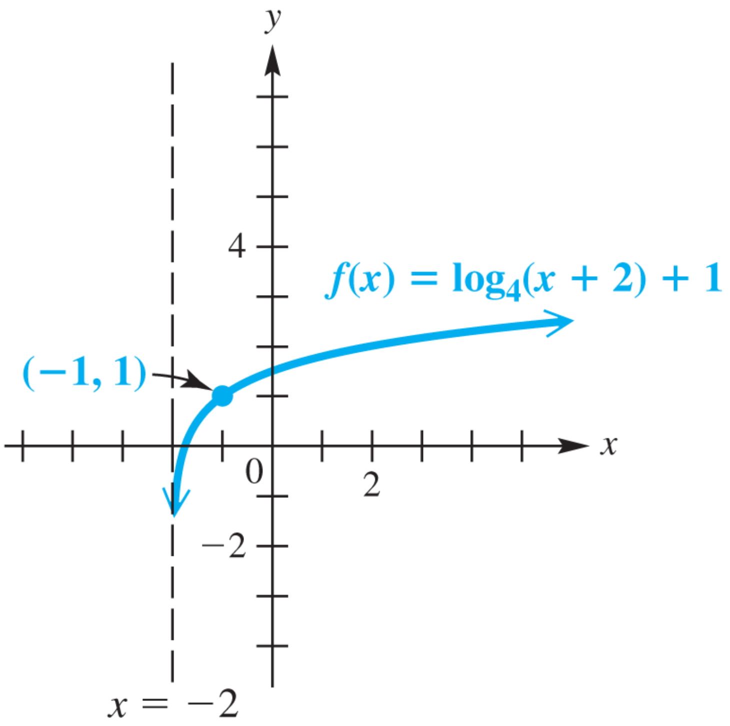
The graph of $f(x) = \log_4(x + 2) + 1$ is obtained by shifting the graph of $y = \log_4 x$ to the left 2 units and up 1 unit. The domain is found by solving $x + 2 > 0$, which yields $(–2, \infty)$. The vertical asymptote has been shifted to the left 2 units as well, and it has equation $x = –2$. The range is unaffected by the vertical shift and remains $(–\infty, \infty)$.

Homework 2 GRAPHING TRANSLATED LOGARITHMIC FUNCTIONS

Graph each function. Give the domain and range.

(c) $f(x) = \log_4(x+2)+1$

Solution



Properties of Logarithms

The properties of logarithms enable us to change the form of logarithmic statements so that products can be converted to sums, quotients can be converted to differences, and powers can be converted to products.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r, the following properties hold.

Property	**Description**
Product Property $\log_a xy = \log_a x + \log_a y$	The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r, the following properties hold.

Property	**Description**
Quotient Property $\log_a \frac{x}{y} = \log_a x - \log_a y$	The logarithm of the quotient of two numbers is equal to the difference between the logarithms of the numbers.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r, the following properties hold.

Property	**Description**
Power Property $\log_a x^r = r \log_a x$	The logarithm of a number raised to a power is equal to the exponent multiplied by the logarithm of the number.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r, the following properties hold.

Property	**Description**
Logarithm of 1 $\log_a 1 = 0$	The base a logarithm of 1 is 0.
Base *a* Logarithm of *a* $\log_a a = 1$	The base a logarithm of a is 1.

Example 3 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_6(7 \cdot 9)$

Solution

$$\log_6(7 \cdot 9) = \log_6 7 + \log_6 9$$ Product property

Example 3 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(b) $\log_9 \dfrac{15}{7}$

Solution

$$\log_9 \frac{15}{7} = \log_9 15 - \log_9 7$$ Quotient property

Example 3 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(c) $\log_5 \sqrt{8}$

Solution

$$\log_5 \sqrt{8} = \log_5 (8^{1/2}) = \frac{1}{2} \log_5 8 \qquad \text{Power property}$$

Example 3 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(d) $\log_a \dfrac{mnq}{p^2t^4}$

Solution

$$\log_a \frac{mnq}{p^2t^4} = \log_a m + \log_a n + \log_a q - (\log_a p^2 + \log_a t^4)$$

$$= \log_a m + \log_a n + \log_a q - (2\log_a p + 4\log_a t)$$

$$= \log_a m + \log_a n + \log_a q - 2\log_a p - 4\log_a t$$

Use parentheses to avoid errors.

Be careful with signs.

Example 3 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(e) $\log_a \sqrt[3]{m^2}$

Solution

$$\log_a \sqrt[3]{m^2} = \log_a m^{2/3} = \frac{2}{3}\log_a m$$

Example 3 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(f) $\log_b \sqrt[n]{\dfrac{x^3 y^5}{z^m}}$

Solution $\log_b \sqrt[n]{\dfrac{x^3 y^5}{z^m}} = \log_b \left(\dfrac{x^3 y^5}{z^m}\right)^{1/n}$ $\quad \sqrt[n]{a} = a^{1/n}$

$= \dfrac{1}{n} \log_b \dfrac{x^3 y^5}{z^m}$ Power property

$= \dfrac{1}{n}\left(\log_b x^3 + \log_b y^5 - \log_b z^m\right)$ Product and quotient properties

Example 3 USING THE PROPERTIES OF LOGARITHMS

Rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(f) $\log_b \sqrt[n]{\dfrac{x^3 y^5}{z^m}}$

Solution

$$= \frac{1}{n}\left(3\log_b x + 5\log_b y - m\log_b z\right)$$ Power property

$$= \frac{3}{n}\log_b x + \frac{5}{n}\log_b y - \frac{m}{n}\log_b z$$ Distributive property

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_3(x+2)+\log_3 x-\log_3 2$

Solution

$$\log_3(x+2)+\log_3 x-\log_3 2=\log_3\frac{(x+2)x}{2}$$

Product and quotient properties

Homework 3 USING THE PROPERTIES OF LOGARITHMS

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(b) $2\log_a m - 3\log_a n$

Solution

$$2\log_a m - 3\log_a n = \log_a m^2 - \log_a n^3$$

Power property

$$= \log_a \frac{m^2}{n^3}$$

Quotient property

Homework 3 USING THE PROPERTIES OF LOGARITHMS

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(c) $\dfrac{1}{2}\log_b m + \dfrac{3}{2}\log_b 2n - \log_b m^2 n$

Solution

$$\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$$

$$= \log_b m^{1/2} + \log_b (2n)^{3/2} - \log_b m^2 n \quad \text{Power property}$$

USING THE PROPERTIES OF LOGARITHMS

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(c) $\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$

Solution

$$= \log_b \frac{m^{1/2}(2n)^{3/2}}{m^2 n}$$

Product and quotient properties

$$= \log_b \frac{2^{3/2} n^{1/2}}{m^{3/2}}$$

Rules for exponents

Homework 3 USING THE PROPERTIES OF LOGARITHMS

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(c) $\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$

Solution

$$= \log_b \left(\frac{2^3 n}{m^3}\right)^{1/2}$$ Rules for exponents

$$= \log_b \sqrt{\frac{8n}{m^3}}$$ Definition of $a^{1/n}$

Caution ***There is no property of logarithms to rewrite a logarithm of a sum or difference.*** That is why, in **Example 6(a),** $\log_3(x + 2)$ was not written as $\log_3 x + \log_3 2$. The distributive property does not apply in a situation like this because $\log_3 (x + y)$ is one term. The abbreviation "log" is a function name, *not* a factor.

Example 4 USING THE PROPERTIES OF LOGARITHMS WITH NUMERICAL VALUES

Assume that $\log_{10} 2 = 0.3010$. Find each logarithm.

(a) $\log_{10} 4$

Solution

$$\log_{10} 4 = \log_{10} 2^2$$

$$= 2 \log_{10} 2$$

$$= 2(0.3010)$$

$$= 0.6020$$

Example 4 USING THE PROPERTIES OF LOGARITHMS WITH NUMERICAL VALUES

Assume that $\log_{10} 2 = 0.3010$. Find each logarithm.

(b) $\log_{10} 5$

Solution

$$\log_{10} 5 = \log_{10} \frac{10}{2}$$

$$= \log_{10} 10 - \log_{10} 2$$

$$= 1 - 0.3010$$

$$= 0.6990$$

Theorem on Inverses

For $a > 0$, $a \neq 1$, the following properties hold.

$$a^{\log_a x} = x \text{ (for } x > 0) \quad \text{and} \quad \log_a a^x = x$$

Theorem on Inverses

The following are examples of applications of this theorem.

$$7^{\log_7 10} = 10, \quad \log_5 5^3 = 3, \quad \text{and} \quad \log_r r^{k+1} = k+1$$

The second statement in the theorem will be useful in **Sections 4.5 and 4.6** when we solve other logarithmic and exponential equations.