Course Mathematics

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Credits

R

Review of Prerequisites

Chapter Outline

Athletes know that in order to optimize their performance they need to pace themselves and be mindful of their target heart rate. For example, a 25-year-old with a maximum heart rate of 195 beats per minute should strive for a target heart rate zone of between 98 and 166 beats per minute. This correlates to between 50% and 85% of the individual's maximum heart rate (Source: American Heart Association, www.americanheart.org). The mathematics involved in finding maximum heart rate and an individual's target heart rate zone use a linear model relating age and resting heart rate. An introduction to modeling is presented here in Chapter R along with the standard order of operations used to carry out these calculations.

Chapter R reviews skills and concepts required for success in college algebra. Just as an athlete must first learn the basics of a sport and build endurance and speed, a student studying mathematics must focus on necessary basic skills to prepare for the challenge ahead. Preparation for algebra is comparable to an athlete preparing for a sporting event. Putting the time and effort into the basics here in Chapter R will be your foundation for success in later chapters.

SECTION R.1 Sets and the Real Number Line

OBJECTIVES

1. Identify Subsets of the Set of Real Numbers
2. Use Inequality Symbols and Interval Notation
3. Find the Union and Intersection of Sets
4. Evaluate Absolute Value Expressions
5. Use Absolute Value to Represent Distance
6. Apply the Order of Operations
7. Simplify Algebraic Expressions
8. Write Algebraic Models

1. Identify Subsets of the Set of Real Numbers

A hybrid vehicle gets 48 mpg in city driving and 52 mpg on the highway. The formula $A = \frac{1}{48}c + \frac{1}{52}h$ gives the amount of gas A (in gal) for c miles of city driving and h miles of highway driving. In the formula, A, c, and h are called **variables** and these represent values that are subject to change. The values $\frac{1}{48}$ and $\frac{1}{52}$ are called **constants** because their values do not change in the formula.

For a trip from Houston, Texas, to Dallas, Texas, a motorist travels 36 mi of city driving and 91 mi of highway driving. The amount of fuel used by this hybrid vehicle is given by

$$\begin{aligned} A &= \frac{1}{48}(36) + \frac{1}{52}(91) \\ &= 2.5 \text{ gal} \end{aligned}$$

The numbers used in day-to-day life such as those used to determine fuel consumption come from the set of real numbers, denoted by $\mathbb{R}$. A **set** is a collection of items called **elements.** The braces { and } are used to enclose the elements of a set. For example, {gold, silver, bronze} represents the set of medals awarded to the top three finishers in an Olympic event. A set that contains no elements is called the **empty set** (or **null set**) and is denoted by { } or $\varnothing$.

When referring to individual elements of a set, the symbol $\in$ means "is an element of," and the symbol $\notin$ means "is not an element of." For example,

$5 \in \{1, 3, 5, 7\}$ is read as "5 is an element of the set of elements 1, 3, 5, and 7."

$6 \notin \{1, 3, 5, 7\}$ is read as "6 is *not* an element of the set of elements 1, 3, 5, and 7."

A set can be defined in several ways. Listing the elements in a set within braces is called the **roster method.** Using the roster method, the set of the even numbers between 0 and 10 is represented by {2, 4, 6, 8}. Another method to define this set is by using **set-builder notation.** This uses a description of the elements of the set. For example,

$$\underset{\text{The set of}}{\{}\ \underset{\text{all } x}{x}\ \underset{\text{such that}}{|}\ \underbrace{x \text{ is an even number between 0 and 10}}_{x \text{ is an even number between 0 and 10}}\}$$

In our study of college algebra, we will often refer to several important **subsets** (parts of) the set of real numbers (Table R-1).

Table R-1 Subsets of the Set of Real Numbers, $\mathbb{R}$

Set	Definition
Natural numbers, $\mathbb{N}$	$\{1, 2, 3, \ldots\}$
Whole numbers, $\mathbb{W}$	$\{0, 1, 2, 3, \ldots\}$
Integers, $\mathbb{Z}$	$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers, $\mathbb{Q}$	$\left\{\frac{p}{q} \,\middle\vert\, p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$ • Rational numbers can be expressed as a ratio of integers where the denominator is not zero. Examples: $-\frac{6}{11}$ (ratio of -6 and 11) and 9 (ratio of 9 and 1). • All terminating and repeating decimals are rational numbers. Examples: 0.71 (ratio of 71 and 100), $0.\overline{6} = 0.666\ldots$ (ratio of 2 and 3).
Irrational numbers, $\mathbb{H}$	Irrational numbers are real numbers that cannot be expressed as a ratio of integers. The decimal form of an irrational number is nonterminating and nonrepeating. Examples: π and $\sqrt{2}$

TIP Notice that the first five letters of the word *ratio*nal spell *ratio*. This will help you remember that a rational number is a *ratio* of integers.

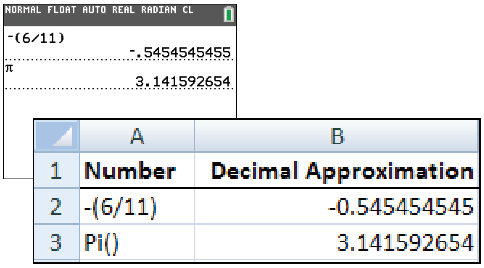
Point of Interest

Computers have approximated π to over 1 trillion digits, but for most applications, the approximation 3.14159 is sufficient. Some mathematics enthusiasts have named the date March 14 ($\pi \approx 3.14$) as World Pi Day.

TECHNOLOGY CONNECTIONS

Approximating Rational and Irrational Numbers

The number $-\frac{6}{11}$ is a rational number, and the number π is an irrational number. It is important to realize that for nonterminating decimals, a calculator or spreadsheet will only give approximate values, not exact values.



```
NORMAL FLOAT AUTO REAL RADIAN CL
-(6/11)
                    -.5454545455
π
                     3.141592654
```

	A	B
1	Number	Decimal Approximation
2	-(6/11)	-0.545454545
3	Pi()	3.141592654

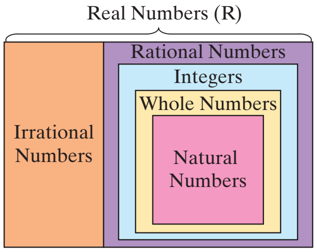


Figure R-1

The relationships among the subsets of real numbers defined in Table R-1 are shown in Figure R-1. In particular, notice that together the elements of the set of rational numbers and the set of irrational numbers make up the set of real numbers.

EXAMPLE 1 Identifying Elements of a Set

Given $A = \left\{\sqrt{3}, 0.\overline{83}, -\frac{19}{7}, 0.39, -16, 0, 11, 0.2020020002\ldots, 0.444\right\}$, determine which elements belong to the following sets.

Solution:

a. $\mathbb{N}$ $11 \in \mathbb{N}$

b. $\mathbb{W}$ $0,\ 11 \in \mathbb{W}$

c. $\mathbb{Z}$ $-16,\ 0,\ 11 \in \mathbb{Z}$

d. $\mathbb{Q}$ $0.\overline{83},\ -\frac{19}{7},\ 0.39,\ -16,\ 0,\ 11,\ 0.444 \in \mathbb{Q}$

e. $\mathbb{H}$ $\sqrt{3},\ 0.2020020002\ldots \in \mathbb{H}$

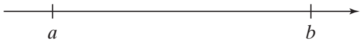
f. $\mathbb{R}$ $\sqrt{3},\ 0.\overline{83},\ -\frac{19}{7},\ 0.39,\ -16,\ 0,\ 11,\ 0.2020020002\ldots,\ 0.444 \in \mathbb{R}$

Skill Practice 1 Given set B, determine which elements belong to the following sets. $B = \left\{-\frac{11}{7}, \sqrt{59}, 4.3, 0, 23, -13, \pi, 4.\overline{9}\right\}$

a. $\mathbb{N}$ **b.** $\mathbb{W}$ **c.** $\mathbb{Z}$ **d.** $\mathbb{Q}$ **e.** $\mathbb{H}$ **f.** $\mathbb{R}$

2. Use Inequality Symbols and Interval Notation

All real numbers can be located on the real number line. We say that a is less than b (written symbolically as $a < b$) if a lies to the left of b on the number line. This is equivalent to saying that b is greater than a (written symbolically as $b > a$) because b lies to the right of a.



$a < b$ is equivalent to $b > a$

In Table R-2, we summarize other symbols used to compare two real numbers.

Answers

1. a. $23 \in \mathbb{N}$ **b.** $0, 23 \in \mathbb{W}$
c. $0, 23, -13 \in \mathbb{Z}$
d. $-\frac{11}{7}, 4.3, 0, 23, -13, 4.\overline{9} \in \mathbb{Q}$
e. $\sqrt{59}, \pi \in \mathbb{H}$
f. $-\frac{11}{7}, \sqrt{59}, 4.3, 0, 23, -13, \pi, 4.\overline{9} \in \mathbb{R}$

Table R-2 Summary of Inequality Symbols and Their Meanings

Inequality	Verbal Interpretation	Other Implied Meanings	Numerical Examples
$a < b$	a is less than b	b exceeds a b is greater than a	$5 < 7$
$a > b$	a is greater than b	a exceeds b b is less than a	$-3 > -6$
$a \le b$	a is less than or equal to b	a is at most b a is no more than b	$4 \le 5$ $5 \le 5$
$a \ge b$	a is greater than or equal to b	a is no less than b a is at least b	$9 \ge 8$ $9 \ge 9$
$a = b$	a is equal to b		$-4.3 = -4.3$
$a \ne b$	a is not equal to b		$-6 \ne -7$
$a \approx b$	a is approximately equal to b		$-12.99 \approx -13$

Point of Interest

The infinity symbol ∞ is called a lemniscate from the Latin *lemniscus* meaning "ribbon." English mathematician John Wallis is credited with introducing the symbol in the seventeenth century. The symbols $-\infty$ and ∞ are not themselves real numbers, but instead refer to quantities without bound or end.

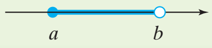
An interval on the real number line can be represented in set-builder notation or in interval notation. In Table R-3, observe that a parenthesis) or (indicates that an endpoint is not included in an interval. A bracket] or [indicates that an endpoint *is* included in the interval. The real number line extends infinitely far to the left and right. We use the symbols $-\infty$ and ∞ to denote the unbounded behavior to the left and right, respectively.

Table R-3 Summary of Interval Notation and Set-Builder Notation

Let a, b, and x represent real numbers.

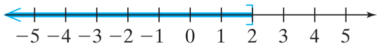
Set-Builder Notation	Verbal Interpretation	Graph	Interval Notation
$\{x \mid x > a\}$	the set of real numbers greater than a		(a, ∞)
$\{x \mid x \ge a\}$	the set of real numbers greater than or equal to a		$[a, \infty)$
$\{x \mid x < b\}$	the set of real numbers less than b		$(-\infty, b)$
$\{x \mid x \le b\}$	the set of real numbers less than or equal to b		$(-\infty, b]$
$\{x \mid a < x < b\}$	the set of real numbers between a and b		(a, b)
$\{x \mid a \le x < b\}$	the set of real numbers greater than or equal to a and less than b		$[a, b)$
$\{x \mid a < x \le b\}$	the set of real numbers greater than a and less than or equal to b		$(a, b]$
$\{x \mid a \le x \le b\}$	the set of real numbers between a and b, inclusive		$[a, b]$
$\{x \mid x$ is a real number$\}$ $\mathbb{R}$	the set of all real numbers		$(-\infty, \infty)$

TIP As an alternative to using parentheses and brackets to represent the endpoints of an interval, an open dot or closed dot may be used. For example, $\{x \mid a \le x < b\}$ would be represented as follows.

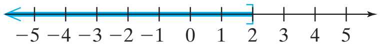
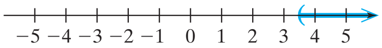
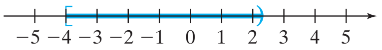


EXAMPLE 2 Expressing Sets in Interval Notation and Set-Builder Notation

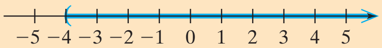
Complete the table.

Graph	Interval Notation	Set-Builder Notation
		
	$\left(\frac{7}{2}, \infty\right)$	
		$\{y \mid -4 \le y < 2.3\}$

Solution:

Graph	Interval Notation	Set-Builder Notation	Comments
	$(-\infty, 2]$	$\{x \mid x \le 2\}$	The bracket at 2 indicates that 2 is included in the set.
	$\left(\frac{7}{2}, \infty\right)$	$\left\{x \mid x > \frac{7}{2}\right\}$	The parenthesis at $\frac{7}{2} = 3.5$ indicates that $\frac{7}{2}$ is *not* included in the set.
	$[-4, 2.3)$	$\{y \mid -4 \le y < 2.3\}$	The set includes the real numbers between -4 and 2.3, including the endpoint -4.

Skill Practice 2



a. Write the set represented by the graph in interval notation and set-builder notation.

b. Given the interval, $\left(-\infty, -\frac{4}{3}\right]$, graph the set and write the set-builder notation.

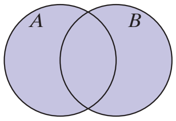
c. Given the set, $\{x \mid 1.6 < x \le 5\}$, graph the set and write the interval notation.

3. Find the Union and Intersection of Sets

Two or more sets can be combined by the operations of union and intersection.

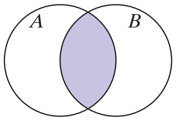
Union and Intersection of Sets

The **union** of sets A and B, denoted $A \cup B$, is the set of elements that belong to set A or to set B or to both sets A and B.



$A \cup B$
A union B
The elements in A or B or both

The **intersection** of sets A and B, denoted $A \cap B$, is the set of elements common to both set A and set B.



$A \cap B$
A intersection B
The elements common to A and B

In Examples 3 and 4, we practice finding the union and intersections of sets.

EXAMPLE 3 Finding the Union and Intersection of Sets

Find the union or intersection of sets as indicated.

$$A = \{-5, -3, -1, 1\} \qquad B = \{-5, 0, 5\} \qquad C = \{-4, -2, 0, 2, 4\}$$

a. $A \cap B$ **b.** $A \cup B$ **c.** $A \cap C$

Solution:

a. $A \cap B = \{-5\}$

The only element common to both A and B is -5.
$A = \{-5, -3, -1, 1\}$, $B = \{-5, 0, 5\}$

b. $A \cup B = \{-5, -3, -1, 0, 1, 5\}$

The union of A and B consists of all elements from A along with all elements from B.

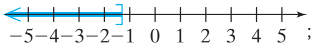
c. $A \cap C = \{\ \}$

Sets A and C have no common elements.

Answers

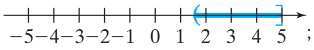
2. a. Interval notation: $(-4, \infty)$;
Set-builder notation: $\{x \mid x > -4\}$

b.



Set-builder notation: $\left\{x \mid x \le -\frac{4}{3}\right\}$

c.



Interval notation: $(1.6, 5]$

Skill Practice 3 From the sets A, B, and C defined in Example 3, find

a. $B \cap C$ **b.** $B \cup C$ **c.** $A \cup C$

EXAMPLE 4 Finding the Union and Intersection of Sets

Find the union or intersection as indicated.

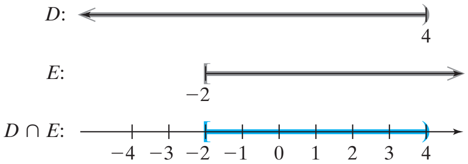
$$D = \{x \mid x < 4\} \qquad E = \{x \mid x \geq -2\} \qquad F = \{x \mid x \leq -3\}$$

a. $D \cap E$ **b.** $D \cup E$ **c.** $D \cup F$ **d.** $E \cap F$

Solution:

Graph each individual set and take the union or intersection.

a.

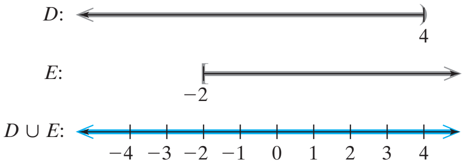


The intersection is the region of overlap.

Set notation: $D \cap E = \{x \mid -2 \leq x < 4\}$

Interval notation: $[-2, 4)$

b.

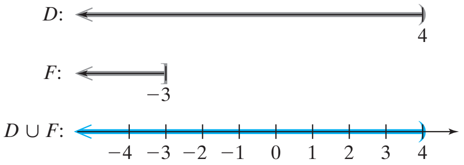


The union contains the elements from D along with those from E.

Set notation: $D \cup E = \mathbb{R}$

Interval notation: $(-\infty, \infty)$

c.

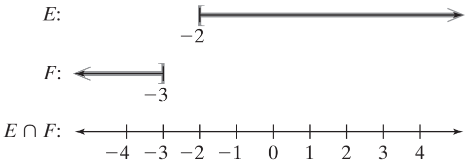


Set F is contained within set D. The union is set D itself.

Set notation: $D \cup F = \{x \mid x < 4\}$

Interval notation: $(-\infty, 4)$

d.



There are no elements common to both sets E and F.

Set notation: $E \cap F = \{\ \}$

Skill Practice 4 Given $X = \{x \mid x \geq -1\}$, $Y = \{x \mid x < 2\}$, and $Z = \{x \mid x < -4\}$, find the union or intersection of sets as indicated.

a. $X \cap Y$ **b.** $X \cup Y$ **c.** $Y \cap Z$ **d.** $X \cap Z$

Answers

3. a. $\{0\}$ **b.** $\{-5, -4, -2, 0, 2, 4, 5\}$
c. $\{-5, -4, -3, -2, -1, 0, 1, 2, 4\}$
4. a. $\{x \mid -1 \leq x < 2\}$; $[-1, 2)$
b. $\mathbb{R}$; $(-\infty, \infty)$
c. $\{x \mid x < -4\}$; $(-\infty, -4)$
d. $\{\ \}$

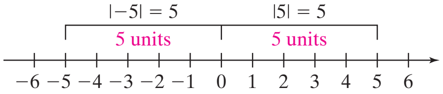
4. Evaluate Absolute Value Expressions

Every real number x has an opposite denoted by $-x$. For example, $-(4)$ is the opposite of 4 and simplifies to -4. Likewise, $-(-2.1)$ is the opposite of -2.1 and simplifies to 2.1.

The **absolute value** of a real number x, denoted by $|x|$, is the distance between x and zero on the number line. For example:

$|-5| = 5$ because -5 is 5 units from zero on the number line.

$|5| = 5$ because 5 is 5 units from zero on the number line.



Notice that if a number is negative, its absolute value is the opposite of the number. If a number is positive, its absolute value is the number itself.

Definition of Absolute Value

Let x be a real number. Then $|x| = \begin{cases} x \text{ if } x \ge 0 \\ -x \text{ if } x < 0 \end{cases}$

Verbal Interpretation	Numerical Example
• If x is positive or zero, then $\lvert x\rvert$ is just x itself.	$\lvert 4\rvert = 4$ $\lvert 0\rvert = 0$
• If x is negative, then $\lvert x\rvert$ is the opposite of x.	$\lvert -4\rvert = -(-4) = 4$

EXAMPLE 5 Removing Absolute Value Symbols

Use the definition of absolute value to rewrite each expression without absolute value bars.

a. $|\sqrt{3} - 3|$ **b.** $|3 - \sqrt{3}|$ **c.** $\dfrac{|x - 4|}{x - 4}$ for $x < 4$

TIP: Calculator approximations can be used to show that $\sqrt{3} - 3 \approx -1.27$ is negative, and $3 - \sqrt{3} \approx 1.27$ is positive.

Solution:

a. $|\sqrt{3} - 3| = -(\sqrt{3} - 3)$
$= -\sqrt{3} + 3$ or $3 - \sqrt{3}$

The value $\sqrt{3} \approx 1.73 < 3$, which implies that $\sqrt{3} - 3 < 0$. Since the expression inside the absolute value bars is negative, take the opposite.

b. $|3 - \sqrt{3}| = 3 - \sqrt{3}$

The value $\sqrt{3} \approx 1.73 < 3$, which implies that $3 - \sqrt{3} > 0$. Since the expression inside the absolute value bars is positive, the simplified form is the expression itself.

c. $\dfrac{|x - 4|}{x - 4}$ for $x < 4$

$$= \frac{-(x-4)}{x-4}$$
$$= -1 \cdot \frac{x-4}{x-4}$$
$$= -1$$

The condition $x < 4$, implies that $x - 4 < 0$. Since the expression inside the absolute value bars is negative, take the opposite.

Skill Practice 5 Use the definition of absolute value to rewrite each expression without absolute value bars.

a. $|5 - \sqrt{7}|$ **b.** $|\sqrt{7} - 5|$ **c.** $\dfrac{x + 6}{|x + 6|}$ for $x > -6$

Answers

5. a. $5 - \sqrt{7}$
b. $5 - \sqrt{7}$
c. 1

5. Use Absolute Value to Represent Distance

Absolute value is also used to denote distance between two points on a number line.

Distance Between Two Points on a Number Line

The distance between two points a and b on a number line is given by

$$|a - b| \quad \text{or} \quad |b - a|$$

That is, the distance between two points on a number line is the absolute value of their difference.

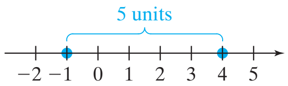
EXAMPLE 6 Determining the Distance Between Two Points

Write an absolute value expression that represents the distance between 4 and −1 on the number line. Then simplify.

Solution:

$$|4 - (-1)| = |5| = 5$$
$$|-1 - 4| = |-5| = 5$$

The distance between 4 and −1 is represented by $|4 - (-1)|$ or by $|-1 - 4|$.



Skill Practice 6 Write an absolute value expression that represents the distance between −9 and 2 on the number line. Then simplify.

6. Apply the Order of Operations

Repeated multiplication can be written by using exponential notation. For example, the product $5 \cdot 5 \cdot 5$ can be written as 5^3. In this case, 5 is called the base of the expression and 3 is the exponent (or power). The exponent indicates how many times the base is used as a factor.

Definition of b^n

Let b be a real number and let n represent a natural number. Then

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{b \text{ is used as a factor } n \text{ times}}$$

b^n is read as "b to the nth-power."
b is the **base** and n is the **exponent** or **power.**

To find a square root of a nonnegative real number, we reverse the process to square a number. For example, a square root of 25 is a number that when squared equals 25. Both 5 and −5 are square roots of 25, because $5^2 = 25$ and $(-5)^2 = 25$. A radical sign $\sqrt{}$ is used to denote the principal square root of a number. The **principal square root** of a nonnegative real number is the square root that is greater than or equal to zero. Therefore, the principal square root of 25, denoted by $\sqrt{25}$, equals 5.

$$\sqrt{25} = 5 \text{ because } 5 \geq 0 \text{ and } 5^2 = 25$$

Answer

6. $|-9 - 2|$ or $|2 - (-9)|$; The distance is 11 units.

TIP Note that the square of any real number is nonnegative. Therefore, there is no real number that is the square root of a negative number. For example,

$\sqrt{-25}$ is not a real number because no real number when squared equals -25.

Note: The value $\sqrt{-25}$ is an imaginary number and will be discussed in Section 1.3.

The symbol $\sqrt[3]{}$ represents the cube root of a number. For example:

$$\sqrt[3]{64} = 4 \text{ because } 4^3 = 64$$

Many expressions involve multiple operations. In such a case, it is important to follow the order of operations.

Order of Operations

Step 1 Simplify expressions within parentheses and other grouping symbols. These include absolute value bars, fraction bars, and radicals. If nested grouping symbols are present, start with the innermost symbols.

Step 2 Evaluate expressions involving exponents.

Step 3 Perform multiplication or division in the order in which they occur from left to right.

Step 4 Perform addition or subtraction in the order in which they occur from left to right.

EXAMPLE 7 Simplifying a Numerical Expression

Simplify. $7 - \{8 + 4[2 - (5 - \sqrt{64})^2]\}$

Solution:

$7 - \{8 + 4[2 - (5 - \sqrt{64})^2]\}$

$= 7 - \{8 + 4[2 - (5 - 8)^2]\}$	Simplify within inner parentheses, $\sqrt{64} = 8$.
$= 7 - \{8 + 4[2 - (-3)^2]\}$	Subtract within the inner parentheses.
$= 7 - [8 + 4(2 - 9)]$	Continue simplifying within the inner parentheses. Simplify $(-3)^2$ to get 9.
$= 7 - [8 + 4(-7)]$	Simplify $(2 - 9)$ to get (-7).
$= 7 - (8 - 28)$	Multiply before adding or subtracting.
$= 7 - (-20)$	Simplify within parentheses.
$= 27$	

Skill Practice 7 Simplify. $50 - \{2 - [\sqrt{121} + 3(-1 - 3)^2]\}$

Answer

7. 107

When simplifying an expression, particular care must be taken with expressions involving division and zero.

Division Involving Zero

To investigate division involving zero, consider the expressions $\frac{5}{0}$, $\frac{0}{5}$, and $\frac{0}{0}$ and their related multiplicative forms.

1. Division by zero is undefined.
   Example: $\frac{5}{0} = n$ implies that $n \cdot 0 = 5$. No number, n, satisfies this requirement.
2. Zero divided by any nonzero number is zero.
   Example: $\frac{0}{5} = 0$ implies that $0 \cdot 5 = 0$ which is a true statement.
3. We say that $\frac{0}{0}$ is **indeterminate** (cannot be uniquely determined). This concept is investigated in detail in a first course in calculus.
   Example: $\frac{0}{0} = n$ implies that $n \cdot 0 = 0$. This is true for any number n. Therefore, the quotient cannot be uniquely determined.

7. Simplify Algebraic Expressions

An algebraic **term** is a product of factors that may include constants and variables. An algebraic **expression** is a single term or the sum of two or more terms. For example, the expression

$$-3xz^2 + \left(-\frac{4}{b}\right) + z\sqrt{x - y} + 5 \text{ has four terms.}$$

The terms $-3xz^2$, $-\frac{4}{b}$, and $z\sqrt{x - y}$ are **variable terms.** The term 5 is not subject to change and is called a **constant term.** The constant factor within each term is called the **coefficient** of the term. Writing the expressions as $-3xz^2 + \left(-4 \cdot \frac{1}{b}\right) + 1z\sqrt{x - y} + 5$, we identify the coefficients of the four terms as -3, -4, 1, and 5, respectively.

The properties of real numbers summarized in Table R-4 here and on page 11 are often helpful when working with algebraic expressions.

Table R-4 Properties of Real Numbers

Let a, b, and c represent real numbers or real-valued expressions.

Property	In Symbols and Words	Examples
Commutative property of addition	$a + b = b + a$ The order in which real numbers are added does not affect the sum.	ex: $4 + (-7) = -7 + 4$ ex: $6 + w = w + 6$
Commutative property of multiplication	$a \cdot b = b \cdot a$ The order in which real numbers are multiplied does not affect the product.	ex: $5 \cdot (-4) = -4 \cdot 5$ ex: $x \cdot 12 = 12x$
Associative property of addition	$(a + b) + c = a + (b + c)$ The order in which real numbers are grouped under addition does not affect the sum.	ex: $(3 + 5) + 2 = 3 + (5 + 2)$ ex: $-9 + (2 + t) = (-9 + 2) + t$ $= -7 + t$
Associative property of multiplication	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ The order in which real numbers are grouped under multiplication does not affect the product.	ex: $(6 \cdot 7) \cdot 3 = 6 \cdot (7 \cdot 3)$ ex: $8 \cdot \left(\frac{1}{8} \cdot y\right) = \left(8 \cdot \frac{1}{8}\right) \cdot y$ $= 1y$
Identity property of addition	$a + 0 = a$ and $0 + a = a$ The number 0 is called the **identity element of addition** because any number plus 0 is the number itself.	ex: $-5 + 0 = -5$ ex: $0 + \sqrt{z} = \sqrt{z}$
Identity property of multiplication	$a \cdot 1 = a$ and $1 \cdot a = a$ The number 1 is called the **identity element of multiplication** because any number times 1 is the number itself.	ex: $\sqrt{2} \cdot 1 = \sqrt{2}$ ex: $1 \cdot (2w + 3) = 2w + 3$

(Continued)

Property	In Symbols and Words	Examples
Inverse property of addition	$a + (-a) = 0 \quad \text{and} \quad (-a) + a = 0$ For any real number a, the value $-a$ is called the **additive inverse of *a*** (also called the **opposite of *a***). The sum of any number and its additive inverse is the identity element for addition, 0.	ex: $4\pi + (-4\pi) = 0$ ex: $-ab^2 + ab^2 = 0$
Inverse property of multiplication	$a \cdot \frac{1}{a} = 1 \quad \text{and} \quad \frac{1}{a} \cdot a = 1 \quad \text{where } a \neq 0$ For any nonzero real number a, the value $\frac{1}{a}$ is called the **multiplicative inverse of *a*** (also called the **reciprocal of *a***). The product of any nonzero number and its multiplicative inverse is the identity element, 1.	ex: $-5 \cdot \left(-\frac{1}{5}\right) = 1$ ex: $\frac{1}{x^2} \cdot x^2 = 1 \quad \text{for } x \neq 0$ *Note*: The number zero does not have a multiplicative inverse (reciprocal).
Distributive property of multiplication over addition	$a \cdot (b + c) = a \cdot b + a \cdot c$ The product of a number and a sum equals the sum of the products of the number and each term in the sum.	ex: $4 \cdot (5 + x) = 4 \cdot 5 + 4 \cdot x$ $= 20 + 4x$ ex: $2(x + \sqrt{3}) = 2x + 2\sqrt{3}$

The distributive property is used to "clear" parentheses when a factor outside parentheses is multiplied by multiple terms inside parentheses.

EXAMPLE 8 Applying the Distributive Property

Apply the distributive property.

a. $4(2x^2 - 5.1x + 3)$ **b.** $-(9y - \sqrt{2})$

Solution:

a. $4(2x^2 - 5.1x + 3)$

$= 4[2x^2 + (-5.1x) + 3]$	Write subtraction in terms of addition.
$= 4 \cdot (2x^2) + 4 \cdot (-5.1x) + 4 \cdot (3)$	Apply the distributive property.
$= 8x^2 - 20.4x + 12$	Simplify.

b. $-(9y - \sqrt{2})$

$= -1[(9y + (-\sqrt{2})]$	The negative sign in front of parentheses can be interpreted as -1 times the expression in parentheses.
$= -9y + \sqrt{2}$	Apply the distributive property.

TIP A negative factor preceding the parentheses will change the signs of the terms within parentheses.

$$-(+9y - \sqrt{2})$$
$$= -9y + \sqrt{2}$$

Skill Practice 8 Apply the distributive property.

a. $5(6y^3 - 4y^2 + 7)$ **b.** $-2(5t - \pi)$

Two terms in an expression are **like terms** if they have the same variables and the corresponding variables are raised to the same power. We can combine like terms by using the distributive property. For example,

$8x^2 + 6x^2 - x^2$

$= 8x^2 + 6x^2 - 1x^2$	Note that the coefficient on the third term is -1.
$= (8 + 6 - 1)x^2$	Apply the distributive property.
$= 13x^2$	Simplify.

Answers

8. a. $30y^3 - 20y^2 + 35$
b. $-10t + 2\pi$

TIP Although the distributive property is used to add and subtract like terms, it is tedious to write each step. Adding or subtracting like terms can also be done by combining the coefficients and leaving the variable factor unchanged.

$8x^2 + 6x^2 - 1x^2 = 13x^2$ This method will be used throughout the text.

In Example 9, we simplify an expression by applying the distributive property to "clear" parentheses and combine like terms.

EXAMPLE 9 Clearing Parentheses and Combining Like Terms

Simplify.

a. $5 - 2(4c - 8d) + 3(1 - d) + c$

b. $-3x^2 - \left[8 + \frac{1}{2}(2x^2 - 6) - 4x^2\right]$

Solution:

TIP After applying the distributive property, the original parentheses are removed. For this reason, we often call this process "*clearing parentheses*."

a. $5 - 2(4c - 8d) + 3(1 - d) + c$ Apply the distributive property to clear parentheses.

$= 5 - 8c + 16d + 3 - 3d + c$ Combine like terms.

$= 8 - 7c + 13d$

b. $-3x^2 - \left[8 + \frac{1}{2}(2x^2 - 6) - 4x^2\right]$ Apply the distributive property.

$= -3x^2 - [8 + x^2 - 3 - 4x^2]$ Combine like terms inside brackets.

$= -3x^2 - [-3x^2 + 5]$ Apply the distributive property.

$= -3x^2 + 3x^2 - 5$ Combine like terms.

$= -5$

Skill Practice 9 Clear parentheses and combine like terms.

a. $12 - 3(5x - 2y) + 5(3 - x) - y$

b. $4w^3 - \left[3 - \frac{1}{4}(4 + 8w^3) - w^3\right] + 2$

8. Write Algebraic Models

An important skill in mathematics and science is to develop mathematical models. Example 10 offers practice writing algebraic expressions based on verbal statements.

EXAMPLE 10 Writing an Algebraic Model

a. The maximum recommended heart rate M for adults is the difference of 220 and the person's age a. Write a model to represent an adult's maximum recommended heart rate in terms of age.

b. After eating at a restaurant, it is customary to leave a tip t for the server for at least 15% of the cost of the meal c. Write a model to represent the amount of the tip based on the cost of the meal.

Answers

9. a. $-20x + 5y + 27$ **b.** $7w^3$

Solution:

a. $M = 220 - a$ The word "difference" implies subtraction in the order given.

b. $t \geq 0.15c$ 15% of c implies multiplication.

Skill Practice 10

a. The sale price S on a lawn mower is the difference of the original price P and the amount of discount D. Write a model to represent the sale price.

b. The amount of simple interest I earned on a certain certificate of deposit is 4.65% of the amount of principal invested P. Write a model for the amount of interest.

Answers

10. a. $S = P - D$
b. $I = 0.0465P$

SECTION R.1 Practice Exercises

Concept Connections

1. A(n) __________ is a collection of items called elements.

2. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is called the set of __________.

3. $\mathbb{R}$ is the notation used to denote the set of __________ __________.

4. A(n) __________ number is a real number that cannot be expressed as a ratio of two integers.

5. Write an absolute value expression to represent the distance between a and b on the real number line.

6. The __________ properties of addition and multiplication indicate that the order in which two real numbers are added or multiplied does not affect the sum or product.

7. The associative property of addition indicates that $a + (b + c) =$ __________, and the associative property of multiplication indicates that $a(bc) =$ __________.

8. The statement $a(b + c) = ab + ac$ represents the __________ property of multiplication over addition.

Objective 1: Identify Subsets of the Set of Real Numbers

For Exercises 9–10, write an English sentence to represent the algebraic statement.

9. a. $3 \in \mathbb{N}$ **b.** $-3.1 \notin \mathbb{W}$

10. a. $\frac{2}{5} \in \mathbb{Q}$ **b.** $\pi \notin \mathbb{Q}$

For Exercises 11–12, determine whether the statement is true or false.

11. a. $-5 \in \mathbb{N}$ **b.** $-5 \in \mathbb{W}$
c. $-5 \in \mathbb{Z}$ **d.** $-5 \in \mathbb{Q}$

12. a. $\frac{1}{3} \in \mathbb{N}$ **b.** $\frac{1}{3} \in \mathbb{W}$
c. $\frac{1}{3} \in \mathbb{Z}$ **d.** $\frac{1}{3} \in \mathbb{Q}$

13. Refer to $A = \left\{\sqrt{5}, 0.\overline{3}, 0.33, -0.9, -12, \frac{11}{4}, 6, \frac{\pi}{6}\right\}$. Determine which elements belong to the given set. (**See Example 1**)

a. $\mathbb{N}$ **b.** $\mathbb{W}$
c. $\mathbb{Z}$ **d.** $\mathbb{Q}$
e. $\mathbb{H}$ **f.** $\mathbb{R}$

14. Refer to $B = \left\{\frac{\pi}{2}, 0, -4, 0.\overline{48}, 1, -\sqrt{13}, 9.4\right\}$. Determine which elements belong to the given set.

a. $\mathbb{N}$ **b.** $\mathbb{W}$
c. $\mathbb{Z}$ **d.** $\mathbb{Q}$
e. $\mathbb{H}$ **f.** $\mathbb{R}$

Objective 2: Use Inequality Symbols and Interval Notation

For Exercises 15–20, write each statement as an inequality.

15. a is at least 5.

16. b is at most -6.

17. $3c$ is no more than 9.

18. $8d$ is no less than 16.

19. The quantity $(m + 4)$ exceeds 70.

20. The quantity $(n - 7)$ is approximately equal to 4.

For Exercises 21–28, determine whether the statement is true or false.

21. $3.14 < \pi$

22. $-7 < -\sqrt{7}$

23. $6.7 \geq 6.7$

24. $-2.1 \leq -2.1$

25. $6.\overline{15} > 6.1\overline{5}$

26. $2.9\overline{3} > 2.\overline{93}$

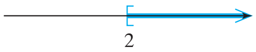
27. $-\frac{9}{7} < -\frac{11}{8}$

28. $-\frac{5}{3} < -\frac{9}{5}$

For Exercises 29–34, write the interval notation and set-builder notation for each given graph. (See Example 2)

29.

30.



31.

32.

33.

34.

For Exercises 35–40, graph the given set and write the corresponding interval notation. (See Example 2)

35. $\{x \mid x \leq 6\}$

36. $\{x \mid x < -4\}$

37. $\left\{x \,\middle|\, -\frac{7}{6} < x \leq \frac{1}{3}\right\}$

38. $\left\{x \,\middle|\, -\frac{4}{3} \leq x < \frac{7}{4}\right\}$

39. $\{x \mid 4 < x\}$

40. $\{x \mid -3 \leq x\}$

For Exercises 41–46, interval notation is given for several sets of real numbers. Graph the set and write the corresponding set-builder notation. (See Example 2)

41. $(-3, 7]$

42. $[-4, -1)$

43. $(-\infty, 6.7]$

44. $(-\infty, -3.2)$

45. $\left[-\frac{3}{5}, \infty\right)$

46. $\left(\frac{7}{8}, \infty\right)$

Objective 3: Find the Union and Intersection of Sets

For Exercises 47–50, refer to sets A, B, C, X, Y, and Z and find the union or intersection of sets as indicated. (See Example 3)

$$A = \{0, 4, 8, 12\}, \quad B = \{0, 3, 6, 9, 12\}, \quad C = \{-2, 4, 8\}$$
$$X = \{1, 2, 3, 4, 5\}, \quad Y = \{1, 2, 3\}, \quad Z = \{6, 7, 8\}$$

47. a. $A \cup B$ **b.** $A \cap B$ **c.** $A \cup C$
d. $A \cap C$ **e.** $B \cup C$ **f.** $B \cap C$

48. a. $X \cup Z$ **b.** $Y \cup Z$ **c.** $Y \cap Z$
d. $X \cup Y$ **e.** $X \cap Y$ **f.** $X \cap Z$

49. a. $A \cup X$ **b.** $A \cap Z$ **c.** $C \cap Y$
d. $B \cap Y$ **e.** $C \cup Z$ **f.** $A \cup Z$

50. a. $A \cap X$ **b.** $B \cup Z$ **c.** $C \cup Y$
d. $B \cap Z$ **e.** $C \cap Z$ **f.** $A \cup Y$

51. Refer to sets C, D, and F and find the union or intersection of sets as indicated. Write the answers in set notation. **(See Example 4)**

$C = \{x|x < 9\}, \quad D = \{x|x \geq -1\}, \quad F = \{x|x < -8\}$

a. $C \cup D$ **b.** $C \cap D$ **c.** $C \cup F$

d. $C \cap F$ **e.** $D \cup F$ **f.** $D \cap F$

52. Refer to sets M, N, and P and find the union or intersection of sets as indicated. Write the answers in set notation.

$M = \{y|y \geq -3\}, \quad N = \{y|y \geq 5\}, \quad P = \{y|y < 0\}$

a. $M \cup N$ **b.** $M \cap N$ **c.** $M \cup P$

d. $M \cap P$ **e.** $N \cup P$ **f.** $N \cap P$

For Exercises 53–56, find the union or intersection of the given intervals. Write the answers in interval notation.

53. a. $(-\infty, 4) \cup (-2, 1]$ **b.** $(-\infty, 4) \cap (-2, 1]$

54. a. $[0, 5) \cup [-1, \infty)$ **b.** $[0, 5) \cap [-1, \infty)$

55. a. $(-\infty, 5) \cup [3, \infty)$ **b.** $(-\infty, 5) \cap [3, \infty)$

56. a. $(-\infty, -1] \cup [-4, \infty)$ **b.** $(-\infty, -1] \cap [-4, \infty)$

Objective 4: Evaluate Absolute Value Expressions

For Exercises 57–68, simplify each expression by writing the expression without absolute value bars. (See Example 5)

57. $|-6|$ **58.** $|-4|$ **59.** $|0|$

60. $|1|$ **61.** $|\sqrt{17} - 5|$ **62.** $|\sqrt{6} - 6|$

63. a. $|\pi - 3|$ **64. a.** $|m - 11|$ for $m \geq 11$ **65. a.** $|x + 2|$ for $x \geq -2$

b. $|3 - \pi|$ **b.** $|m - 11|$ for $m < 11$ **b.** $|x + 2|$ for $x < -2$

66. a. $|t + 6|$ for $t < -6$ **67. a.** $\dfrac{|z - 5|}{z - 5}$ for $z > 5$ **68. a.** $\dfrac{7 - x}{|7 - x|}$ for $x < 7$

b. $|t + 6|$ for $t \geq -6$ **b.** $\dfrac{|z - 5|}{z - 5}$ for $z < 5$ **b.** $\dfrac{7 - x}{|7 - x|}$ for $x > 7$

Objective 5: Use Absolute Value to Represent Distance

For Exercises 69–74, write an absolute value expression to represent the distance between the two points on the number line. Then simplify without absolute value bars. (See Example 6)

69. 1 and 6 **70.** 2 and 9 **71.** 3 and -4

72. -8 and 2 **73.** 6 and 2π **74.** 3 and π

Objective 6: Apply the Order of Operations

For Exercises 75–88, simplify the expressions. (See Example 7)

75. a. 4^2 **b.** $(-4)^2$ **c.** -4^2 **d.** $\sqrt{4}$ **e.** $-\sqrt{4}$ **f.** $\sqrt{-4}$

76. a. 9^2 **b.** $(-9)^2$ **c.** -9^2 **d.** $\sqrt{9}$ **e.** $-\sqrt{9}$ **f.** $\sqrt{-9}$

77. a. $\sqrt[3]{8}$ **b.** $\sqrt[3]{-8}$ **c.** $-\sqrt[3]{8}$ **d.** $\sqrt{100}$ **e.** $\sqrt{-100}$ **f.** $-\sqrt{100}$

78. a. $\sqrt[3]{27}$ **b.** $\sqrt[3]{-27}$ **c.** $-\sqrt[3]{27}$ **d.** $\sqrt{49}$ **e.** $\sqrt{-49}$ **f.** $-\sqrt{49}$

79. $20 - 12(36 \div 3^2 \div 2)$ **80.** $200 - 2^2(6 \div \frac{1}{2} \cdot 4)$ **81.** $6 - \{-12 + 3[(1 - 6)^2 - 18]\}$

82. $-5 - \{4 - 6[(2 - 8)^2 - 31]\}$ **83.** $-4 \cdot \left(\dfrac{2}{5} - \dfrac{7}{10}\right)^2$ **84.** $6 \cdot \left[\left(\dfrac{1}{3}\right)^2 - \left(\dfrac{1}{2}\right)^2\right]$

85. $9 - (6 + ||3 - 7| - 8|) \div \sqrt{25}$ **86.** $8 - 2(4 + ||2 - 5| - 5|) \div \sqrt{9}$ **87.** $\dfrac{|11 - 13| - 4 \cdot 2}{\sqrt{12^2 + 5^2} - 3 - 10}$

88. $\dfrac{(4 - 9)^2 + 2^2 - 3^2}{|-7 + 4| + (-12) \div 4}$

Objective 7: Simplify Algebraic Expressions

For Exercises 89–92, apply the commutative property of addition.

89. $7 + x$

90. $9 + z$

91. $-3 + w$

92. $-11 + p$

For Exercises 93–96, apply the associative property of addition or multiplication. Then simplify if possible.

93. $(t + 3) + 9$

94. $(c + 4) + 5$

95. $\frac{1}{5}(5w)$

96. $-\frac{4}{9}\left(-\frac{9}{4}p\right)$

For Exercises 97–102, combine like terms.

97. $-14w^3 - 3w^3 + w^3$

98. $12t^5 - t^5 - 6t^5$

99. $3.9x^3y - 2.2xy^3 + 5.1x^3y - 4.7xy^3$

100. $0.004m^4n - 0.005m^3n^2 - 0.01m^4n + 0.007m^3n^2$

101. $\frac{1}{3}c^7d + \frac{1}{2}cd^7 - \frac{2}{5}c^7d - 2cd^7$

102. $\frac{1}{10}yz^4 - \frac{3}{4}y^4z + yz^4 + \frac{3}{2}y^4z$

For Exercises 103–108, apply the distributive property. (See Example 8)

103. $-(4x - \pi)$

104. $-\left(\frac{1}{2}k - \sqrt{7}\right)$

105. $-8(3x^2 + 2x - 1)$

106. $-6(-5y^2 - 3y + 4)$

107. $\frac{2}{3}(-6x^2y - 18yz^2 + 2z^3)$

108. $\frac{3}{4}(12p^5q - 8p^4q^2 - 6p^3q)$

For Exercises 109–116, simplify each expression. (See Example 9)

109. $2(4w + 8) + 7(2w - 4) + 12$

110. $3(2z - 4) + 8(z - 9) + 84$

111. $-(4u - 8v) - 3(7u - 2v) + 2v$

112. $-(10x - z) - 2(8x - 4z) - 3x$

113. $12 - 4[(8 - 2v) + 5(-3w - 4v)] - w$

114. $6 - 2[(9z + 6y) - 8(y - z)] - 11$

115. $2y^2 - \left[13 - \frac{2}{3}(6y^2 - 9) - 10\right] + 9$

116. $6 - \left[5t^2 - \frac{3}{4}(12 - 8t^2) + 5\right] + 11t^2$

Objective 8: Write Algebraic Models

117. Jake is 1 yr younger than Charlotte.

a. Write a model for Jake's age *J* in terms of Charlotte's age *C*.

b. Write a model for Charlotte's age *C* in terms of Jake's age *J*. **(See Example 10)**

118. For a recent NFL season, Aaron Rodgers had 9 more touchdown passes than Joe Flacco.

a. Write a model for the number of touchdown passes *R* thrown by Rodgers in terms of the number thrown by Flacco *F*.

b. Write a model for the number of touchdown passes *F* thrown by Flacco in terms of the number thrown by Rodgers *R*.

119. At the end of the summer, a store discounts an outdoor grill for at least 25% of the original price. If the original price is *P*, write a model for the amount of the discount *D*. **(See Example 10)**

120. When Ms. Celano has excellent service at a restaurant, the amount she leaves for a tip *t* is at least 20% of the cost of the meal *c*. Write a model representing the amount of the tip.

121. Suppose that an object is dropped from a height *h*. Its velocity *v* at impact with the ground is given by the square root of twice the product of the acceleration due to gravity *g* and the height *h*. Write a model to represent the velocity of the object at impact.

122. The height of a sunflower plant can be determined by the time *t* in weeks after the seed has germinated. Write a model to represent the height *h* if the height is given by the product of 8 and the square root of *t*.

123. A power company charges one household \$0.12 per kilowatt-hour (kWh) and \$14.89 in monthly taxes.

a. Write a formula for the monthly charge C for this household if it uses k kilowatt-hours.

b. Compute the monthly charge if the household uses 1200 kWh.

125. The cost C (in \$) to rent an apartment is \$640 per month, plus a \$500 nonrefundable security deposit, plus a \$200 nonrefundable deposit for each dog or cat.

a. Write a formula for the total cost to rent an apartment for m months with n cats/dogs.

b. Determine the cost to rent the apartment for 12 months, with 2 cats and 1 dog.

127. A hotel charges \$159 per night plus an 11% nightly room tax.

a. Write a formula to represent the total cost C for n nights in the hotel. (*Hint*: The total cost is the cost for n nights, plus the tax on the cost for n nights.)

b. Determine the cost to stay in the hotel for four nights.

124. A utility company charges a base rate for water usage of \$19.50 per month, plus \$4.58 for every additional 1000 gal of water used over a 2000-gal base.

a. Write a formula for the monthly charge C for a household that uses n thousand gallons over the 2000-gal base.

b. Compute the cost for a family that uses a total of 6000 gal of water for a given month.

126. For a certain college, the cost C (in \$) for taking classes the first semester is \$105 per credit-hour, \$35 for each lab, plus a one-time admissions fee of \$40.

a. Write a formula for the total cost to take n credit-hours and L labs the first semester.

b. Determine the cost for the first semester if a student takes 12 credit-hours with 2 labs.

128. A hotel charges \$149 per night plus a 16% nightly room tax. In addition, there is a one-time parking fee of \$40.

a. Write a formula to represent the total cost C for n nights in the hotel.

b. Determine the cost to stay in the hotel for two nights.

Mixed Exercises

For Exercises 129–134, evaluate each expression for the given values of the variables.

129. $\dfrac{-b}{2a}$ for $a = -1, b = -6$

131. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ for $x_1 = 2, x_2 = -1, y_1 = -4, y_2 = 1$

133. $\frac{1}{3}\pi r^2 h$ for $\pi \approx \frac{22}{7}, r = 7, h = 6$

135. Under selected conditions, a sports car gets 15 mpg in city driving and 25 mpg for highway driving. The model $G = \frac{1}{15}c + \frac{1}{25}h$ represents the amount of gasoline used (in gal) for c miles driven in the city and h miles driven on the highway. Determine the amount of gas required to drive 240 mi in the city and 500 mi on the highway.

130. $\sqrt{b^2 - 4ac}$ for $a = 2, b = -6, c = 4$

132. $\dfrac{y_2 - y_1}{x_2 - x_1}$ for $x_1 = -1.4, x_2 = 2, y_1 = 3.1, y_2 = -3.7$

134. $\frac{4}{3}\pi r^3$ for $\pi \approx \frac{22}{7}, r = 3$

136. Under selected conditions, a sedan gets 22 mpg in city driving and 32 mpg for highway driving. The model $G = \frac{1}{22}c + \frac{1}{32}h$ represents the amount of gasoline used (in gal) for c miles driven in the city and h miles driven on the highway. Determine the amount of gas required to drive 220 mi in the city and 512 mi on the highway.

Write About It

137. When is a parenthesis used when writing interval notation?

139. Explain the difference between the commutative property of addition and the associative property of addition.

138. When is a bracket used when writing interval notation?

140. Explain why 0 has no multiplicative inverse.

Expanding Your Skills

141. If $n > 0$, then $n - |n| =$ ______. **142.** If $n < 0$, then $n - |n| =$ ______. **143.** If $n > 0$, then $n + |n| =$ ______.

144. If $n < 0$, then $n + |n| =$ ______. **145.** If $n > 0$, then $-|n| =$ ______. **146.** If $n < 0$, then $-|n| =$ ______.

For Exercises 147–150, determine the sign of the expression. Assume that *a*, *b*, and *c* are real numbers and $a < 0$, $b > 0$, and $c < 0$.

147. $\dfrac{ab^2}{c^3}$ **148.** $\dfrac{a^2c}{b^4}$ **149.** $\dfrac{b(a+c)^3}{a^2}$ **150.** $\dfrac{(a+b)^2(b+c)^4}{b}$

For Exercises 151–154, write the set as a single interval.

151. $(-\infty, 2) \cap (-3, 4] \cap [1, 3]$ **152.** $(-\infty, 5) \cap (-1, \infty) \cap [0, 3)$

153. $[(-\infty, -2) \cup (4, \infty)] \cap [-5, 3)$ **154.** $[(-\infty, 6) \cup (10, \infty)] \cap [8, 12)$

Technology Connections

For Exercises 155–158, use a calculator to approximate the expression to 2 decimal places.

155. $5000\left(1 + \dfrac{0.06}{12}\right)^{(12)(5)}$ **156.** $8500\left(1 + \dfrac{0.05}{4}\right)^{(4)(30)}$

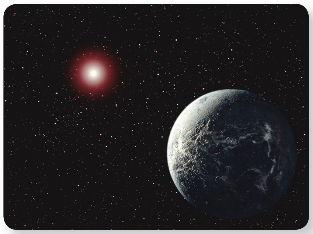
157. $\dfrac{-3 + 5\sqrt{2}}{7}$ **158.** $\dfrac{6 - 3\sqrt{5}}{4}$

SECTION R.3 Rational Exponents and Radicals

OBJECTIVES

1. Evaluate *n*th-Roots
2. Simplify Expressions of the Forms $a^{1/n}$ and $a^{m/n}$
3. Simplify Expressions with Rational Exponents
4. Simplify Radicals
5. Multiply Single-Term Radical Expressions
6. Add and Subtract Radicals

As scientists search for life beyond our solar system, they look for planets on which liquid water can exist. This means that for a planet with atmospheric pressure similar to Earth, the temperature of the planet must be greater than 0°C (the temperature at which water turns to ice) but less than 100°C (the temperature at which water turns to steam).



The following model approximates the surface temperature T_p (in °C) of an Earth-like planet based on its distance d (in km) from its primary star, the radius r (in km) of the star, and the temperature T_s (in °C) of the star.

$$T_p = 0.7(T_s + 273)\left(\frac{r}{d}\right)^{1/2} - 273$$

The expression on the right contains an exponent that is a rational number. In this section, we learn how to evaluate and simplify such expressions.

1. Evaluate *n*th-Roots

In Section R.2 we defined b^n, where b is a real number and n is an integer. In this section, we want to extend this definition to expressions in which the exponent, n, is a rational number.

First we need to understand the relationship between nth-powers and nth-roots. From Section R.1, we know that for $a \geq 0$, $\sqrt{a} = b$ if $b^2 = a$ and $b \geq 0$. Square roots are a special case of nth-roots.

Definition of an *n*th-Root

For a positive integer $n > 1$, the principal nth-root of a, denoted by $\sqrt[n]{a}$, is a number b such that

$$\sqrt[n]{a} = b \text{ means that } b^n = a.$$

If n is even, then we require that $a \geq 0$ and $b \geq 0$.

For the expression $\sqrt[n]{a}$, the symbol $\sqrt[n]{}$ is called a **radical sign,** the value a is called the **radicand,** and n is called the **index.**

EXAMPLE 1 Simplifying *n*th-Roots

Simplify.

a. $\sqrt[5]{32}$ **b.** $\sqrt{\dfrac{49}{64}}$ **c.** $\sqrt[3]{-0.008}$ **d.** $\sqrt[4]{-1}$ **e.** $-\sqrt[4]{1}$

Solution:

a. $\sqrt[5]{32} = 2$ — $\sqrt[5]{32} = 2$ because $2^5 = 32$.

b. $\sqrt{\dfrac{49}{64}} = \dfrac{7}{8}$ — $\sqrt{\dfrac{49}{64}} = \dfrac{7}{8}$ because $\dfrac{7}{8} \geq 0$ and $\left(\dfrac{7}{8}\right)^2 = \dfrac{49}{64}$.

c. $\sqrt[3]{-0.008} = -0.2$ — $\sqrt[3]{-0.008} = -0.2$ because $(-0.2)^3 = -0.008$.

d. $\sqrt[4]{-1}$ is not a real number — $\sqrt[4]{-1}$ is not a real number because no real number when raised to the fourth power equals -1.

e.
$$\begin{aligned} -\sqrt[4]{1} &= -1 \cdot \sqrt[4]{1} \\ &= -1 \cdot 1 \\ &= -1 \end{aligned}$$
$-\sqrt[4]{1}$ is interpreted as $-1\sqrt[4]{1}$. The factor of -1 is outside the radical.

Skill Practice 1 Simplify.

a. $\sqrt[3]{-125}$ **b.** $\sqrt{\dfrac{144}{121}}$ **c.** $\sqrt[5]{0.00001}$ **d.** $\sqrt[6]{-64}$ **e.** $-\sqrt[6]{64}$

2. Simplify Expressions of the Forms $a^{1/n}$ and $a^{m/n}$

Next, we want to define an expression of the form a^n, where n is a rational number. Furthermore, we want a definition for which the properties of integer exponents can be extended to rational exponents. For example, we want

$$(\underset{\uparrow}{25^{1/2}})^2 = 25^{(1/2)\cdot 2} = 25^1 = 25$$

$25^{1/2}$ must be a square root of 25, because when squared, it equals 25.

Definition of an $a^{1/n}$ and $a^{m/n}$

Let m and n be integers such that m/n is a rational number in lowest terms and $n > 1$. Then,

$$a^{1/n} = \sqrt[n]{a} \qquad \text{and} \qquad a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

If n is even, we require that $a \geq 0$.

The definition of $a^{m/n}$ indicates that $a^{m/n}$ can be written as a radical whose index is the denominator of the rational exponent. The order in which the nth-root and exponent m are performed within the radical does not affect the outcome. For example:

Take the 4th root first:
$$\begin{aligned} 16^{3/4} &= (\sqrt[4]{16})^3 \\ &= (2)^3 \\ &= 8 \end{aligned}$$

or

Cube 16 first:
$$\begin{aligned} 16^{3/4} &= \sqrt[4]{16^3} \\ &= \sqrt[4]{4096} \\ &= 8 \end{aligned}$$

Answers

1. **a.** -5 **b.** $\dfrac{12}{11}$ **c.** 0.1
   d. Not a real number
   e. -2

EXAMPLE 2 Simplifying Expressions of the Form $a^{1/n}$ and $a^{m/n}$

Write the expressions using radical notation and simplify if possible.

a. $25^{1/2}$ **b.** $\left(\frac{64}{27}\right)^{1/3}$ **c.** $(-81)^{1/4}$ **d.** $32^{3/5}$ **e.** $(-27)^{2/3}$

Solution:

a. $25^{1/2} = \sqrt{25} = 5$

b. $\left(\frac{64}{27}\right)^{1/3} = \sqrt[3]{\frac{64}{27}} = \frac{4}{3}$

c. $(-81)^{1/4}$ is undefined because $\sqrt[4]{-81}$ is not a real number.

d. $32^{3/5} = \sqrt[5]{32^3} = (\sqrt[5]{32})^3 = (2)^3 = 8$

e. $(-27)^{2/3} = \sqrt[3]{(-27)^2} = (\sqrt[3]{-27})^2 = (-3)^2 = 9$

Skill Practice 2 Simplify the expressions if possible.

a. $36^{1/2}$ **b.** $\left(\frac{1}{125}\right)^{1/3}$ **c.** $(-9)^{1/2}$ **d.** $(-1)^{4/3}$ **e.** $(16)^{3/4}$

3. Simplify Expressions with Rational Exponents

The properties of integer exponents learned in Section R.2 can be extended to expressions with rational exponents.

EXAMPLE 3 Simplifying Expressions with Rational Exponents

Simplify. Assume that all variables represent positive real numbers.

a. $\frac{x^{4/7}x^{2/7}}{x^{1/7}}$ **b.** $\left(\frac{5c^{3/4}}{d^{1/2}}\right)^2\left(\frac{d^{5/3}}{2c^{1/2}}\right)^3$ **c.** $81^{-3/4}$

Solution:

a. $\frac{x^{4/7}x^{2/7}}{x^{1/7}} = \frac{x^{4/7+2/7}}{x^{1/7}} = \frac{x^{6/7}}{x^{1/7}}$ Add the exponents in the numerator.

$= x^{6/7-1/7} = x^{5/7}$ Subtract exponents.

b. $\left(\frac{5c^{3/4}}{d^{1/2}}\right)^2\left(\frac{d^{5/3}}{2c^{1/2}}\right)^3 = \frac{5^2c^{(3/4)\cdot 2}}{d^{(1/2)\cdot 2}} \cdot \frac{d^{(5/3)\cdot 3}}{2^3c^{(1/2)\cdot 3}}$ Raise each factor inside parentheses to the power outside parentheses.

$= \frac{25c^{3/2}}{d} \cdot \frac{d^5}{8c^{3/2}}$ Multiply exponents.

$= \frac{25d^4}{8}$ Simplify.

c. $81^{-3/4} = \frac{1}{81^{3/4}}$ Rewrite the expression to remove the negative exponent.

$= \frac{1}{(\sqrt[4]{81})^3} = \frac{1}{(3)^3} = \frac{1}{27}$ Rewrite $81^{3/4}$ using radical notation.

Answers

2 a. 6 **b.** $\frac{1}{5}$
c. Undefined (not a real number)
d. 1 **e.** 8

Skill Practice 3 Simplify. Assume that all variables represent positive real numbers.

a. $\dfrac{c^{3/4}c^{7/4}}{c^{1/4}}$ **b.** $\left(\dfrac{4a^{2/3}}{b^{1/6}}\right)^3\left(\dfrac{b^{1/4}}{3a^{3/4}}\right)^2$ **c.** $(-125)^{-2/3}$

4. Simplify Radicals

In Example 3, we simplified several expressions with rational exponents. Next, we want to simplify radical expressions. First consider expressions of the form $\sqrt[n]{a^n}$. The value of $\sqrt[n]{a^n}$ is not necessarily a. Since $\sqrt[n]{a}$ represents the principal nth-root of a, then $\sqrt[n]{a}$ must be nonnegative for even values of n. For example:

$$\sqrt{(5)^2} = 5 \quad \text{and} \quad \sqrt{(-5)^2} = |-5| = 5$$

The absolute value is needed here to guarantee a nonnegative result.

$$\sqrt[4]{(2)^4} = 2 \quad \text{and} \quad \sqrt[4]{(-2)^4} = |-2| = 2$$

In Table R-6, we generalize this result and give three other important properties of radicals.

Table R-6 **Properties of Radicals**

Let $n > 1$ be an integer and a and b be real numbers. The following properties are true provided that the given radicals are real numbers.

Property	Examples	
1. If n is even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{x^2} = \lvert x \rvert$	$\sqrt[4]{x^8} = \lvert x^2 \rvert = x^2$
2. If n is odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{x^3} = x$	$\sqrt[5]{(y+4)^5} = y + 4$
3. Product Property $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt[3]{7} \cdot \sqrt[3]{x} = \sqrt[3]{7x}$	$\sqrt{75} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$
4. Quotient Property $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$	$\dfrac{\sqrt{125}}{\sqrt{5}} = \sqrt{\dfrac{125}{5}} = \sqrt{25} = 5$	
5. Nested Radical Property $\sqrt[m]{\sqrt[n]{a}} = \sqrt[m \cdot n]{a}$	$\sqrt[4]{\sqrt[3]{x}} = (x^{1/3})^{1/4} = x^{(1/3)(1/4)} = x^{1/12} = \sqrt[12]{x}$	

Properties 1–5 follow from the definition of $a^{1/n}$ and the properties of exponents. We use these properties to simplify radical expressions and must address four specific criteria.

Simplified Form of a Radical

Suppose that the radicand of a radical is written as a product of prime factors. Then the radical is simplified if all of the following conditions are met.

1. The radicand has no factor other than 1 that is a perfect nth-power. This means that all exponents in the radicand must be less than the index.
2. No fractions may appear in the radicand.
3. No denominator of a fraction may contain a radical.
4. The exponents in the radicand may not all share a common factor with the index.

Answers

3. a. $c^{9/4}$ **b.** $\dfrac{64a^{1/2}}{9}$ **c.** $\dfrac{1}{25}$

In Example 4, we simplify expressions that fail condition 1. Also notice that the expressions in Example 4 are assumed to have positive radicands. This eliminates the need to insert absolute value bars around the simplified form of $\sqrt[n]{a^n}$.

EXAMPLE 4 Simplifying Radicals Using the Product Property

Simplify each expression. Assume that all variables represent positive real numbers.

a. $\sqrt[3]{c^5}$ **b.** $\sqrt{50}$ **c.** $\sqrt[4]{32x^9y^6}$

Solution:

a. $\sqrt[3]{c^5} = \sqrt[3]{c^3 \cdot c^2}$ Write the radicand as a product of a perfect cube and another factor.

$= \sqrt[3]{c^3} \cdot \sqrt[3]{c^2}$ Apply the product property of radicals.

$= c\sqrt[3]{c^2}$ Simplify $\sqrt[3]{c^3}$ as c.

b. $\sqrt{50} = \sqrt{5^2 \cdot 2}$ Factor the radicand. The radical is not simplified because the radicand has a perfect square.

$= \sqrt{5^2} \cdot \sqrt{2}$ Apply the product property of radicals.

$= 5\sqrt{2}$ Simplify.

c. $\sqrt[4]{32x^9y^6}$

$= \sqrt[4]{2^5x^9y^6}$ Factor the radicand.

$= \sqrt[4]{(2^4x^8y^4)(2xy^2)}$ Write the radicand as the product of a perfect 4th power and another factor.

$= \sqrt[4]{2^4x^8y^4} \cdot \sqrt[4]{2xy^2}$ Apply the product property of radicals.

$= 2x^2y\sqrt[4]{2xy^2}$ Simplify the first radical.

Skill Practice 4 Simplify each expression. Assume that all variables represent positive real numbers.

a. $\sqrt[4]{d^7}$ **b.** $\sqrt{45}$ **c.** $\sqrt[3]{54x^{13}y^8}$

In Example 5, we will use the quotient property of radicals to simplify expressions that fail conditions 2 and 3 for a simplified radical. Removing a radical from the denominator of a fraction is called **rationalizing the denominator.**

EXAMPLE 5 Applying the Quotient Property of Radicals

Simplify the expressions. Assume that x and y are nonzero real numbers.

a. $\sqrt{\dfrac{x^3}{9}}$ **b.** $\dfrac{\sqrt[3]{3x^7y}}{\sqrt[3]{81xy^4}}$

Solution:

a. $\sqrt{\dfrac{x^3}{9}} = \dfrac{\sqrt{x^3}}{\sqrt{9}}$ Apply the quotient property of radicals.

$= \dfrac{\sqrt{x^2 \cdot x}}{3}$ Write the radicand as a product of a perfect square and another factor.

$= \dfrac{\sqrt{x^2} \cdot \sqrt{x}}{3}$ Apply the product property of radicals.

$= \dfrac{x\sqrt{x}}{3}$ Simplify.

Answers

4. a. $d\sqrt[4]{d^3}$ **b.** $3\sqrt{5}$

c. $3x^4y^2\sqrt[3]{2xy^2}$

TIP In Example 5(b), the purpose of writing the quotient of two radicals as a single radical is to simplify the resulting fraction in the radicand.

b. $\dfrac{\sqrt[3]{3x^7y}}{\sqrt[3]{81xy^4}} = \sqrt[3]{\dfrac{3x^7y}{81xy^4}}$ Apply the quotient property of radicals to write the expression as a single radical.

$= \sqrt[3]{\dfrac{x^6}{27y^3}}$ The numerator and denominator share common factors. Simplify the fraction.

$= \dfrac{x^2}{3y}$ Simplify.

Skill Practice 5 Simplify the expressions. Assume that x and y are nonzero real numbers.

a. $\sqrt{\dfrac{y^5}{49}}$ **b.** $\dfrac{\sqrt[3]{625c^2d^{10}}}{\sqrt[3]{5c^5d}}$

5. Multiply Single-Term Radical Expressions

In Example 6, we use the product property of radicals to multiply two radical expressions. We can simplify a product of radicals, provided the indices are the same.

EXAMPLE 6 Multiplying Single-Term Radicals

Multiply. Assume that x represents a positive real number.

a. $\sqrt{6} \cdot \sqrt{10}$ **b.** $(2\sqrt[4]{x^3})(5\sqrt[4]{x^7})$

Solution:

TIP When multiplying radicals, we have the option of factoring the individual radicands before multiplying. For example:

$\sqrt{6} \cdot \sqrt{10}$

$= \sqrt{2 \cdot 3} \cdot \sqrt{2 \cdot 5}$

$= \sqrt{2^2 \cdot 3 \cdot 5}$

a. $\sqrt{6} \cdot \sqrt{10} = \sqrt{60}$ The radicals have the same index. Apply the product property of radicals.

$= \sqrt{2^2 \cdot 3 \cdot 5}$ Factor the radicand.

$= \sqrt{(2^2) \cdot (3 \cdot 5)}$ Write the radicand as the product of a perfect square and another factor.

$= \sqrt{2^2} \cdot \sqrt{3 \cdot 5}$ Apply the product property of radicals.

$= 2\sqrt{15}$ Simplify.

b. $(2\sqrt[4]{x^3})(5\sqrt[4]{x^7}) = 2 \cdot 5\sqrt[4]{x^3 \cdot x^7}$ Regroup factors, and apply the product property of radicals.

$= 10\sqrt[4]{x^{10}}$ Simplify the radicand.

$= 10\sqrt[4]{x^8 \cdot x^2}$ Write the radicand as the product of a perfect 4th power and another factor.

Avoiding Mistakes

The product property of radicals can be applied only if the radicals have the same index.

$= 10\sqrt[4]{x^8} \cdot \sqrt[4]{x^2}$ Apply the product property of radicals.

$= 10x^2\sqrt[4]{x^2}$ The expression $\sqrt[4]{x^2}$ fails condition 4 for a simplified radical. (The exponents in the radicand cannot all share a common factor with the index.)

$= 10x^2\sqrt[2]{x^1}$

$= 10x^2\sqrt{x}$ So, $\sqrt[4]{x^2} = x^{2/4} = x^{1/2} = \sqrt{x}$.

Skill Practice 6 Multiply. Assume that y represents a positive real number.

a. $\sqrt{15} \cdot \sqrt{21}$ **b.** $(3\sqrt[6]{y^5})(4\sqrt[6]{y^{11}})$

Answers

5. a. $\dfrac{y^2\sqrt{y}}{7}$ **b.** $\dfrac{5d^3}{c}$

6. a. $3\sqrt{35}$ **b.** $12y^2\sqrt[3]{y^2}$

6. Add and Subtract Radicals

We can use the distributive property to add or subtract radical expressions. However, the radicals must be like radicals. This means that the radicands must be the same and the indices must be the same. For example:

$3\sqrt{2x}$ and $-5\sqrt{2x}$ are like radicals.
$3\sqrt{2x}$ and $-5\sqrt[3]{2x}$ are not like radicals because the indices are different.
$3\sqrt{2x}$ and $-5\sqrt{2y}$ are not like radicals because the radicands are different.

EXAMPLE 7 Adding and Subtracting Radicals

Add or subtract as indicated. Assume that all variables represent positive real numbers.

a. $5\sqrt[3]{7t^2} - 2\sqrt[3]{7t^2} + \sqrt[3]{7t^2}$
b. $x\sqrt{98x^3y} + 5\sqrt{18x^5y}$
c. $3\sqrt{5x} + 2x\sqrt{5x}$

Solution:

a. $5\sqrt[3]{7t^2} - 2\sqrt[3]{7t^2} + 1\sqrt[3]{7t^2}$	The radicals are like radicals. They have the same radicand and same index.
$= (5 - 2 + 1)\sqrt[3]{7t^2}$	Apply the distributive property.
$= 4\sqrt[3]{7t^2}$	Simplify.
b. $x\sqrt{98x^3y} + 5\sqrt{18x^5y}$	Each radical can be simplified. $x\sqrt{98x^3y} = x\sqrt{(7^2x^2)\cdot(2xy)} = 7x^2\sqrt{2xy}$ $5\sqrt{18x^5y} = 5\sqrt{(3^2x^4)(2xy)} = 15x^2\sqrt{2xy}$
$= 7x^2\sqrt{2xy} + 15x^2\sqrt{2xy}$	The terms are like terms.
$= (7 + 15)x^2\sqrt{2xy}$	Apply the distributive property.
$= 22x^2\sqrt{2xy}$	Simplify.
c. $3\sqrt{5x} + 2x\sqrt{5x}$	The radicals are like radicals.
$= (3 + 2x)\sqrt{5x}$	Apply the distributive property. The expression cannot be further simplified because the terms within parentheses are not like terms.

Skill Practice 7 Add or subtract as indicated. Assume that all variables represent positive real numbers.

a. $-4\sqrt[3]{5w} + 9\sqrt[3]{5w} - 11\sqrt[3]{5w}$ **b.** $\sqrt{75cd^4} + 6d\sqrt{27cd^2}$
c. $8\sqrt{7z} + 3z\sqrt{7z}$

Answers
7. a. $-6\sqrt[3]{5w}$ **b.** $23d^2\sqrt{3c}$
c. $(8 + 3z)\sqrt{7z}$

SECTION R.3 Practice Exercises

Concept Connections

1. b is an nth-root of a if $b^{\square} = a$.

2. Given the expression $\sqrt[n]{a}$, the value a is called the ______________ and n is called the ______________.

3. The expression $a^{m/n}$ can be written in radical notation as ______________, provided that $\sqrt[n]{a}$ is a real number.

4. The expression $a^{1/n}$ can be written in radical notation as ______________, provided that $\sqrt[n]{a}$ is a real number.

5. If x represents any real number, then $\sqrt{x^2} =$ ______________.

6. If x represents any real number, then $\sqrt[3]{x^3} =$ ______________.

7. The product property of radicals indicates that $\sqrt[n]{a} \cdot \sqrt[n]{b} =$ ______________ provided that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ represent real numbers.

8. Removing a radical from the denominator of a fraction is called ______________ the denominator.

Objective 1: Evaluate *n*th-Roots

For Exercises 9–20, simplify the expression. (See Example 1)

9. $\sqrt[4]{81}$ **10.** $\sqrt[3]{125}$ **11.** $\sqrt{\dfrac{4}{49}}$ **12.** $\sqrt{\dfrac{9}{121}}$

13. $\sqrt{0.09}$ **14.** $\sqrt{0.16}$ **15.** $\sqrt[4]{-81}$ **16.** $\sqrt[4]{-625}$

17. $-\sqrt[4]{81}$ **18.** $-\sqrt[4]{625}$ **19.** $\sqrt[3]{-\dfrac{1}{8}}$ **20.** $\sqrt[3]{-\dfrac{64}{125}}$

Objective 2: Simplify Expressions of the Forms $a^{1/n}$ and $a^{m/n}$

For Exercises 21–30, simplify each expression. (See Example 2)

21. a. $25^{1/2}$ **b.** $(-25)^{1/2}$ **c.** $-25^{1/2}$

22. a. $36^{1/2}$ **b.** $(-36)^{1/2}$ **c.** $-36^{1/2}$

23. a. $27^{1/3}$ **b.** $(-27)^{1/3}$ **c.** $-27^{1/3}$

24. a. $125^{1/3}$ **b.** $(-125)^{1/3}$ **c.** $-125^{1/3}$

25. a. $\left(\dfrac{121}{169}\right)^{1/2}$ **b.** $\left(\dfrac{121}{169}\right)^{-1/2}$

26. a. $\left(\dfrac{49}{144}\right)^{1/2}$ **b.** $\left(\dfrac{49}{144}\right)^{-1/2}$

27. a. $16^{3/4}$ **b.** $16^{-3/4}$ **c.** $-16^{3/4}$
d. $-16^{-3/4}$ **e.** $(-16)^{3/4}$ **f.** $(-16)^{-3/4}$

28. a. $81^{3/4}$ **b.** $81^{-3/4}$ **c.** $-81^{3/4}$
d. $-81^{-3/4}$ **e.** $(-81)^{3/4}$ **f.** $(-81)^{-3/4}$

29. a. $64^{2/3}$ **b.** $64^{-2/3}$ **c.** $-64^{2/3}$
d. $-64^{-2/3}$ **e.** $(-64)^{2/3}$ **f.** $(-64)^{-2/3}$

30. a. $8^{2/3}$ **b.** $8^{-2/3}$ **c.** $-8^{2/3}$
d. $-8^{-2/3}$ **e.** $(-8)^{2/3}$ **f.** $(-8)^{-2/3}$

For Exercises 31–32, write the expression using radical notation. Assume that all variables represent positive real numbers.

31. a. $y^{4/11}$ **b.** $6y^{4/11}$ **c.** $(6y)^{4/11}$

32. a. $z^{3/10}$ **b.** $8z^{3/10}$ **c.** $(8z)^{3/10}$

For Exercises 33–40, write the expression using rational exponents. Assume that all variables represent positive real numbers.

33. $\sqrt[5]{a^3}$

34. $\sqrt[7]{z^4}$

35. $\sqrt{6x}$

36. $\sqrt{11t}$

37. $6\sqrt{x}$

38. $11\sqrt{t}$

39. $\sqrt[5]{a^5 + b^5}$

40. $\sqrt[3]{m^3 + n^3}$

Objective 3: Simplify Expressions with Rational Exponents

For Exercises 41–50, simplify each expression. Assume that all variable expressions represent positive real numbers. (See Example 3)

41. $\dfrac{a^{2/3}a^{5/3}}{a^{1/3}}$

42. $\dfrac{y^{7/5}y^{4/5}}{y^{1/5}}$

43. $\dfrac{3w^{-2/3}}{y^{-1/3}}$

44. $\dfrac{8d^{-5/7}}{c^{-3/4}}$

45. $(16x^{-8}y^{1/5})^{3/4}$

46. $(125a^6b^{-7/5})^{1/3}$

47. $\left(\dfrac{2m^{2/3}}{n^{3/4}}\right)^{12}\left(\dfrac{n^{1/5}}{2m^{1/2}}\right)^{10}$

48. $\left(\dfrac{3x^{1/2}}{y^{3/8}}\right)^{4}\left(\dfrac{y^{1/2}}{3x^{4/3}}\right)^{3}$

49. $\left(\dfrac{m^2}{m+n}\right)^{-1}\left(\dfrac{m^2}{m+n}\right)^{1/2}$

50. $\left(\dfrac{c^2}{c-d}\right)^{-2}\left(\dfrac{c^2}{c-d}\right)^{3/2}$

Objective 4: Simplify Radicals

51. a. For what values of t will the statement be true? $\sqrt{t^2} = t$

b. For what value of t will the statement be true? $\sqrt{t^2} = |t|$

52. a. For what values of c will the statement be true? $\sqrt[4]{(c+8)^4} = c + 8$

b. For what value of c will the statement be true? $\sqrt[4]{(c+8)^4} = |c + 8|$

For Exercises 53–60, simplify each expression.

53. $\sqrt{y^2}$

54. $\sqrt[4]{y^4}$

55. $\sqrt[3]{y^3}$

56. $\sqrt[5]{y^5}$

57. $\sqrt[4]{(2x-5)^4}$

58. $\sqrt{(3z+2)^2}$

59. $\sqrt{w^{12}}$

60. $\sqrt[4]{c^{32}}$

For Exercises 61–76, simplify each expression. Assume that all variable expressions represent positive real numbers. (See Examples 4–5)

61. a. $\sqrt{c^7}$ **b.** $\sqrt[3]{c^7}$ **c.** $\sqrt[4]{c^7}$ **d.** $\sqrt[9]{c^7}$

62. a. $\sqrt{d^{11}}$ **b.** $\sqrt[3]{d^{11}}$ **c.** $\sqrt[4]{d^{11}}$ **d.** $\sqrt[12]{d^{11}}$

63. a. $\sqrt{24}$ **b.** $\sqrt[3]{24}$

64. a. $\sqrt{54}$ **b.** $\sqrt[3]{54}$

65. $\sqrt[3]{250x^2y^6z^{11}}$

66. $\sqrt[3]{40ab^{13}c^{17}}$

67. $\sqrt[4]{96p^{14}q^7}$

68. $\sqrt[4]{243m^{19}n^{10}}$

69. $\sqrt{84(y-2)^3}$

70. $\sqrt{18(w-6)^3}$

71. $\sqrt{\dfrac{p^7}{36}}$

72. $\sqrt{\dfrac{q^{11}}{4}}$

73. $4\sqrt[3]{\dfrac{w^3z^5}{8}}$

74. $8\sqrt[3]{\dfrac{c^6d^7}{64}}$

75. $\dfrac{\sqrt[3]{5x^5y}}{\sqrt[3]{625x^2y^4}}$

76. $\dfrac{\sqrt[3]{2m^2n^7}}{\sqrt[3]{16m^{14}n^4}}$

5. Multiply Single-Term Radical Expressions

For Exercises 77–84, simplify each expression. Assume that all variables represent positive real numbers. (See Example 6)

77. $\sqrt{10} \cdot \sqrt{14}$

78. $\sqrt{6} \cdot \sqrt{21}$

79. $\sqrt[3]{xy^2} \cdot \sqrt[3]{x^2y}$

80. $\sqrt[4]{a^3b} \cdot \sqrt[4]{ab^3}$

81. $(3\sqrt[4]{a^3})(-5\sqrt[4]{a^3})$

82. $(7\sqrt[6]{t^5})(-2\sqrt[6]{t^5})$

83. $\left(-\dfrac{1}{2}\sqrt[3]{6a^2b^2c}\right)\left(\dfrac{4}{3}\sqrt[3]{4a^2c^2}\right)$

84. $\left(-\dfrac{3}{4}\sqrt[3]{9m^2n^5p}\right)\left(\dfrac{1}{6}\sqrt[3]{6m^2np^4}\right)$

6. Add and Subtract Radicals

For Exercises 85–94, add or subtract as indicated. Assume that all variables represent positive real numbers. (See Example 7)

85. $3\sqrt[3]{2y^2} - 9\sqrt[3]{2y^2} + \sqrt[3]{2y^2}$

86. $8\sqrt[4]{3z^3} - \sqrt[4]{3z^3} + 2\sqrt[4]{3z^3}$

87. $\frac{1}{5}\sqrt{50} - \frac{7}{3}\sqrt{18} + \frac{5}{6}\sqrt{72}$

88. $\frac{2}{5}\sqrt{75} - \frac{2}{3}\sqrt{27} - \frac{1}{2}\sqrt{12}$

89. $-3x\sqrt[3]{16xy^4} + xy\sqrt[3]{54xy} - 5\sqrt[3]{250x^4y^4}$

90. $8\sqrt[4]{32a^5b^6} - 5b\sqrt[4]{2a^5b^2} - ab\sqrt[4]{162ab^2}$

91. $12\sqrt{2y} + 5y\sqrt{2y}$

92. $-8\sqrt{3w} + 3w\sqrt{3w}$

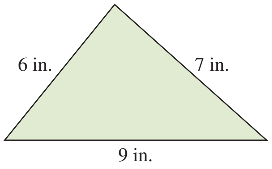
93. $-\frac{1}{2}\sqrt{8z^3} + \frac{3}{7}\sqrt{98z}$

94. $\frac{2}{3}\sqrt{45c} + \frac{1}{2}\sqrt{20c^3}$

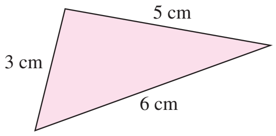
Mixed Exercises

For Exercises 95–96, use Heron's formula to determine the area A of a triangle with sides of length a, b, and c. Write each answer as a simplified radical. Heron's formula: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$.

95.

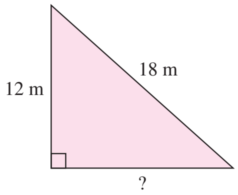


96.

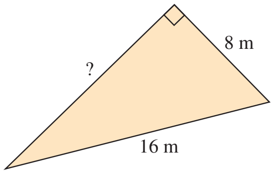


For Exercises 97–98, use the Pythagorean theorem to determine the length of the missing side. Write the answer as a simplified radical.

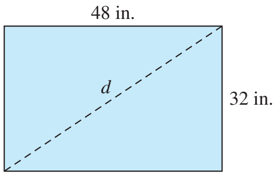
97.



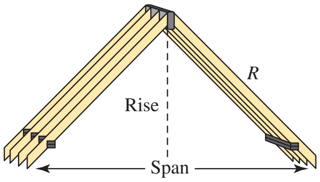
98.



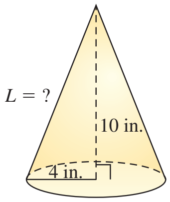
99. The size of a television is identified by the length of the diagonal. If Lynn's television is 48 in. across and 32 in. high, what size television does she have? Give the exact value and a decimal approximation to the nearest inch.



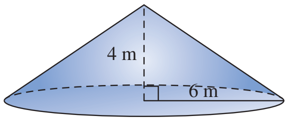
100. If the span of a roof is 36 ft and the rise is 12 ft, determine the length of the rafter R. Give the exact value and a decimal approximation to the nearest tenth of a foot.



101. The slant length L for a right circular cone is given by $L = \sqrt{r^2 + h^2}$, where r and h are the radius and height of the cone. Find the slant length of a cone with radius 4 in. and height 10 in. Determine the exact value and a decimal approximation to the nearest tenth of an inch.



102. The lateral surface area A of a right circular cone is given by $A = \pi r\sqrt{r^2 + h^2}$, where r and h are the radius and height of the cone. Determine the exact value (in terms of π) of the lateral surface area of a cone with radius 6 m and height 4 m. Then give a decimal approximation to the nearest square meter.



103. The depreciation rate for a car is given by $r = 1-\left(\frac{S}{C}\right)^{1/n}$, where S is the value of the car after n years, and C is the initial cost. Determine the depreciation rate for a car that originally cost \$22,990 and was valued at \$11,500 after 4 yr. Round to the nearest tenth of a percent.

104. For a certain oven, the baking time t (in hr) for a turkey that weighs x pounds can be approximated by the model $t = 0.84x^{3/5}$. Determine the baking time for a 15-lb turkey. Round to 1 decimal place.

Write About It

105. Explain the similarity in simplifying the given expressions.

a. $2x + 3x$
b. $2\sqrt{x} + 3\sqrt{x}$
c. $2\sqrt[3]{x} + 3\sqrt[3]{x}$

106. Explain why the given expressions cannot be simplified further.

a. $2x + 3y$
b. $2\sqrt{x} + 3\sqrt{y}$
c. $2\sqrt[3]{x} + 3\sqrt{x}$

Expanding Your Skills

For Exercises 107–112, write each expression as a single radical for positive values of the variable. (*Hint*: Write the radicals as expressions with rational exponents and simplify. Then convert back to radical form.)

107. $\sqrt[5]{x^3y^2} \cdot \sqrt[4]{x}$
108. $\sqrt[4]{a^2b} \cdot \sqrt[3]{ab^2}$
109. $\sqrt[6]{m\sqrt[3]{m^2}}$
110. $\sqrt[5]{y\sqrt[4]{y^3}}$
111. $\sqrt{x\sqrt{x\sqrt{x}}}$
112. $\sqrt[3]{y\sqrt[3]{y\sqrt[3]{y}}}$

For Exercises 113–114, evaluate the expression without the use of a calculator.

113. $\sqrt{\dfrac{8.0 \times 10^{12}}{2.0 \times 10^4}}$

114. $\sqrt{\dfrac{1.44 \times 10^{16}}{9.0 \times 10^{10}}}$

For Exercises 115–116, simplify the expression.

115. $\sqrt{\sqrt[3]{6} + \sqrt[4]{16}} + \sqrt{\sqrt{25} + \sqrt{16}} + \sqrt{9}$

116. $\sqrt{\sqrt[4]{11} + \sqrt[3]{125}} + \sqrt{\sqrt{81} + \sqrt[3]{1000} + \sqrt{36}} + \sqrt{25}$

The mean surface temperature T_p (in °C) of an Earth-like planet can be approximated based on its distance from its primary star d (in km), the radius of the star r (in km), and the temperature of the star T_s (in °C) by the following formula.

$$T_p = 0.7(T_s + 273)\left(\frac{r}{d}\right)^{1/2} - 273$$

For Exercises 117–118, use the model to find T_p.

117. The star Altair is relatively close to the Earth (16.8 light-years) and has a mean surface temperature of approximately 7700°C. Although not completely spherical in shape, Altair has a mean radius of approximately 1.26×10^6 km. If a planet with an atmosphere similar to that of the Earth is 4.3×10^8 km away from Altair, will the temperature on the surface of the planet be suitable for liquid water to exist? (Recall that under pressure similar to that at sea level on Earth, water freezes at 0°C and turns to steam at 100°C.)

118. Suppose the Sun has a mean surface temperature of 5700°C and a radius of approximately 7.0×10^5 km. If the Earth is a distance of 1.49×10^8 km from the Sun, approximate the mean surface temperature for the Earth.

SECTION R.6 Rational Expressions and More Operations on Radicals

OBJECTIVES

1. Determine Restricted Values for a Rational Expression
2. Simplify Rational Expressions
3. Multiply and Divide Rational Expressions
4. Add and Subtract Rational Expressions
5. Simplify Complex Fractions
6. Rationalize the Denominator of a Radical Expression

1. Determine Restricted Values for a Rational Expression

Suppose that an object that is originally 35°C is placed in a freezer. The temperature T (in °C) of the object t hours after being placed in the freezer can be approximated by the model

$$T = \frac{350}{t^2 + 3t + 10}$$

For example, 2 hr after being placed in the freezer the temperature of the object is

$$T = \frac{350}{(2)^2 + 3(2) + 10} = 17.5°\text{C}$$

The expression $\frac{350}{t^2 + 3t + 10}$ is called a rational expression. A **rational expression** is a ratio of two polynomials. Since a rational expression may have a variable in the denominator, we must be careful to exclude values of the variable that make the denominator zero.

EXAMPLE 1 Determining Restricted Values for a Rational Expression

Determine the restrictions on the variable for each rational expression.

a. $\frac{x-3}{x+2}$ **b.** $\frac{x}{x^2-49}$ **c.** $\frac{4}{5x^2y}$

Solution:

a. $$\frac{x-3}{x+2}$$ $\boxed{x \neq -2}$

Division by zero is undefined.

For this expression $x \neq -2$. If -2 were substituted for x, the denominator would be zero. $-2 + 2 = 0$

b. $$\frac{x}{x^2-49} = \frac{x}{(x+7)(x-7)}$$ $\boxed{x \neq -7}$ $\boxed{x \neq 7}$

For this expression, $x \neq -7$ and $x \neq 7$. If x were -7, then $-7 + 7 = 0$. If x were 7, then $7 - 7 = 0$. In either case, the denominator would be zero.

c. $$\frac{4}{5x^2y}$$ $\boxed{x \neq 0}$ $\boxed{y \neq 0}$

For this expression, $x \neq 0$ and $y \neq 0$. If either x or y were zero, then the product would be zero.

Skill Practice 1 Determine the restrictions on the variable.

a. $\frac{x+4}{x-3}$ **b.** $\frac{5}{c^2-16}$ **c.** $\frac{6}{7ab^3}$

2. Simplify Rational Expressions

A rational expression is simplified (in lowest terms) if the only factors shared by the numerator and denominator are 1 or -1. To simplify a rational expression, factor the numerator and denominator, then apply the property of equivalent algebraic fractions.

Answers

1. **a.** $x \neq 3$ **b.** $c \neq -4$ and $c \neq 4$
c. $a \neq 0$ and $b \neq 0$

Equivalent Algebraic Fractions

If a, b, and c represent real-valued expressions, then

$$\frac{ac}{bc} = \frac{a}{b} \quad \text{for } b \neq 0 \text{ and } c \neq 0 \qquad \underline{\text{Example}}: \quad \frac{10x}{15x} = \frac{2 \cdot \overset{1}{\cancel{5 \cdot x}}}{3 \cdot \underset{1}{\cancel{5 \cdot x}}} = \frac{2}{3}$$

EXAMPLE 2 Simplifying Rational Expressions

Simplify.

a. $\dfrac{x^2 - 16}{x^2 - x - 12}$ **b.** $\dfrac{8 + 2\sqrt{7}}{4}$

Solution:

a. $\dfrac{x^2 - 16}{x^2 - x - 12} = \dfrac{(x + 4)(x - 4)}{(x + 3)(x - 4)}$ Factor the numerator and denominator. We have the restrictions that $x \neq -3$, $x \neq 4$.

$= \dfrac{(x + 4)\overset{1}{\cancel{(x - 4)}}}{(x + 3)\underset{1}{\cancel{(x - 4)}}}$ Divide out common factors that form a ratio of $\frac{1}{1}$.

$= \dfrac{x + 4}{x + 3}$ for $x \neq -3$, $x \neq 4$ The same restrictions for the original expression also apply to the simplified expression.

b. $\dfrac{8 + 2\sqrt{7}}{4} = \dfrac{2\left(4 + \sqrt{7}\right)}{2 \cdot 2} = \dfrac{\overset{1}{\cancel{2}}\left(4 + \sqrt{7}\right)}{\underset{1}{\cancel{2}} \cdot 2} = \dfrac{4 + \sqrt{7}}{2}$

Skill Practice 2 Simplify.

a. $\dfrac{x^2 - 8x}{x^2 - 7x - 8}$ **b.** $\dfrac{3 + 9\sqrt{5}}{6}$

In Example 2(a), the expressions $\dfrac{x^2 - 16}{x^2 - x - 12}$ and $\dfrac{x + 4}{x + 3}$ are equal for all values of x for which *both* expressions are defined. This excludes the values $x = -3$ and $x = 4$.

The property of equivalent algebraic fractions tells us that we can divide out common factors that form a ratio of 1. We can also divide out factors that form a ratio of -1. For example:

$$\frac{4 - x}{x - 4} \xrightarrow{\text{Factor out } -1 \text{ from the numerator.}} \frac{-1(-4 + x)}{x - 4} = \frac{-1(x - 4)}{x - 4} = -1$$

Numerator and denominator are opposite polynomials. Their ratio is -1.

TIP The expressions $4 - x$ and $x - 4$ are opposite polynomials (the signs of their terms are opposites). The ratio of two opposite factors is -1.

EXAMPLE 3 Simplifying Rational Expressions

Simplify. $\dfrac{14 - 2x}{x^2 - 7x}$

Answers

2. a. $\dfrac{x}{x + 1}$; $x \neq 8, x \neq -1$

b. $\dfrac{1 + 3\sqrt{5}}{2}$

Solution:

$$\frac{14-2x}{x^2-7x}=\frac{2(7-x)}{x(x-7)}$$

Factor the numerator and denominator. We have the restrictions that $x \neq 0$, $x \neq 7$.

$$=\frac{2\overset{(-1)}{\cancel{(7-x)}}}{x\cancel{(x-7)}}$$

The factors $(7-x)$ and $(x-7)$ are opposites. Their ratio is -1.

Divide out opposite factors that form a ratio of -1.

$$=-\frac{2}{x} \quad \text{for } x \neq 0, x \neq 7$$

The same restrictions for the original expression also apply to the simplified expression.

TIP The negative sign in the answer to Example 3 can be placed in the numerator, denominator, or out in front of the fraction.

$$\frac{-2}{x}=\frac{2}{-x}=-\frac{2}{x}$$

Skill Practice 3 Simplify. $\dfrac{30-10m}{m^2-9}$

3. Multiply and Divide Rational Expressions

To multiply and divide rational expressions, use the following properties.

Multiplication and Division of Algebraic Fractions

Let a, b, c, and d be real-valued expressions.

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd} \quad \text{for } b \neq 0, d \neq 0$$

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc} \quad \text{for } b \neq 0, c \neq 0, d \neq 0$$

To multiply or divide rational expressions, factor the numerator and denominator of each fraction completely. Then apply the multiplication and division properties for algebraic fractions.

Restriction Agreement

Operations on rational expressions are valid for all values of the variable for which the rational expressions are defined. In Examples 4–10 and in the exercises, we will perform operations on rational expressions without explicitly stating the restrictions. Instead, the restrictions on the variables will be implied.

EXAMPLE 4 Multiplying Rational Expressions

Multiply. $\dfrac{2xy}{x^2y+3xy}\cdot\dfrac{x^2+6x+9}{4x+12}$

Solution:

$$\frac{2xy}{x^2y+3xy}\cdot\frac{x^2+6x+9}{4x+12}=\frac{2xy}{xy(x+3)}\cdot\frac{(x+3)^2}{4(x+3)}$$

$$=\frac{\overset{1}{\cancel{2}}\,\overset{1}{\cancel{xy}}\cdot\overset{1}{\cancel{(x+3)}}\overset{1}{\cancel{(x+3)}}}{\cancel{xy}\cancel{(x+3)}\cdot\cancel{2}\cdot 2\cancel{(x+3)}}$$

$$=\frac{1}{2}$$

Avoiding Mistakes

All factors from the numerator divided out, leaving a factor of 1. Do not forget to write 1 in the numerator.

Skill Practice 4 Multiply. $\dfrac{7x-7y}{x^2-2xy+y^2}\cdot\dfrac{x^2-xy}{21x}$

Answers

3. $-\dfrac{10}{m+3}$; $m \neq 3$, $m \neq -3$

4. $\dfrac{1}{3}$

EXAMPLE 5 Dividing Rational Expressions

Divide. $\dfrac{x^3 - 8}{4 - x^2} \div \dfrac{3x^2 + 6x + 12}{x^2 - x - 6}$

Solution:

$$\frac{x^3 - 8}{4 - x^2} \div \frac{3x^2 + 6x + 12}{x^2 - x - 6}$$

$$= \frac{x^3 - 8}{4 - x^2} \cdot \frac{x^2 - x - 6}{3x^2 + 6x + 12}$$

Multiply the first fraction by the reciprocal of the second fraction.

$$= \frac{(x - 2)(x^2 + 2x + 4)}{(2 - x)(2 + x)} \cdot \frac{(x - 3)(x + 2)}{3(x^2 + 2x + 4)}$$

Factor. Note that $x^3 - 8$ is a difference of cubes.

$$= \frac{\overset{(-1)}{\cancel{(x - 2)}}\overset{1}{\cancel{(x^2 + 2x + 4)}} \cdot (x - 3)\overset{1}{\cancel{(x + 2)}}}{\cancel{(2 - x)}\cancel{(2 + x)} \cdot 3\cancel{(x^2 + 2x + 4)}}$$

Note that $(x - 2)$ and $(2 - x)$ are opposite polynomials, and their ratio is -1.

$$= -\frac{x - 3}{3}$$

Avoiding Mistakes

For the expression $-\dfrac{x - 3}{3}$ do not be tempted to "divide out" the 3 in the numerator with the 3 in the denominator. The 3's are *terms*, not factors. Only common *factors* can be divided out.

Skill Practice 5 Divide. $\dfrac{y^3 - 27}{9 - y^2} \div \dfrac{5y^2 + 15y + 45}{y^2 + 8y + 15}$

4. Add and Subtract Rational Expressions

Recall that fractions can be added or subtracted if they have a common denominator.

Addition and Subtraction of Algebraic Fractions

Let a, b, c, and d be real-valued expressions.

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a - c}{b} \quad \text{for } b \neq 0$$

If two rational expressions have different denominators, then it is necessary to convert the expressions to equivalent expressions with the same denominator. We do this by applying the property of equivalent fractions.

Property of Equivalent Fractions

Let a, b, and c be real-valued expressions.

$\dfrac{a}{b} = \dfrac{ac}{bc}$, where $b \neq 0$ and $c \neq 0$ Example: $\dfrac{5}{x} = \dfrac{5 \cdot xy}{x \cdot xy} = \dfrac{5xy}{x^2y}$

When adding or subtracting numerical fractions or rational expressions, it is customary to use the least common denominator (LCD) of the original expressions and to follow these guidelines.

Answer

5. $-\dfrac{y + 5}{5}$

Adding and Subtracting Rational Expressions

Step 1 Factor the denominators and determine the LCD of all expressions. The LCD is the product of unique prime factors where each factor is raised to the greatest power to which it appears in any denominator.

Step 2 Write each expression as an equivalent expression with the LCD as its denominator.

Step 3 Add or subtract the numerators as indicated and write the result over the LCD.

Step 4 Simplify if possible.

EXAMPLE 6 Adding Rational Expressions

Add the rational expressions and simplify the result. $\dfrac{7}{4a} + \dfrac{11}{10a^2}$

Solution:

$$\frac{7}{4a} + \frac{11}{10a^2} = \frac{7}{2^2 a} + \frac{11}{2 \cdot 5a^2}$$

Step 1: Factor the denominators. The LCD is $2^2 \cdot 5a^2$ or $20a^2$.

$$= \frac{7 \cdot (5a)}{2^2 a \cdot (5a)} + \frac{11 \cdot (2)}{2 \cdot 5a^2 \cdot (2)}$$

Step 2: Multiply numerator and denominator of each expression by the factors missing from the denominators.

$$= \frac{35a}{20a^2} + \frac{22}{20a^2}$$

Step 3: Add the numerators and write the result over the common denominator.

$$= \frac{35a + 22}{20a^2}$$

Step 4: The expression is already simplified.

Skill Practice 6 Add the rational expressions and simplify the result.

$$\frac{8}{9y^2} + \frac{1}{15y}$$

EXAMPLE 7 Subtracting Rational Expressions

Subtract the rational expressions and simplify the result.

$$\frac{3x + 5}{x^2 + 4x + 3} - \frac{x - 5}{x^2 + 2x - 3}$$

Solution:

$$\frac{3x + 5}{x^2 + 4x + 3} - \frac{x - 5}{x^2 + 2x - 3}$$

$$= \frac{3x + 5}{(x + 3)(x + 1)} - \frac{x - 5}{(x + 3)(x - 1)}$$

Factor the denominators. The LCD is $(x + 3)(x + 1)(x - 1)$.

$$= \frac{(3x + 5)(x - 1)}{(x + 3)(x + 1)(x - 1)} - \frac{(x - 5)(x + 1)}{(x + 3)(x - 1)(x + 1)}$$

Answer

6. $\dfrac{40 + 3y}{45y^2}$

$$= \frac{(3x + 5)(x - 1) - (x - 5)(x + 1)}{(x + 3)(x + 1)(x - 1)}$$

Subtract the numerators and write the result over the common denominator.

$$= \frac{3x^2 + 2x - 5 - (x^2 - 4x - 5)}{(x + 3)(x + 1)(x - 1)}$$

Avoiding Mistakes

It is very important to use parentheses around the second trinomial in the numerator. This will ensure that all terms that follow will be subtracted.

$$= \frac{3x^2 + 2x - 5 - x^2 + 4x + 5}{(x + 3)(x + 1)(x - 1)}$$

$$= \frac{2x^2 + 6x}{(x + 3)(x + 1)(x - 1)}$$

$$= \frac{2x\overset{1}{\cancel{(x + 3)}}}{\cancel{(x + 3)}(x + 1)(x - 1)}$$

Factor the numerator and denominator and simplify.

$$= \frac{2x}{(x + 1)(x - 1)}$$

Skill Practice 7 Subtract the rational expressions and simplify the result.

$$\frac{t}{t^2 + 5t + 6} - \frac{2}{t^2 + 3t + 2}$$

5. Simplify Complex Fractions

A **complex fraction** (also called a **compound fraction**) is an expression that contains one or more fractions in the numerator or denominator. We present two methods to simplify a complex fraction. Method I is an application of the order of operations.

Simplifying a Complex Fraction: Order of Operations (Method I)

Step 1 Add or subtract the fractions in the numerator to form a single fraction. Add or subtract the fractions in the denominator to form a single fraction.

Step 2 Divide the resulting expressions.

Step 3 Simplify if possible.

EXAMPLE 8 Simplifying a Complex Fraction (Method I)

Simplify. $$\frac{\frac{x}{4} - \frac{4}{x}}{\frac{1}{4} + \frac{1}{x}}$$

Answer

7. $\frac{t - 3}{(t + 1)(t + 3)}$

Solution:

$$\frac{\frac{x}{4}-\frac{4}{x}}{\frac{1}{4}+\frac{1}{x}} = \frac{\frac{x \cdot x}{4 \cdot x}-\frac{4 \cdot 4}{x \cdot 4}}{\frac{1 \cdot x}{4 \cdot x}+\frac{1 \cdot 4}{x \cdot 4}} = \frac{\frac{x^2-16}{4x}}{\frac{x+4}{4x}}$$

Step 1: Subtract the fractions in the numerator. Add the fractions in the denominator.

$$= \frac{x^2-16}{4x} \cdot \frac{4x}{x+4}$$

Step 2: Multiply the rational expression from the numerator by the reciprocal of the expression from the denominator.

$$= \frac{(x-4)\overset{1}{\cancel{(x+4)}}}{\cancel{4x}} \cdot \frac{\overset{1}{\cancel{4x}}}{\cancel{x+4}}$$

Step 3: Simplify by factoring and dividing out common factors.

$$= x - 4$$

Skill Practice 8 Simplify. $\dfrac{\frac{1}{7}+\frac{1}{y}}{\frac{y}{7}-\frac{7}{y}}$

In Example 9 we demonstrate another method (Method II) to simplify a complex fraction.

Simplifying a Complex Fraction: Multiply by the LCD (Method II)

Step 1 Multiply the numerator and denominator of the complex fraction by the LCD of all individual fractions.

Step 2 Apply the distributive property and simplify the numerator and denominator.

Step 3 Simplify the resulting expression if possible.

EXAMPLE 9 Simplifying a Complex Fraction (Method II)

Simplify. $\dfrac{d^{-2}-c^{-2}}{d^{-1}-c^{-1}}$

Solution:

$$\frac{d^{-2}-c^{-2}}{d^{-1}-c^{-1}} = \frac{\frac{1}{d^2}-\frac{1}{c^2}}{\frac{1}{d}-\frac{1}{c}}$$

First write the expression with positive exponents.

$$= \frac{c^2d^2 \cdot \left(\frac{1}{d^2}-\frac{1}{c^2}\right)}{c^2d^2 \cdot \left(\frac{1}{d}-\frac{1}{c}\right)}$$

Step 1: Multiply numerator and denominator by the LCD of all four individual fractions: c^2d^2.

Answer

8. $\dfrac{1}{y-7}$

$$= \frac{\frac{c^2d^2}{1}\cdot\frac{1}{d^2} - \frac{c^2d^2}{1}\cdot\frac{1}{c^2}}{\frac{c^2d^2}{1}\cdot\frac{1}{d} - \frac{c^2d^2}{1}\cdot\frac{1}{c}}$$

Step 2: Apply the distributive property.

$$= \frac{c^2 - d^2}{c^2d - cd^2}$$

$$= \frac{\cancel{(c - d)}(c + d)}{cd\cancel{(c - d)}} = \frac{c + d}{cd}$$

Step 3: Simplify by factoring and dividing out common factors.

Skill Practice 9 Simplify. $\dfrac{4 - 6x^{-1}}{2x^{-1} - 3x^{-2}}$

EXAMPLE 10 Simplifying a Complex Fraction (Method II)

Simplify. $\dfrac{\frac{2}{1 + h} - 2}{h}$

Solution:

$$\frac{\frac{2}{1 + h} - 2}{h} = \frac{\frac{2}{1 + h} - \frac{2}{1}}{h} = \frac{(1 + h)\cdot\left(\frac{2}{1 + h} - \frac{2}{1}\right)}{(1 + h)\cdot(h)}$$

Step 1: Multiply numerator and denominator by the LCD, which is $(1 + h)$.

$$= \frac{\frac{(1 + h)}{1}\cdot\left(\frac{2}{1 + h}\right) - \frac{(1 + h)}{1}\cdot\left(\frac{2}{1}\right)}{(1 + h)\cdot(h)}$$

Step 2: Apply the distributive property.

$$= \frac{2 - 2(1 + h)}{h(1 + h)}$$

Step 3: Simplify.

$$= \frac{2 - 2 - 2h}{h(1 + h)} = \frac{-2\cancel{h}}{\cancel{h}(1 + h)} = \frac{-2}{1 + h} \quad \text{or} \quad -\frac{2}{1 + h}$$

Skill Practice 10 Simplify. $\dfrac{\frac{5}{1 + h} - 5}{h}$

TIP The expression given in Example 10 is a pattern we see in a first semester calculus course.

6. Rationalize the Denominator of a Radical Expression

The same principle that applies to simplifying rational expressions also applies to simplifying algebraic fractions. For example, $\frac{5}{\sqrt{x}}$ is an algebraic fraction, but not a rational expression because the denominator is not a polynomial.

From Section R.3, we outlined the criteria for a radical expression to be simplified. Conditions 2 and 3 are stated here.

- No fraction may appear in the radicand.
- No denominator of a fraction may contain a radical.

Answers

9. $2x$

10. $-\dfrac{5}{1 + h}$

In Example 11, we use the property of equivalent fractions to remove a radical from the denominator of a fraction. This is called **rationalizing the denominator.**

EXAMPLE 11 Rationalizing the Denominator

Simplify. Assume that x is a positive real number.

a. $\dfrac{5}{\sqrt{x}}$ **b.** $\dfrac{4}{\sqrt{7}-\sqrt{5}}$

Solution:

a. $$\frac{5}{\sqrt{x}} = \frac{5 \cdot \sqrt{x}}{\sqrt{x} \cdot \sqrt{x}}$$

Multiply numerator and denominator by $\sqrt{x}$ so that the radicand in the denominator is a perfect square.

$$= \frac{5\sqrt{x}}{\sqrt{x^2}}$$

Apply the product property of radicals.

$$= \frac{5\sqrt{x}}{x}$$

Simplify the radical in the denominator.

b. $$\frac{4}{\sqrt{7}-\sqrt{5}} = \frac{4 \cdot \left(\sqrt{7}+\sqrt{5}\right)}{\left(\sqrt{7}-\sqrt{5}\right) \cdot \left(\sqrt{7}+\sqrt{5}\right)}$$

Multiply numerator and denominator by the conjugate of the denominator.

$$= \frac{4\left(\sqrt{7}+\sqrt{5}\right)}{\left(\sqrt{7}\right)^2 - \left(\sqrt{5}\right)^2}$$

Recall that $(a-b)(a+b) = a^2 - b^2$.

$$= \frac{4\left(\sqrt{7}+\sqrt{5}\right)}{7-5}$$

Simplify the radicals in the denominator.

$$= \frac{\overset{2}{\cancel{4}}\left(\sqrt{7}+\sqrt{5}\right)}{\cancel{2}}$$

Simplify the fraction.

$$= 2\left(\sqrt{7}+\sqrt{5}\right) \quad \text{or} \quad 2\sqrt{7}+2\sqrt{5}$$

TIP Keep the numerator in factored form until the denominator is simplified completely. By so doing, it will be easier to identify common factors in the numerator and denominator.

Skill Practice 11 Simplify.

a. $\dfrac{\sqrt{5}}{\sqrt{7}}$ **b.** $\dfrac{12}{\sqrt{13}-\sqrt{10}}$

In Example 11(b), we multiplied the numerator and denominator by the conjugate of the denominator. The rationale is that product $(a+b)(a-b)$ results in a difference of squares $a^2 - b^2$. If either a or b has a square root factor, then the product will simplify to an expression without square roots.

Answers

11. a. $\dfrac{\sqrt{35}}{7}$

b. $4\left(\sqrt{13}+\sqrt{10}\right)$ or $4\sqrt{13}+4\sqrt{10}$

SECTION R.6 Practice Exercises

Concept Connections

1. A _________ expression is a ratio of two polynomials.

2. The restricted values of the variable for a rational expression are those that make the denominator equal to _________.

3. The expression $\dfrac{5(x+2)}{(x+2)(x-1)}$ equals $\dfrac{5}{x-1}$ provided that $x \neq$ _________ and $x \neq$ _________.

4. The ratio of a polynomial and its opposite equals _________.

5. A _______________ fraction is an expression that contains one or more fractions in the numerator or denominator.

6. The process to remove a radical from the denominator of a fraction is called _________ the denominator.

Objective 1: Determine Restricted Values for a Rational Expression

For Exercises 7–14, determine the restrictions on the variable. (See Example 1)

7. $\dfrac{x-4}{x+7}$

8. $\dfrac{y-1}{y+10}$

9. $\dfrac{a}{a^2-81}$

10. $\dfrac{t}{t^2-16}$

11. $\dfrac{a}{a^2+81}$

12. $\dfrac{t}{t^2+16}$

13. $\dfrac{6c}{7a^3b^2}$

14. $\dfrac{11z}{8x^5y}$

Objective 2: Simplify Rational Expressions

15. Determine which expressions are equal to $-\dfrac{5}{x-3}$.

a. $\dfrac{-5}{x-3}$ **b.** $\dfrac{5}{3-x}$

c. $-\dfrac{5}{3-x}$ **d.** $\dfrac{-5}{3-x}$

16. Determine which expressions are equal to $\dfrac{-2}{a+b}$.

a. $\dfrac{-2}{a-b}$ **b.** $-\dfrac{2}{a+b}$

c. $\dfrac{2}{-a-b}$ **d.** $\dfrac{2}{a-b}$

For Exercises 17–26, simplify the expression and state the restrictions on the variable. (See Examples 2–3)

17. $\dfrac{x^2-9}{x^2-4x-21}$

18. $\dfrac{y^2-64}{y^2-7y-8}$

19. $-\dfrac{12a^2bc}{3ab^5}$

20. $-\dfrac{15tu^5v}{3t^3u}$

21. $\dfrac{10-5\sqrt{6}}{15}$

22. $\dfrac{12+4\sqrt{3}}{8}$

23. $\dfrac{2y^2-16y}{64-y^2}$

24. $\dfrac{81-t^2}{7t^2-63t}$

25. $\dfrac{4b-4a}{ax-xb-2a+2b}$

26. $\dfrac{2z-2y}{xy-xz+3y-3z}$

Objective 3: Multiply and Divide Rational Expressions

For Exercises 27–34, multiply or divide as indicated. The restrictions on the variables are implied. (See Examples 4–5)

27. $\dfrac{3a^5b^7}{a-5b} \cdot \dfrac{2a-10b}{12a^4b^{10}}$

28. $\dfrac{8x-3y}{x^3y^4} \cdot \dfrac{6xy^8}{24x-9y}$

29. $\dfrac{c^2-d^2}{cd^{11}} \div \dfrac{8c^2+4cd-4d^2}{8c^4d^{10}}$

30. $\dfrac{m^{11}n^2}{m^2-n^2} \div \dfrac{18m^9n^5}{9m^2+6mn-15n^2}$

31. $\dfrac{2a^2b-ab^2}{8b^2+ab} \cdot \dfrac{a^2+16ab+64b^2}{2a^2+15ab-8b^2}$

32. $\dfrac{2c^2-2cd}{3c^2d+2c^3} \cdot \dfrac{4c^2+12cd+9d^2}{2c^2+cd-3d^2}$

33. $\dfrac{x^3-64}{16x-x^3} \div \dfrac{2x^2+8x+32}{x^2+2x-8}$

34. $\dfrac{3y^2+21y+147}{25y-y^3} \div \dfrac{y^3-343}{y^2-12y+35}$

Objective 4: Add and Subtract Rational Expressions

For Exercises 35–40, identify the least common denominator for each pair of expressions.

35. $\dfrac{7}{6x^5yz^4}$ and $\dfrac{3}{20xy^2z^3}$

36. $\dfrac{12}{35b^4cd^3}$ and $\dfrac{8}{25b^2c^3d}$

37. $\dfrac{2t+1}{(3t+4)^3(t-2)}$ and $\dfrac{4}{t(3t+4)^2(t-2)}$

38. $\dfrac{5y-7}{y(2y-5)(y+6)^4}$ and $\dfrac{6}{(2y-5)^3(y+6)^2}$

39. $\dfrac{x+3}{x^2+20x+100}$ and $\dfrac{3}{2x^2+20x}$

40. $\dfrac{z-4}{4z^2-20z+25}$ and $\dfrac{5}{12z^2-30z}$

For Exercises 41–56, add or subtract as indicated. (See Examples 6–7)

41. $\dfrac{m^2}{m+3}+\dfrac{6m+9}{m+3}$

42. $\dfrac{n^2}{n+5}+\dfrac{7n+10}{n+5}$

43. $\dfrac{2}{9c}+\dfrac{7}{15c^3}$

44. $\dfrac{6}{25x}+\dfrac{7}{10x^4}$

45. $\dfrac{9}{2x^2y^4}-\dfrac{11}{xy^5}$

46. $\dfrac{-2}{3m^3n}-\dfrac{5}{m^2n^4}$

47. $\dfrac{2}{x+3}-\dfrac{7}{x}$

48. $\dfrac{4}{m-2}-\dfrac{3}{m}$

49. $\dfrac{1}{x^2+xy}-\dfrac{2}{x^2-y^2}$

50. $\dfrac{4}{4a^2-b^2}-\dfrac{1}{2a^2-ab}$

51. $\dfrac{5}{y}+\dfrac{2}{y+1}-\dfrac{6}{y^2}$

52. $\dfrac{5}{t^2}+\dfrac{4}{t+2}-\dfrac{3}{t}$

53. $\dfrac{3w}{w-4}+\dfrac{2w+4}{4-w}$

54. $\dfrac{2x-1}{x-7}+\dfrac{x+6}{7-x}$

55. $\dfrac{4}{x^2+6x+5}-\dfrac{3}{x^2+7x+10}$

56. $\dfrac{3}{x^2-4x-5}-\dfrac{2}{x^2-6x+5}$

Objective 5: Simplify Complex Fractions

For Exercises 57–68, simplify the complex fraction. (See Examples 8–10)

57. $\dfrac{\dfrac{1}{27x}+\dfrac{1}{9}}{\dfrac{1}{3}+\dfrac{1}{9x}}$

58. $\dfrac{\dfrac{1}{8x}+\dfrac{1}{4}}{\dfrac{1}{2}+\dfrac{1}{4x}}$

59. $\dfrac{\dfrac{x}{6}-\dfrac{5x+14}{6x}}{\dfrac{1}{6}-\dfrac{7}{6x}}$

60. $\dfrac{\dfrac{x}{3}-\dfrac{2x+3}{3x}}{\dfrac{1}{3}+\dfrac{1}{3x}}$

61. $\dfrac{2a^{-1}-b^{-1}}{4a^{-2}-b^{-2}}$

62. $\dfrac{3u^{-1}-v^{-1}}{9u^{-2}-v^{-2}}$

63. $\dfrac{\dfrac{3}{1+h}-3}{h}$

64. $\dfrac{\dfrac{4}{1+h}-4}{h}$

65. $\dfrac{\dfrac{7}{x+h}-\dfrac{7}{x}}{h}$

66. $\dfrac{\dfrac{8}{x+h}-\dfrac{8}{x}}{h}$

67. $\dfrac{\dfrac{3}{x-1}-\dfrac{1}{x+1}}{\dfrac{6}{x^2-1}}$

68. $\dfrac{\dfrac{1}{x+1}}{\dfrac{-5}{x^2-3x-4}+\dfrac{1}{x-4}}$

Objective 6: Rationalize the Denominator of a Radical Expression

For Exercises 69–84, simplify the expression. Assume that the variable expressions represent positive real numbers. (See Example 11)

69. $\dfrac{4}{\sqrt{y}}$

70. $\dfrac{7}{\sqrt{z}}$

71. $\dfrac{4}{\sqrt[3]{y}}$

72. $\dfrac{7}{\sqrt[4]{z}}$

73. $\dfrac{\sqrt{12}}{\sqrt{x+1}}$

74. $\dfrac{\sqrt{50}}{\sqrt{x-2}}$

75. $\dfrac{8}{\sqrt{15}-\sqrt{11}}$

76. $\dfrac{12}{\sqrt{6}-\sqrt{2}}$

77. $\dfrac{x - 5}{\sqrt{x} + \sqrt{5}}$

78. $\dfrac{y - 3}{\sqrt{y} + \sqrt{3}}$

79. $\dfrac{2\sqrt{10} + 3\sqrt{5}}{4\sqrt{10} + 2\sqrt{5}}$

80. $\dfrac{3\sqrt{3} + \sqrt{6}}{5\sqrt{3} - 2\sqrt{6}}$

81. $\dfrac{7}{\sqrt{3x}} + \dfrac{\sqrt{3x}}{x}$

82. $\dfrac{4}{\sqrt{11y}} + \dfrac{\sqrt{11y}}{y}$

83. $\dfrac{5}{w\sqrt{7}} - \dfrac{\sqrt{7}}{w}$

84. $\dfrac{13}{t\sqrt{2}} - \dfrac{\sqrt{2}}{t}$

Mixed Exercises

85. The average round trip speed S (in mph) of a vehicle traveling a distance of d miles each way is given by $S = \dfrac{2d}{\dfrac{d}{r_1} + \dfrac{d}{r_2}}$.

In this formula, r_1 is the average speed going one way, and r_2 is the average speed on the return trip.

a. Simplify the complex fraction.

b. If a plane flies 400 mph from Orlando to Albuquerque and 460 mph on the way back, compute the average speed of the round trip. Round to 1 decimal place.

86. The formula $R = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$ gives the total electrical resistance R (in ohms, Ω) when two resistors of resistance R_1 and R_2 are connected in parallel.

a. Simplify the complex fraction.

b. Find the total resistance when $R_1 = 12\ \Omega$ and $R_2 = 20\ \Omega$.

87. The concentration C (in ng/mL) of a drug in the bloodstream t hours after ingestion is modeled by $C = \dfrac{600t}{t^3 + 125}$.

a. Determine the concentration at 1 hr, 12 hr, 24 hr, and 48 hr. Round to 1 decimal place.

b. What appears to be the limiting concentration for large values of t?

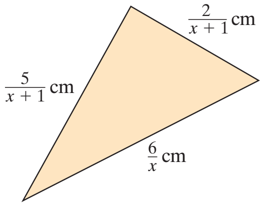
88. An object that is originally 35°C is placed in a freezer. The temperature T (in °C) of the object can be approximated by the model $T = \dfrac{350}{t^2 + 3t + 10}$, where t is the time in hours after the object is placed in the freezer.

a. Determine the temperature at 2 hr, 4 hr, 12 hr, and 24 hr. Round to 1 decimal place.

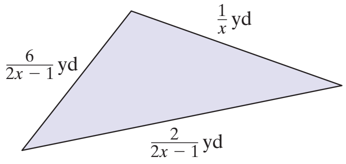
b. What appears to be the limiting temperature for large values of t?

For Exercises 89–92, write a simplified expression for the perimeter or area as indicated.

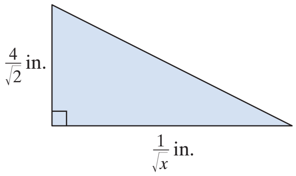
89. Perimeter



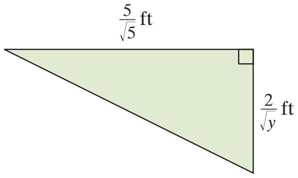
90. Perimeter



91. Area



92. Area



For Exercises 93–108, simplify the expression.

93. $\dfrac{2x^3y}{x^2y + 3xy} \cdot \dfrac{x^2 + 6x + 9}{2x + 6} \div 5xy^4$

94. $\dfrac{2y^2 + 20y + 50}{12 - 4y} \cdot \dfrac{y - 3}{y^2 + 12y + 35} \div (3y + 15)$

95. $\left(\dfrac{4}{2t + 1} - \dfrac{t}{2t^2 + 17t + 8}\right)(t + 8)$

96. $\left(\dfrac{2m}{6m + 3} - \dfrac{1}{m + 4}\right)(2m + 1)$

97. $\dfrac{n-2}{n-4}+\dfrac{2n^2-15n+12}{n^2-16}-\dfrac{2n-5}{n+4}$

98. $\dfrac{c^2+13c+18}{c^2-9}+\dfrac{c+1}{c+3}-\dfrac{c+8}{c-3}$

99. $\dfrac{1-a^{-1}-6a^{-2}}{1-4a^{-1}+3a^{-2}}$

100. $\dfrac{1+t^{-1}-12t^{-2}}{1-4t^{-1}+3t^{-2}}$

101. $\dfrac{34}{2\sqrt{5}-\sqrt{3}}$

102. $\dfrac{13}{2\sqrt{7}+\sqrt{2}}$

103. $\dfrac{8-\sqrt{48}}{6}$

104. $\dfrac{10-\sqrt{50}}{15}$

105. $\dfrac{45+9x-5x^2-x^3}{x^3-3x^2-25x+75}$

106. $\dfrac{98-49x-2x^2+x^3}{7x^2-x^3-28+4x}$

107. $\dfrac{t+6}{1+\dfrac{2}{t}}-t-4$

108. $\dfrac{m-4}{1-\dfrac{2}{m}}-m+2$

For Exercises 109–116, write the expression as a single term, factored completely. Do not rationalize the denominator.

109. $1-x^{-2}-2x^{-3}$

110. $1-8x^{-5}+30x^{-7}$

111. $\dfrac{3}{2\sqrt{3x}}+\sqrt{3x}$

112. $\dfrac{2}{\sqrt{x}}+\sqrt{x}$

113. $\dfrac{\sqrt{x^2+1}+\dfrac{x^2}{\sqrt{x^2+1}}}{x^2+1}$

114. $\dfrac{\dfrac{x^2}{\sqrt{x^2+9}}-\sqrt{x^2+9}}{x^2}$

115. $2\sqrt{4x^2+9}+\dfrac{8x^2}{\sqrt{4x^2+9}}$

116. $3\sqrt{9x^2+1}+\dfrac{27x}{\sqrt{9x^2+1}}$

Write About It

117. Explain why the expression $\dfrac{x}{x-y}$ is not defined for $x = y$.

118. Is the statement $\dfrac{3(x-4)}{(x+2)(x-4)}=\dfrac{3}{x+2}$ true for all values of x? Explain why or why not.

Expanding Your Skills

119. The numbers 1, 2, 4, 5, 10, and 20 are natural numbers that are factors of 20. There are other factors of 20 within the set of rational numbers and the set of irrational numbers. For example:

a. Show that $\dfrac{14}{3}$ and $\dfrac{30}{7}$ are factors of 20 over the set of rational numbers.

b. Show that $(5-\sqrt{5})$ and $(5+\sqrt{5})$ are factors of 20 over the set of irrational numbers.

120. a. Show that $\dfrac{15}{2}$ and $\dfrac{4}{5}$ are factors of 6 over the set of rational numbers.

b. Show that $(3-\sqrt{3})$ and $(3+\sqrt{3})$ are factors of 6 over the set of irrational numbers.

For Exercises 121–128, simplify the expression.

121. $\dfrac{w^{3n+1}-w^{3n}z}{w^{n+2}-w^{n}z^{2}}$

122. $\dfrac{x^{2n+1}-x^{2n}y}{x^{n+3}-x^{n}y^{3}}$

123. $\sqrt{\dfrac{x-y}{x+y}}$

124. $\sqrt{\dfrac{m-3}{m+3}}$

125. $\dfrac{\sqrt{5}}{\sqrt[3]{2}}$

126. $\dfrac{\sqrt{7}}{\sqrt[4]{3}}$

127. $\dfrac{a-b}{\sqrt[3]{a}-\sqrt[3]{b}}$ (*Hint*: Factor the numerator as a difference of cubes over the set of irrational numbers.)

128. $\dfrac{x+y}{\sqrt[3]{x}+\sqrt[3]{y}}$

For Exercises 129–130, rationalize the numerator by multiplying numerator and denominator by the conjugate of the *numerator*.

129. $\dfrac{\sqrt{4+h}-2}{h}$

130. $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

SECTION 1.3 Complex Numbers

OBJECTIVES

1. Simplify Imaginary Numbers
2. Write Complex Numbers in the Form $a + bi$
3. Perform Operations on Complex Numbers

1. Simplify Imaginary Numbers

In our study of algebra thus far, we have worked exclusively with real numbers. However, as we encounter new types of equations, we need to look outside the set of real numbers to find solutions. For example, the equation $x^2 = 1$ has two solutions: 1 and -1. But what about the equation $x^2 = -1$? There is no real number x for which $x^2 = -1$. For this reason, mathematicians defined a new number i such that $i^2 = -1$. The number i is called an *imaginary number* and is used to represent $\sqrt{-1}$. Furthermore, the square root of any negative real number is an imaginary number that can be expressed in terms of i.

The Imaginary Number *i*

- $i = \sqrt{-1}$ and $i^2 = -1$
- If b is a positive real number, then $\sqrt{-b} = i\sqrt{b}$.

EXAMPLE 1 Writing Imaginary Numbers in Terms of *i*

Write each expression in terms of i.

a. $\sqrt{-25}$ **b.** $\sqrt{-12}$ **c.** $\sqrt{-13}$

Solution:

a. $\sqrt{-25} = i\sqrt{25} = 5i$

b. $\sqrt{-12} = i\sqrt{12} = i \cdot 2\sqrt{3}$
$= 2i\sqrt{3}$ or $2\sqrt{3}i$

The value $i \cdot 2\sqrt{3}$ can be written as $2i\sqrt{3}$ or as $2\sqrt{3}i$. Note, however, that the factor i is written *outside* the radical.

c. $\sqrt{-13} = i\sqrt{13}$ or $\sqrt{13}i$

Skill Practice 1 Write each expression in terms of i.

a. $\sqrt{-81}$ **b.** $\sqrt{-50}$ **c.** $\sqrt{-11}$

Answers

1. a. $9i$ **b.** $5i\sqrt{2}$ **c.** $i\sqrt{11}$

In Example 2, we multiply and divide the square roots of negative real numbers. However, note that the multiplication and division properties of radicals can be used only if the radicals represent real-valued expressions.

$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ provided that the roots represent real numbers.

$\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ provided that the roots represent real numbers.

For this reason, in Example 2 it is important to write the radical expressions in terms of i first, before applying the multiplication or division property of radicals.

EXAMPLE 2 Simplifying Expressions in Terms of *i*

Multiply or divide as indicated.

a. $\sqrt{-9} \cdot \sqrt{-25}$ **b.** $\sqrt{-15} \cdot \sqrt{-3}$ **c.** $\dfrac{\sqrt{-50}}{\sqrt{-2}}$

Solution:

a. $\sqrt{-9} \cdot \sqrt{-25} = i\sqrt{9} \cdot i\sqrt{25}$ Write each radical in terms of i first, *before* multiplying.

$= 3i \cdot 5i$ Simplify the radicals.

$= 15i^2$ Multiply.

$= 15(-1)$ By definition, $i^2 = -1$.

$= -15$

b. $\sqrt{-15} \cdot \sqrt{-3} = i\sqrt{15} \cdot i\sqrt{3}$ Write each radical in terms of i first, *before* multiplying.

$= i^2\sqrt{45}$ Apply the multiplication property of radicals.

$= (-1)\sqrt{3^2 \cdot 5}$ Simplify $i^2 = -1$.

$= -3\sqrt{5}$

c. $\dfrac{\sqrt{-50}}{\sqrt{-2}} = \dfrac{i\sqrt{50}}{i\sqrt{2}}$ Write each radical in terms of i first, *before* dividing.

$= \dfrac{\overset{1}{\cancel{i}}\sqrt{50}}{\cancel{i}\sqrt{2}}$ Simplify the ratio of common factors to 1.

$= \sqrt{\dfrac{50}{2}}$ Apply the division property of radicals.

$= \sqrt{25} = 5$ Simplify.

Skill Practice 2 Multiply or divide as indicated.

a. $\sqrt{-16} \cdot \sqrt{-49}$ **b.** $\sqrt{-10}\sqrt{-2}$ **c.** $\dfrac{\sqrt{-48}}{\sqrt{-3}}$

2. Write Complex Numbers in the Form *a* + *bi*

We now define a new set of numbers that includes the real numbers and imaginary numbers. This is called the set of complex numbers.

Answers

2. a. -28 **b.** $-2\sqrt{5}$ **c.** 4

Complex Numbers

Given real numbers a and b, a number written in the form $a + bi$ is called a **complex number.** The value a is called the **real part** of the complex number and the value b is called the **imaginary part.**

$$5 - 7i = \overset{\text{Real part: }5}{5} + \overset{\text{Imaginary part: }-7}{(-7)}i$$

Notes	Examples
• If $b = 0$, then $a + bi$ equals the real number a. This tells us that all real numbers are complex numbers.	$4 + 0i$ is generally written as the real number 4.
• If $a = 0$ and $b \neq 0$, then $a + bi$ equals bi, which we say is **pure imaginary.**	The number $0 + 8i$ is a pure imaginary number and is generally written as simply $8i$.

A complex number written in the form $a + bi$ is said to be in **standard form.** That being said, we sometimes write $a - bi$ in place of $a + (-b)i$. Furthermore, a number such as $5 + \sqrt{3}i$ is sometimes written as $5 + i\sqrt{3}$ to emphasize that the factor of i is not under the radical. In Example 3, we practice writing complex numbers in standard form.

EXAMPLE 3 Writing Complex Numbers in Standard Form

Simplify each expression and write the result in the form $a + bi$.

a. $3 - \sqrt{-100}$ **b.** $\dfrac{2 + 7i}{5}$ **c.** $\dfrac{-6 + \sqrt{-18}}{9}$

Solution:

a. $3 - \sqrt{-100} = 3 - 10i$	Simplify the expression.
$= 3 + (-10)i$	Although $3 + (-10)i$ is written in standard form, $3 - 10i$ is also acceptable.
b. $\dfrac{2 + 7i}{5} = \dfrac{2}{5} + \dfrac{7}{5}i$	Write the fraction as two separate terms.
c. $\dfrac{-6 + \sqrt{-18}}{9} = \dfrac{-6 + 3i\sqrt{2}}{9}$	Simplify the radical. $\sqrt{-18} = i\sqrt{18} = i\sqrt{3^2 \cdot 2} = 3i\sqrt{2}$
$= \dfrac{-6}{9} + \dfrac{3i\sqrt{2}}{9}$	Write the fraction as two separate terms.
$= -\dfrac{2}{3} + \dfrac{\sqrt{2}}{3}i$	Simplify each fraction and write the result in the form $a + bi$.

Skill Practice 3 Simplify each expression and write the result in the form $a + bi$.

a. $4 + \sqrt{-49}$ **b.** $\dfrac{3 - 8i}{7}$ **c.** $\dfrac{10 + \sqrt{-75}}{20}$

Answers

3. a. $4 + 7i$ **b.** $\frac{3}{7} + \left(-\frac{8}{7}\right)i$

c. $\frac{1}{2} + \frac{\sqrt{3}}{4}i$

3. Perform Operations on Complex Numbers

By definition, $i^2 = -1$, but what about other powers of i? Consider the following pattern.

$$\begin{array}{ll}
i^1 = i & \left.\begin{array}{l} i^1 = i \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{array}\right\} \text{Pattern: } i, -1, -i, 1 \\
i^2 = -1 & \\
i^3 = i^2 \cdot i = (-1)i = -i & \\
i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 & \\
i^5 = i^4 \cdot i = (1)i = i & \left.\begin{array}{l} i^5 = i \\ i^6 = -1 \\ i^7 = -i \\ i^8 = 1 \end{array}\right\} \text{Pattern repeats: } i, -1, -i, 1, \ldots \\
i^6 = i^4 \cdot i^2 = (1)(-1) = -1 & \\
i^7 = i^4 \cdot i^2 \cdot i = (1)(-1)i = -i & \\
i^8 = i^4 \cdot i^4 = (1)(1) = 1 &
\end{array}$$

TIP Notice that even powers of i simplify to 1 or -1.

- If the exponent is a multiple of 4, then the expression equals 1.
- If the exponent is even but *not* a multiple of 4, then the expression equals -1.

Notice that the fourth powers of i (i^4, i^8, i^{12}, ...) equal the real number 1. For other powers of i, we can write the expression as a product of a fourth power of i and a factor of i, i^2, or i^3, which equals i, -1, or $-i$, respectively.

EXAMPLE 4 Simplifying Powers of *i*

Simplify.

a. i^{48} **b.** i^{23} **c.** i^{50} **d.** i^{-19}

Solution:

a. $i^{48} = (i^4)^{12} = (1)^{12} = 1$

The exponent 48 is a multiple of 4. Thus, i^{48} is equal to 1.

b. $i^{23} = i^{20} \cdot i^3$
$= (1) \cdot i^3 = -i$

Write i^{23} as a product of the largest fourth power of i and a remaining factor.

c. $i^{50} = i^{48} \cdot i^2$
$= (1)(-1) = -1$

d. $i^{-19} = i^{-20} \cdot i^1$
$= (i^4)^{-5} \cdot i = (1) \cdot i = i$

TIP To simplify i^n, divide the exponent, n, by 4. The remainder is the exponent of the remaining factor of i once the fourth power of i has been extracted.

Example: $i^{50} = i^{48} \cdot i^2 = (1)i^2$

$$\begin{array}{r} 12 \\ 4\overline{)50} \\ \underline{48} \\ 2 \end{array}$$

So $i^{50} = (1) \cdot i^2 = -1$

Skill Practice 4 Simplify.

a. i^{13} **b.** i^{103} **c.** i^{64} **d.** i^{-30}

To add or subtract complex numbers, add or subtract their real parts, and add or subtract their imaginary parts. That is,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

EXAMPLE 5 Adding and Subtracting Complex Numbers

Add or subtract as indicated. Write the answer in the form $a + bi$.

a. $(-2 - 4i) + (5 + 2i) - (3 - 6i)$

b. Subtract $\left(\frac{1}{2} + \frac{2}{3}i\right)$ from $\left(\frac{3}{4} + \frac{9}{5}i\right)$.

Answers

4. a. i **b.** $-i$ **c.** 1 **d.** -1

Solution:

a. $(-2 - 4i) + (5 + 2i) - (3 - 6i)$ $= (-2 + 5 - 3) + [-4 + 2 - (-6)]i$	Combine the real parts and combine the imaginary parts.
$= 0 + 4i$	Write the result in the form $a + bi$.
b. Subtract $\left(\frac{1}{2} + \frac{2}{3}i\right)$ from $\left(\frac{3}{4} + \frac{9}{5}i\right)$.	
$\left(\frac{3}{4} + \frac{9}{5}i\right) - \left(\frac{1}{2} + \frac{2}{3}i\right)$	The statement "subtract x from y" is equivalent to $y - x$. The order is important.
$= \left(\frac{3}{4} - \frac{1}{2}\right) + \left(\frac{9}{5} - \frac{2}{3}\right)i$	Subtract the real parts. Subtract the imaginary parts.
$= \left(\frac{3}{4} - \frac{2}{4}\right) + \left(\frac{27}{15} - \frac{10}{15}\right)i$	Write using common denominators.
$= \frac{1}{4} + \frac{17}{15}i$	Write the result in the form $a + bi$.

Skill Practice 5 Add or subtract as indicated. Write the answer in the form $a + bi$.

a. $(8 - 3i) - (2 + 4i) + (5 + 7i)$ **b.** Subtract $\left(\frac{1}{10} + \frac{1}{3}i\right)$ from $\left(\frac{3}{5} + \frac{5}{6}i\right)$

In Examples 6 and 7, we multiply complex numbers using a process similar to multiplying polynomials.

EXAMPLE 6 Multiplying Complex Numbers

Multiply. Write the results in the form $a + bi$.

a. $-\frac{1}{2}i(4 + 6i)$ **b.** $(-2 + 6i)(4 - 3i)$

Solution:

a. $-\frac{1}{2}i(4 + 6i) = -2i - 3i^2$	Apply the distributive property.
$= -2i - 3(-1)$	Recall that $i^2 = -1$.
$= 3 - 2i$ or $3 + (-2)i$	Write the result in the form $a + bi$.
b. $(-2 + 6i)(4 - 3i)$	Apply the distributive property.
$= -2(4) + (-2)(-3i) + 6i(4) + 6i(-3i)$	
$= -8 + 6i + 24i - 18i^2$	
$= -8 + 30i - 18(-1)$	Recall that $i^2 = -1$.
$= -8 + 30i + 18$	
$= 10 + 30i$	Write the result in the form $a + bi$.

Skill Practice 6 Multiply. Write the result in the form $a + bi$.

a. $-\frac{1}{3}i(9 - 15i)$ **b.** $(-5 + 4i)(3 - i)$

Answers

5. a. $11 + 0i$ **b.** $\frac{1}{2} + \frac{1}{2}i$

6. a. $-5 + (-3)i$ **b.** $-11 + 17i$

In Example 7, we make use of the special case products:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \quad \text{and} \quad (a + b)(a - b) = a^2 - b^2$$

EXAMPLE 7 Evaluating Special Products with Complex Numbers

Multiply. Write the results in the form $a + bi$.

a. $(3 + 4i)^2$ **b.** $(5 + 2i)(5 - 2i)$

Solution:

a. $(3 + 4i)^2 = (3)^2 + 2(3)(4i) + (4i)^2$ Apply the property $(a + b)^2 = a^2 + 2ab + b^2$.

$= 9 + 24i + 16i^2$

$= 9 + 24i + 16(-1)$

$= 9 + 24i - 16$

$= -7 + 24i$ Write the result in the form $a + bi$.

b. $(5 + 2i)(5 - 2i) = (5)^2 - (2i)^2$ Apply the property $(a + b)(a - b) = a^2 - b^2$.

$= 25 - 4i^2$

$= 25 - 4(-1)$

$= 25 + 4$

$= 29$ or $29 + 0i$ Write the result in the form $a + bi$.

Skill Practice 7 Multiply. Write the result in the form $a + bi$.

a. $(4 - 7i)^2$ **b.** $(10 - 3i)(10 + 3i)$

In Section R.4 we noted that the expressions $(a + b)$ and $(a - b)$ are conjugates. Similarly, the expressions $(a + bi)$ and $(a - bi)$ are called **complex conjugates.** Furthermore, as illustrated in Example 7(b), the product of complex conjugates is a real number.

$$\begin{aligned}(a + bi)(a - bi) &= (a)^2 - (bi)^2\\ &= a^2 - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

Product of Complex Conjugates

If a and b are real numbers, then $(a + bi)(a - bi) = a^2 + b^2$.

Number	Standard Form	Conjugate	Product
$3 + 7i$	$3 + 7i$	$3 - 7i$	$(3 + 7i)(3 - 7i) = (3)^2 + (7)^2 = 58$
$\sqrt{-5}$	$0 + \sqrt{5}i$	$0 - \sqrt{5}i$	$(0 + \sqrt{5}i)(0 - \sqrt{5}i) = (0)^2 + (\sqrt{5})^2 = 5$

In Example 8, we demonstrate division of complex numbers such as $\frac{8 + 2i}{3 - 5i}$. The goal is to make the denominator a real number so that the quotient can be written in standard form $a + bi$. This can be accomplished by multiplying the denominator by its complex conjugate. Of course, this means that we must also multiply the numerator by the same quantity.

Answers

7. a. $-33 + (-56)i$ **b.** 109

EXAMPLE 8 Dividing Complex Numbers

Divide. Write the result in the form $a + bi$.

a. $\dfrac{8 + 2i}{3 - 5i}$ **b.** $(2 + \sqrt{3}i)^{-1}$ **c.** $\dfrac{-2}{5i}$

Solution:

a. $\dfrac{8 + 2i}{3 - 5i} = \dfrac{(8 + 2i) \cdot (3 + 5i)}{(3 - 5i) \cdot (3 + 5i)}$ Multiply numerator and denominator by the conjugate of the denominator.

$= \dfrac{24 + 40i + 6i + 10i^2}{(3)^2 + (5)^2}$ Apply the distributive property in the numerator. Multiply conjugates in the denominator.

$= \dfrac{24 + 46i + 10(-1)}{9 + 25}$ Replace i^2 by -1.

$= \dfrac{14 + 46i}{34}$

$= \dfrac{14}{34} + \dfrac{46}{34}i = \dfrac{7}{17} + \dfrac{23}{17}i$ Write the result in the form $a + bi$.

b. $(2 + \sqrt{3}i)^{-1} = \dfrac{1 \cdot (2 - \sqrt{3}i)}{(2 + \sqrt{3}i) \cdot (2 - \sqrt{3}i)}$ Multiply numerator and denominator by the conjugate of the denominator.

$= \dfrac{2 - \sqrt{3}i}{(2)^2 + (\sqrt{3})^2}$

$= \dfrac{2 - \sqrt{3}i}{4 + 3} = \dfrac{2}{7} - \dfrac{\sqrt{3}}{7}i$ Simplify.

c. $\dfrac{-2}{5i} = \dfrac{-2 \cdot i}{5i \cdot i}$ In this example, it is sufficient to multiply numerator and denominator by i (rather than by the conjugate $-5i$) to produce a real number in the denominator.

$= \dfrac{-2i}{5i^2} = \dfrac{-2i}{5(-1)} = \dfrac{-2i}{-5} = \dfrac{2}{5}i$

$= 0 + \dfrac{2}{5}i$ Write the result in the form $a + bi$.

TIP In Example 8(b) we left the answer as $\frac{2}{7} - \frac{\sqrt{3}}{7}i$ rather than as $\frac{2}{7} + \left(-\frac{\sqrt{3}}{7}\right)i$ because the expression written using addition is more cumbersome. Both answers are acceptable.

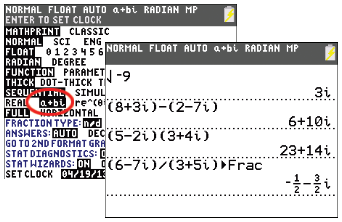
Skill Practice 8 Divide. Write the result in the form $a + bi$.

a. $\dfrac{5 + 6i}{2 - 7i}$ **b.** $(5 + \sqrt{7}i)^{-1}$ **c.** $\dfrac{-7}{10i}$

TECHNOLOGY CONNECTIONS

Operations on Complex Numbers

Most graphing calculators and some scientific calculators can perform operations on complex numbers. A graphing calculator may have two different modes: one for operations over the set of real numbers and one for operations over the set of complex numbers. Choose the "$a + bi$" mode on your calculator. Then evaluate the expressions.



```
NORMAL FLOAT AUTO a+bi RADIAN MP
ENTER TO SET CLOCK
MATHPRINT  CLASSIC
NORMAL  SCI  ENG
FLOAT  0 1 2 3 4 5 6
RADIAN  DEGREE
FUNCTION  PARAMET
THICK  DOT-THICK  T
SEQUENTIAL  SIMUL
REAL  a+bi  re^(θ
FULL  HORIZONTAL
FRACTION TYPE: n/d
ANSWERS: AUTO  DEC
GO TO 2ND FORMAT GRA
STAT DIAGNOSTICS: O
STAT WIZARDS: ON  O
SET CLOCK  04/19/13
```

```
NORMAL FLOAT AUTO a+bi RADIAN MP
√-9
                        3i
(8+3i)-(2-7i)
                     6+10i
(5-2i)(3+4i)
                    23+14i
(6-7i)/(3+5i)▶Frac
                 -1/2-3/2i
```

Answers

8. a. $-\dfrac{32}{53} + \dfrac{47}{53}i$ **b.** $\dfrac{5}{32} - \dfrac{\sqrt{7}}{32}i$

c. $0 + \dfrac{7}{10}i$

SECTION 1.3 Practice Exercises

Prerequisite Review

R.1. Simplify the radical. $\sqrt{28}$ **R.2.** Simplify the radical. $4\sqrt{150}$

R.3. Simplify the radical. $\dfrac{7\sqrt{48}}{16}$ **R.4.** Multiply. $(p+\sqrt{3})(p-\sqrt{3})$

R.5. Rationalize the denominator. $\dfrac{9}{\sqrt{18}}$ **R.6.** Rationalize the denominator. $\dfrac{-33}{\sqrt{5}-4}$

Concept Connections

1. The imaginary number i is defined so that $i = \sqrt{-1}$ and $i^2 =$ ________.

2. For a positive real number b, the value $\sqrt{-b} =$ ________.

3. Given a complex number $a + bi$, the value of a is called the ________ part and the value of b is called the ___________ part.

4. Given a complex number $a + bi$, the expression $a - bi$ is called the complex _________.

Objective 1: Simplify Imaginary Numbers

For Exercises 5–22, write each expression in terms of *i* and simplify. (See Examples 1–2)

5. $\sqrt{-121}$ **6.** $\sqrt{-100}$ **7.** $\sqrt{-98}$ **8.** $\sqrt{-63}$

9. $\sqrt{-19}$ **10.** $\sqrt{-23}$ **11.** $-\sqrt{-16}$ **12.** $-\sqrt{-25}$

13. $\sqrt{-4}\sqrt{-9}$ **14.** $\sqrt{-1}\sqrt{-36}$ **15.** $\sqrt{-10}\sqrt{-5}$ **16.** $\sqrt{-6}\sqrt{-15}$

17. $\sqrt{-6}\sqrt{-14}$ **18.** $\sqrt{-10}\sqrt{-15}$ **19.** $\dfrac{\sqrt{-98}}{\sqrt{-2}}$ **20.** $\dfrac{\sqrt{-45}}{\sqrt{-5}}$

21. $\dfrac{\sqrt{-63}}{\sqrt{7}}$ **22.** $\dfrac{\sqrt{-80}}{\sqrt{5}}$

Objective 2: Write Complex Numbers in the Form *a* + *bi*

For Exercises 23–28, determine the real and imaginary parts of the complex number.

23. $3 - 7i$ **24.** $2 - 4i$ **25.** $19i$

26. $40i$ **27.** $-\dfrac{1}{4}$ **28.** $-\dfrac{4}{7}$

For Exercises 29–40, simplify each expression and write the result in standard form, *a* + *bi*. (See Example 3)

29. $4\sqrt{-4}$ **30.** $2\sqrt{-144}$ **31.** $2 + \sqrt{-12}$ **32.** $6 - \sqrt{-24}$

33. $\dfrac{8+3i}{14}$ **34.** $\dfrac{4+5i}{6}$ **35.** $\dfrac{-4-6i}{-2}$ **36.** $\dfrac{9-15i}{-3}$

37. $\dfrac{-18+\sqrt{-48}}{4}$ **38.** $\dfrac{-20+\sqrt{-50}}{-10}$ **39.** $\dfrac{14-\sqrt{-98}}{-7}$ **40.** $\dfrac{-10+\sqrt{-125}}{5}$

Objective 3: Perform Operations on Complex Numbers

For Exercises 41–44, simplify the powers of *i*. (See Example 4)

41. a. i^{20} **b.** i^{29} **c.** i^{50} **d.** i^{-41}

42. a. i^{32} **b.** i^{47} **c.** i^{66} **d.** i^{-27}

43. a. i^{37} **b.** i^{-37} **c.** i^{82} **d.** i^{-82}

44. a. i^{103} **b.** i^{-103} **c.** i^{52} **d.** i^{-52}

For Exercises 45–68, perform the indicated operations. Write the answers in standard form, *a* + *bi*. (See Examples 5–7)

45. $(2 - 7i) + (8 - 3i)$

46. $(6 - 10i) + (8 + 4i)$

47. $(15 + 21i) - (18 - 40i)$

48. $(250 + 100i) - (80 + 25i)$

49. $\left(\frac{1}{2} + \frac{2}{3}i\right) - \left(\frac{5}{6} + \frac{1}{12}i\right)$

50. $\left(\frac{3}{5} - \frac{1}{8}i\right) - \left(\frac{7}{10} + \frac{1}{6}i\right)$

51. $(2.3 + 4i) - (8.1 - 2.7i) + (4.6 - 6.7i)$

52. $(0.05 - 0.03i) + (-0.12 + 0.08i) - (0.07 + 0.05i)$

53. $-\frac{1}{8}(16 + 24i)$

54. $-\frac{1}{6}(60 - 30i)$

55. $2i(5 + i)$

56. $4i(6 + 5i)$

57. $\sqrt{-3}\left(\sqrt{11} - \sqrt{-7}\right)$

58. $\sqrt{-2}\left(\sqrt{13} + \sqrt{-5}\right)$

59. $(3 - 6i)(10 + i)$

60. $(2 - 5i)(8 + 2i)$

61. $(3 - 7i)^2$

62. $(10 - 3i)^2$

63. $\left(3 - \sqrt{-5}\right)\left(4 + \sqrt{-5}\right)$

64. $\left(2 + \sqrt{-7}\right)\left(10 + \sqrt{-7}\right)$

65. $4(6 + 2i) - 5i(3 - 7i)$

66. $-3(8 - 3i) - 6i(2 + i)$

67. $(2 - i)^2 + (2 + i)^2$

68. $(3 - 2i)^2 + (3 + 2i)^2$

For Exercises 69–72, for each given number, (a) identify the complex conjugate and (b) determine the product of the number and its conjugate.

69. $3 - 6i$ **70.** $4 - 5i$ **71.** $8i$ **72.** $9i$

For Exercises 73–88, perform the indicated operations. Write the answers in standard form, *a* + *bi*. (See Examples 7–8)

73. $(10 - 4i)(10 + 4i)$

74. $(3 - 9i)(3 + 9i)$

75. $(7i)(-7i)$

76. $(-5i)(5i)$

77. $\left(\sqrt{2} + \sqrt{3}i\right)\left(\sqrt{2} - \sqrt{3}i\right)$

78. $\left(\sqrt{5} + \sqrt{7}i\right)\left(\sqrt{5} - \sqrt{7}i\right)$

79. $\frac{6 + 2i}{3 - i}$

80. $\frac{5 + i}{4 - i}$

81. $\frac{8 - 5i}{13 + 2i}$

82. $\frac{10 - 3i}{11 + 4i}$

83. $\left(6 + \sqrt{5}i\right)^{-1}$

84. $\left(4 - \sqrt{3}i\right)^{-1}$

85. $\frac{5}{13i}$ **86.** $\frac{6}{7i}$ **87.** $\frac{-1}{\sqrt{-3}}$ **88.** $\frac{-2}{\sqrt{-11}}$

Mixed Exercises

For Exercises 89–92, evaluate $\sqrt{b^2 - 4ac}$ for the given values of *a*, *b*, and *c*, and simplify.

89. $a = 2$, $b = 4$, and $c = 6$

90. $a = 5$, $b = -5$, and $c = 10$

91. $a = 2$, $b = -6$, and $c = 5$

92. $a = 2$, $b = 4$, and $c = 4$

For Exercises 93–96, verify by substitution that the given values of *x* are solutions to the given equation.

93. $x^2 + 25 = 0$

a. $x = 5i$

b. $x = -5i$

94. $x^2 + 49 = 0$

a. $x = 7i$

b. $x = -7i$

95. $x^2 - 4x + 7 = 0$

a. $x = 2 + i\sqrt{3}$

b. $x = 2 - i\sqrt{3}$

96. $x^2 - 6x + 11 = 0$

a. $x = 3 + i\sqrt{2}$

b. $x = 3 - i\sqrt{2}$

97. Prove that $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

98. Prove that $(a + bi)^2 = (a^2 - b^2) + (2ab)i$.

Write About It

99. Explain the flaw in the following logic.
$\sqrt{-9} \cdot \sqrt{-4} = \sqrt{(-9)(-4)} = \sqrt{36} = 6$

100. Discuss the difference between the products $(a + b)(a - b)$ and $(a + bi)(a - bi)$.

101. Give an example of a complex number that is its own conjugate.

102. Give an example of two complex numbers that are not real numbers, but whose product is a real number.

Expanding Your Skills

The variable z is often used to denote a complex number and $\bar{z}$ is used to denote its conjugate. If $z = a + bi$, simplify the expressions in Exercises 103–104.

103. $z \cdot \bar{z}$

104. $z^2 - \bar{z}^2$

For Exercises 105–110, factor the expressions over the set of complex numbers. For assistance, consider these examples.

- In Chapter R we saw that some expressions factor over the set of integers. For example: $x^2 - 4 = (x + 2)(x - 2)$.
- Some expressions factor over the set of irrational numbers. For example: $x^2 - 5 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$.
- To factor an expression such as $x^2 + 4$, we need to factor over the set of complex numbers. For example, verify that $x^2 + 4 = (x + 2i)(x - 2i)$.

105. a. $x^2 - 9$
b. $x^2 + 9$

106. a. $x^2 - 100$
b. $x^2 + 100$

107. a. $x^2 - 64$
b. $x^2 + 64$

108. a. $x^2 - 25$
b. $x^2 + 25$

109. a. $x^2 - 3$
b. $x^2 + 3$

110. a. $x^2 - 11$
b. $x^2 + 11$

Technology Connections

For Exercises 111–114, use a calculator to perform the indicated operations.

111. a. $\sqrt{-16}$
b. $(4 - 5i) - (2 + 3i)$
c. $(12 - 15i)(-2 + 9i)$

112. a. $\sqrt{-169}$
b. $(-11 - 2i) + (-4 + 9i)$
c. $(8 + 12i)(-3 - 7i)$

113. a. $(4 - 9i)^2$
b. $\dfrac{7}{2i}$
c. $\dfrac{14 + 8i}{3 - i}$

114. a. $(11 + 4i)^2$
b. $\dfrac{11}{10i}$
c. $\dfrac{5 + 7i}{6 + 8i}$

SECTION 1.1 Linear Equations and Rational Equations

OBJECTIVES

1. Solve Linear Equations in One Variable
2. Identify Conditional Equations, Identities, and Contradictions
3. Solve Rational Equations
4. Solve Literal Equations for a Specified Variable

1. Solve Linear Equations in One Variable

Whether to lease an automobile, to buy an automobile, or to use public transportation is an important financial decision that affects a family's monthly budget. As part of an informed decision, it is useful to create a mathematical model of the cost for each option.

Suppose that a couple has an option to buy a used car for \$8800 and that they expect it to last 3 yr without major repair. Another option is to lease a new automobile for an initial down payment of \$2500 followed by 36 monthly payments of \$225. After how many months will the cost to lease the new car equal the cost to buy the used car? The answer can be found by solving a type of equation called a linear equation in one variable (see Example 3).

Definition of a Linear Equation in One Variable

A **linear equation in one variable** is an equation that can be written in the form $ax + b = 0$, where a and b are real numbers, $a \neq 0$, and x is the variable.

A linear equation in one variable is also called a **first-degree equation** because the degree of the variable term must be exactly one.

Linear equation in one variable	**Not a linear equation in one variable**	
$5x + 35 = 0$	$5x^2 + 35 = 0$	(not first degree)
$\frac{x}{4} - 5 = 0$	$\frac{4}{x} - 5 = 0$	(not first degree because $\frac{4}{x}$ is $4x^{-1}$)
$3x + 4 = 7$	$3x + 4y = 7$	(contains two variables)
$0.7x - 0.8 = 0.1$	$0.7x - 0.8 - 0.1$	(This is an expression, not an equation.)

A **solution** to an equation is a value of the variable that makes the equation a true statement. The set of all solutions to an equation is called the **solution set** of the equation. **Equivalent equations** have the same solution set. To solve a linear equation in one variable, we form simpler, equivalent equations until we obtain an equation whose solution is obvious. The properties used to produce equivalent equations include the addition and multiplication properties of equality.

TIP Given $x + 4 = 9$, we can add the opposite of 4 to both sides or subtract 4 from both sides. Likewise, given $4x = 24$, we can multiply both sides by $\frac{1}{4}$ or divide both sides by 4.

Properties of Equality

Let a, b, and c be real-valued expressions.

Property Name	**Statement of Property**	**Example**
Addition property of equality	$a = b$ is equivalent to $a + c = b + c$.	$x + 4 = 9$ $x + 4 + (-4) = 9 + (-4)$ $x = 5$
Multiplication property of equality	$a = b$ is equivalent to $ac = bc$ $(c \neq 0)$.	$4x = 24$ $\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot 24$ $x = 6$

To solve a linear equation in one variable, isolate the variable by following these guidelines.

Solving a Linear Equation in One Variable

Step 1 Simplify both sides of the equation.
- Use the distributive property to clear parentheses.
- Combine like terms.
- Consider clearing fractions or decimals by multiplying both sides of the equation by the least common denominator (LCD) of all terms.

Step 2 Use the addition property of equality to collect the variable terms on one side of the equation and the constant terms on the other side.

Step 3 Use the multiplication property of equality to make the coefficient of the variable term equal to 1.

Step 4 Check the potential solution in the original equation.

Step 5 Write the solution set.

EXAMPLE 1 Solving a Linear Equation

Solve. $-3(w - 4) + 5 = 10 - (w + 1)$

Solution:

$-3(w - 4) + 5 = 10 - (w + 1)$	
$-3w + 12 + 5 = 10 - w - 1$	Apply the distributive property.
$-3w + 17 = 9 - w$	Combine like terms.
$-3w + w + 17 = 9 - w + w$	Add w to both sides of the equation.
$-2w + 17 = 9$	Combine like terms.
$-2w + 17 - 17 = 9 - 17$	Subtract 17 from both sides.
$-2w = -8$	Combine like terms.
$\frac{-2w}{-2} = \frac{-8}{-2}$	Divide both sides by -2 to obtain a coefficient of 1 for w.
$w = 4$	Check: $-3(w - 4) + 5 = 10 - (w + 1)$
	$-3[(4) - 4] + 5 \stackrel{?}{=} 10 - [(4) + 1]$
	$-3(0) + 5 \stackrel{?}{=} 10 - (5)$
	$5 \stackrel{?}{=} 5$ ✓ true

The solution set is $\{4\}$.

Skill Practice 1 Solve. $5(v - 4) - 2 = 2(v + 7) - 3$

If a linear equation contains fractions, it is often helpful to clear the equation of fractions. This is done by multiplying both sides of the equation by the least common denominator (LCD) of all terms in the equation.

EXAMPLE 2 Solving a Linear Equation by Clearing Fractions

Solve. $\frac{m-2}{5} - \frac{m-4}{2} = \frac{m+5}{15} + 2$

Solution:

TIP To find the least common multiple of 5, 2, and 15, first factor each number into prime factors.

$$\begin{aligned} 5 &= 5 \cdot 1 \\ 2 &= 2 \cdot 1 \\ 15 &= 3 \cdot 5 \\ \text{LCD} &= 2 \cdot 3 \cdot 5 = 30 \end{aligned}$$

$$\frac{m-2}{5} - \frac{m-4}{2} = \frac{m+5}{15} + 2$$

The least common denominator is 30.

$$30\left(\frac{m-2}{5} - \frac{m-4}{2}\right) = 30\left(\frac{m+5}{15} + 2\right)$$

Clear fractions by multiplying both sides by the LCD, 30.

$$\frac{\cancel{30}^{6}}{1}\left(\frac{m-2}{\cancel{5}_1}\right) - \frac{\cancel{30}^{15}}{1}\left(\frac{m-4}{\cancel{2}_1}\right) = \frac{\cancel{30}^{2}}{1}\left(\frac{m+5}{\cancel{15}_1}\right) + \frac{30}{1}\left(\frac{2}{1}\right)$$

Apply the distributive property.

$$6(m-2) - 15(m-4) = 2(m+5) + 60$$

$$6m - 12 - 15m + 60 = 2m + 10 + 60$$

Apply the distributive property.

$$-9m + 48 = 2m + 70$$

Combine like terms.

$$-9m - 2m + 48 - 48 = 2m - 2m + 70 - 48$$

Subtract $2m$ and 48 from both sides.

$$-11m = 22$$

Combine like terms.

$$\frac{-11m}{-11} = \frac{22}{-11}$$

Divide both sides by -11 to obtain a coefficient of 1 for m.

$$m = -2$$

The value -2 checks in the original equation.

The solution set is $\{-2\}$.

Skill Practice 2 Solve. $\frac{y+5}{2} - \frac{y-2}{4} = \frac{y+7}{3} + 1$

In Example 3, we use a linear equation to solve the application given at the beginning of the section.

EXAMPLE 3 Using a Linear Equation in an Application

A couple must decide whether to buy a used car for \$8800 or lease a new car for an initial down payment of \$2500 followed by 36 monthly payments of \$225.

a. Write a model for the cost C (in \$) to lease the car for t months.

b. After how many months will the cost to lease the new car be equal to the cost to buy the new car?

Answers

1. {11}

2. {−4}

Solution:

a. The cost to lease the new car for t months includes the monthly payments plus the fixed down payment.

$$C = 225t + 2500$$

b.

$$\begin{aligned} C &= 225t + 2500 \\ 8800 &= 225t + 2500 && \text{Substitute 8800 for the cost } C. \\ 8800 - 2500 &= 225t + 2500 - 2500 && \text{Subtract 2500 from both sides.} \\ 6300 &= 225t \\ \frac{6300}{225} &= \frac{225t}{225} && \text{Divide both sides by 225.} \\ 28 &= t \end{aligned}$$

In 28 months, the cost to lease a new car will be equal to the cost to buy a used car.

Skill Practice 3 To rent a storage unit, a customer must pay a fixed deposit of \$150 plus \$52.50 in rent each month.

a. Write a model for the cost C (in \$) to rent the unit for t months.

b. If Winston has \$1200 budgeted for storage, for how many months can he rent the unit?

2. Identify Conditional Equations, Identities, and Contradictions

The linear equations examined thus far have all had exactly one solution. These equations are examples of conditional equations. A **conditional equation** is true for some values of the variable and false for other values.

An equation that is true for all values of the variable for which the expressions in an equation are defined is called an **identity.** An equation that is false for all values of the variable is called a **contradiction.** Example 4 presents each of these three types of equations.

EXAMPLE 4 Identifying Conditional Equations, Contradictions, and Identities

Identify each equation as a conditional equation, a contradiction, or an identity. Then give the solution set.

a. $3(2x - 1) = 2(3x - 2)$ **b.** $3(2x - 1) = 2(3x - 2) + 1$

c. $3(2x - 1) = 5x - 4$

Solution:

a.

$$\begin{aligned} 3(2x - 1) &= 2(3x - 2) \\ 6x - 3 &= 6x - 4 && \text{Apply the distributive property.} \\ -3 &= -4 && \text{Subtract } 6x \text{ from both sides. Contradiction} \end{aligned}$$

This equation is a contradiction.
The solution set is the empty set { }.

TIP No real number substituted for x will make $-3 = -4$.

Answers

3. a. $C = 150 + 52.50t$
b. 20 months

TIP The statement 0 = 0 is true regardless of the value of *x*. All real numbers are solutions.

b. $3(2x - 1) = 2(3x - 2) + 1$

$6x - 3 = 6x - 4 + 1$ Apply the distributive property.

$6x - 3 = 6x - 3$ Combine like terms.

$0 = 0$ Subtract 6*x* from both sides. Add 3 to both sides.

This equation is an identity.
The solution set is the set of all real numbers, $\mathbb{R}$.

c. $3(2x - 1) = 5x - 4$

$6x - 3 = 5x - 4$ Apply the distributive property.

$x = -1$ Subtract 5*x* from both sides. Add 3 to both sides.

This is a conditional equation.
The solution set is $\{-1\}$.

Conditional equation. The statement is true only under the condition that $x = -1$.

Skill Practice 4 Identify each equation as a conditional equation, a contradiction, or an identity. Then give the solution set.

a. $4x + 1 - x = 6x - 2$ **b.** $2(-5x - 1) = 2x - 12x + 6$
c. $2(3x - 1) = 6(x + 1) - 8$

3. Solve Rational Equations

One of the powerful features of mathematics is that methods used to solve one type of equation can sometimes be adapted to solve other types of equations. Two equations are shown here. The equation on the left is a linear equation. The equation on the right is a rational equation. A **rational equation** is an equation in which each term contains a rational expression. All linear equations are rational equations, but not all rational equations are linear. For example:

Linear Equation with Constants in the Denominator	**Rational Equation with Variables in the Denominator**
$\frac{x}{2} = \frac{2x}{3} - 1$	$\frac{12}{x} = \frac{6}{2x} + 3$

The linear equation can be solved by first multiplying both sides of the equation by the least common denominator of all the fractions. This is the same strategy used in Example 5 to solve a rational equation with a variable in the denominator. However, when a variable appears in the denominator of a fraction, we must restrict the values of the variable to avoid division by zero.

EXAMPLE 5 Solving a Rational Equation

Solve the equation and check the solution. $\frac{12}{x} = \frac{6}{2x} + 3$

Solution:

$$\frac{12}{x} = \frac{6}{2x} + 3$$

Restrict *x* so that $x \neq 0$.

$$2x\left(\frac{12}{x}\right) = 2x\left(\frac{6}{2x} + \frac{3}{1}\right)$$

Clear fractions by multiplying both sides by the LCD, 2*x*. Since $x \neq 0$, this will produce an equivalent equation.

Answers

4. **a.** Conditional equation; {1}
   b. Contradiction; { }
   c. Identity; $\mathbb{R}$

$$\frac{2\cancel{x}}{1}\left(\frac{12}{\cancel{x}}\right) = \frac{\cancel{2x}}{1}\left(\frac{6}{\cancel{2x}}\right) + \frac{2x}{1}\left(\frac{3}{1}\right)$$ Apply the distributive property.

$24 = 6 + 6x$ Simplify.

$18 = 6x$ Subtract 6 from both sides.

$3 = x$

Check: $\frac{12}{x} = \frac{6}{2x} + \frac{3}{1}$

$\frac{12}{(3)} \overset{?}{=} \frac{6}{2(3)} + \frac{3}{1}$

$4 \overset{?}{=} 1 + 3$ ✓ true

The solution set is $\{3\}$.

Skill Practice 5 Solve the equation and check the solution. $\frac{15}{y} = \frac{21}{3y} + 2$

In Example 6, we demonstrate the importance of determining restricted values of the variable in an equation and checking the potential solutions. You will see that for some equations, a potential solution does not check.

EXAMPLE 6 Solving a Rational Equation

Solve the equation and check the solution. $\frac{x}{x-4} = \frac{4}{x-4} - \frac{4}{5}$

Solution:

$$\frac{x}{x-4} = \frac{4}{x-4} - \frac{4}{5}$$ Restrict x so that $x \neq 4$.

$$5(x-4)\left(\frac{x}{x-4}\right) = 5(x-4)\left(\frac{4}{x-4} - \frac{4}{5}\right)$$ Clear fractions by multiplying both sides by the LCD $5(x-4)$.

$$\frac{5(\cancel{x-4})}{1}\left(\frac{x}{\cancel{x-4}}\right) = \frac{5(\cancel{x-4})}{1}\left(\frac{4}{\cancel{x-4}}\right) + \frac{\cancel{5}(x-4)}{1}\left(-\frac{4}{\cancel{5}}\right)$$ Apply the distributive property.

$5x = 20 - 4(x-4)$ Simplify.

$5x = 20 - 4x + 16$ Apply the distributive property.

$9x = 36$ Combine like terms.

$x \xcancel{=} 4$ This is a restricted value of x. Substituting 4 for x in the original equation results in division by 0.

Check: $\frac{x}{x-4} = \frac{4}{x-4} - \frac{4}{5}$

$$\underbrace{\frac{(4)}{(4)-4}}_{\text{undefined}} \overset{?}{=} \underbrace{\frac{4}{(4)-4}}_{\text{undefined}} - \frac{4}{5}$$

The solution set is $\{\ \}$.

Skill Practice 6 Solve the equation and check the solution.

$$\frac{y}{y+5} = \frac{-5}{y+5} + \frac{5}{4}$$

Answers

5. $\{4\}$

6. $\{\ \}$; the value -5 does not check.

EXAMPLE 7 Solving a Rational Equation

Solve the equation and check the solution. $\dfrac{6}{y^2 + 8y + 15} - \dfrac{2}{y + 3} = \dfrac{-4}{y + 5}$

Solution:

$$\frac{6}{y^2 + 8y + 15} - \frac{2}{y + 3} = \frac{-4}{y + 5}$$

$$\frac{6}{(y + 3)(y + 5)} - \frac{2}{y + 3} = \frac{-4}{y + 5}$$

Restrict y so that $y \neq -3, y \neq -5$.
Clear fractions by multiplying both sides by the LCD $(y + 3)(y + 5)$.

$$(y + 3)(y + 5)\left(\frac{6}{(y + 3)(y + 5)} - \frac{2}{y + 3}\right) = (y + 3)(y + 5)\left(\frac{-4}{y + 5}\right)$$

$$\frac{\cancel{(y + 3)(y + 5)}}{1}\left(\frac{6}{\cancel{(y + 3)(y + 5)}}\right) - \frac{\cancel{(y + 3)}(y + 5)}{1}\left(\frac{2}{\cancel{y + 3}}\right)$$

$$= \frac{(y + 3)\cancel{(y + 5)}}{1}\left(\frac{-4}{\cancel{y + 5}}\right)$$

$$6 - 2(y + 5) = -4(y + 3)$$

Apply the distributive property.

$$6 - 2y - 10 = -4y - 12$$

$$-2y - 4 = -4y - 12$$

Combine like terms.

$$2y = -8$$

$$y = -4$$

The value -4 is *not* a restricted value.

Check: $\dfrac{6}{y^2 + 8y + 15} - \dfrac{2}{y + 3} = \dfrac{-4}{y + 5}$

$$\frac{6}{(-4)^2 + 8(-4) + 15} - \frac{2}{(-4) + 3} \stackrel{?}{=} \frac{-4}{(-4) + 5}$$

$-6 + 2 \stackrel{?}{=} -4$ ✓ true

The solution set is $\{-4\}$.

Skill Practice 7 Solve the equation.

$$\frac{11}{x^2 + 5x + 4} - \frac{3}{x + 4} = \frac{1}{x + 1}$$

4. Solve Literal Equations for a Specified Variable

Sometimes an equation contains multiple variables. For example, $d = rt$ relates the distance that an object travels to the rate of travel and time of travel. Such an equation is called a literal equation (an equation with many letters). We often want to manipulate a literal equation to solve for a specified variable. In such a case, we use the same techniques as we would with an equation containing one variable.

Answer

7. {1}

EXAMPLE 8 Solving an Equation for a Specified Variable

Solve for the indicated variable.

a. $d = rt$ for t **b.** $3x + 2y = 6$ for y

c. $A = \frac{1}{2}h(B + b)$ for B

Solution:

a. $d = rt$ for t

$$\frac{d}{r} = \frac{rt}{r}$$

$$\frac{d}{r} = t \quad \text{or} \quad t = \frac{d}{r}$$

The relationship between r and t is multiplication. Therefore, perform the inverse operation. Divide both sides by r.

b. $3x + 2y = 6$ for y

$$2y = -3x + 6$$

Subtract $3x$ from both sides to isolate the y term on one side of the equation.

$$\frac{2y}{2} = \frac{-3x + 6}{2}$$

Divide both sides by 2 to isolate y.

$$y = \frac{-3x + 6}{2} \quad \text{or} \quad y = -\frac{3}{2}x + 3$$

c. $A = \frac{1}{2}h(B + b)$ for B

First note that letters in algebra are case sensitive. The letters b and B represent different variables.

$$2(A) = 2\left[\frac{1}{2}h(B + b)\right]$$

Multiply by 2 to clear fractions.

$$2A = h(B + b)$$

$$\frac{2A}{h} = \frac{h(B + b)}{h}$$

Divide by h.

$$\frac{2A}{h} = B + b$$

$$\frac{2A}{h} - b = B \quad \textbf{or} \quad B = \frac{2A}{h} - b$$

Subtract b from both sides to isolate B.

TIP Alternatively, after clearing fractions we can solve for B by first clearing parentheses.

$$2A = h(B + b)$$
$$2A = hB + hb$$
$$2A - hb = hB$$
$$\frac{2A - hb}{h} = B$$

Skill Practice 8 Solve for the indicated variable.

a. $I = Prt$ for t **b.** $4x + 3y = 12$ for y

c. $A = \frac{1}{2}h(B + b)$ for b

Answers

8. a. $t = \frac{I}{Pr}$

b. $y = \frac{-4x + 12}{3}$ or $y = -\frac{4}{3}x + 4$

c. $b = \frac{2A}{h} - B$

In Example 9, multiple occurrences of the variable x appear within the equation. Factoring is required to combine x terms so that we can isolate x.

EXAMPLE 9 Solving an Equation for a Specified Variable

Solve the equation for x. $ax + by = cx + d$

Solution:

$ax + by = cx + d$ Subtract cx from both sides to combine the x terms on one side.

$ax - cx = d - by$ Subtract by from both sides to combine the non-x terms on the other side.

$x(a - c) = d - by$ Factor out x as the GCF on the left side of the equation.

$\frac{x(a - c)}{(a - c)} = \frac{d - by}{(a - c)}$ Divide by $(a - c)$.

$x = \frac{d - by}{a - c}$

Skill Practice 9 Solve the equation for x. $3x - w = ax + z$

Answer

9. $x = \frac{w + z}{3 - a}$ or $x = -\frac{w + z}{a - 3}$

TIP In Example 9, the answer can be expressed in different forms. For example, if we had isolated x on the right side of the equation, the solution for x would be

$$ax + by = cx + d$$
$$by - d = cx - ax$$
$$\frac{by - d}{(c - a)} = \frac{x(c - a)}{(c - a)}$$
$$x = \frac{by - d}{c - a}$$

To show that $\frac{by - d}{c - a} = \frac{d - by}{a - c}$ multiply either expression by $\frac{-1}{-1}$.

$$\frac{(by - d)}{(c - a)} \cdot \frac{(-1)}{(-1)} = \frac{-by + d}{-c + a} = \frac{d - by}{a - c}$$

SECTION 1.1 Practice Exercises

Prerequisite Review

R.1. Clear parentheses and combine like terms.

$-3(-6 + 2w) - 9w + 2(w - 1)$

R.2. Simplify. $(x + 9)^2$

R.3. Factor completely. $2p^2 + p - 15$

For Exercises R.4–R.6, find the least common denominator (LCD).

R.4. $\frac{y}{6x}; \frac{y^2}{21}$

R.5. $\frac{n}{(5n - 2)(n - 2)}; \frac{7}{(5n - 2)(n + 7)}$

R.6. $\frac{1}{a - 5}; \frac{a - 6}{a^2 - 7a + 10}$

Concept Connections

1. An equation that can be written in the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$, is called a __________ equation in one variable.

2. A linear equation is also called a __________ -degree equation because the degree of the variable is 1.

3. A ________________ to an equation is the value of the variable that makes the equation a true statement.

4. A ________________ equation is one that is true for some values of the variable and false for others.

5. An ________________ is an equation that is true for all values of the variable for which the expressions in the equation are defined.

6. A ________________ is an equation that is false for all values of the variable.

7. A ________________ equation is an equation in which each term contains a rational expression.

8. If an equation has no solution, then the solution set is the ________________ set and is denoted by ________________.

Objective 1: Solve Linear Equations in One Variable

For Exercises 9–10, determine if the equation is linear or nonlinear. If the equation is linear, find the solution set.

9. a. $-2x = 8$ **b.** $\dfrac{-2}{x} = 8$ **c.** $-\dfrac{1}{2}x = 8$

d. $-2|x| = 8$ **e.** $x - 2 = 8$

10. a. $12 = 4x$ **b.** $12 = \dfrac{4}{x}$ **c.** $12 = \dfrac{1}{4}x$

d. $12 = 4\sqrt{x}$ **e.** $12 = 4 + x$

For Exercises 11–30, solve the equation. (See Examples 1–2)

11. $-6x - 4 = 20$

12. $-8y + 6 = 22$

13. $4 = 7 - 3(4t + 1)$

14. $11 = 7 - 2(5p - 2)$

15. $-6(v - 2) + 3 = 9 - (v + 4)$

16. $-5(u - 4) + 2 = 11 - (u - 3)$

17. $2.3 = 4.5x + 30.2$

18. $9.4 = 3.5p - 0.4$

19. $0.05y + 0.02(6000 - y) = 270$

20. $0.06x + 0.04(10{,}000 - x) = 520$

21. $2(5x - 6) = 4[x - 3(x - 10)]$

22. $4(y - 3) = 3[y + 2(y - 2)]$

23. $\dfrac{1}{4}x - \dfrac{3}{2} = 2$

24. $\dfrac{1}{6}x - \dfrac{5}{3} = 1$

25. $\dfrac{1}{2}w - \dfrac{3}{4} = \dfrac{2}{3}w + 2$

26. $\dfrac{2}{5}p - \dfrac{3}{10} = \dfrac{7}{15}p - 1$

27. $\dfrac{y - 1}{5} + \dfrac{y}{4} = \dfrac{y + 3}{2} + 1$

28. $\dfrac{x - 6}{3} + \dfrac{x}{7} = \dfrac{x + 1}{3} + 2$

29. $\dfrac{n + 3}{4} - \dfrac{n - 2}{5} = \dfrac{n + 1}{10} - 1$

30. $\dfrac{t - 2}{3} - \dfrac{t + 7}{5} = \dfrac{t - 4}{10} + 2$

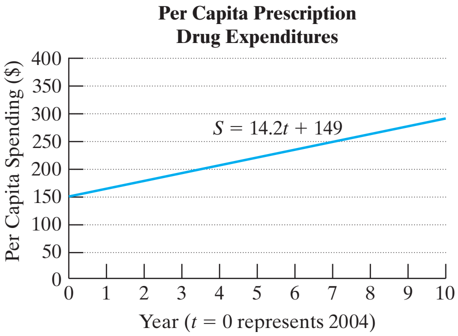
31. In the mid-nineteenth century, explorers used the boiling point of water to estimate altitude. The boiling temperature of water T (in °F) can be approximated by the model $T = -1.83a + 212$, where a is the altitude in thousands of feet.

- **a.** Determine the temperature at which water boils at an altitude of 4000 ft. Round to the nearest degree.
- **b.** Two campers hiking in Colorado boil water for tea. If the water boils at 193°F, approximate the altitude of the campers. Give the result to the nearest hundred feet.

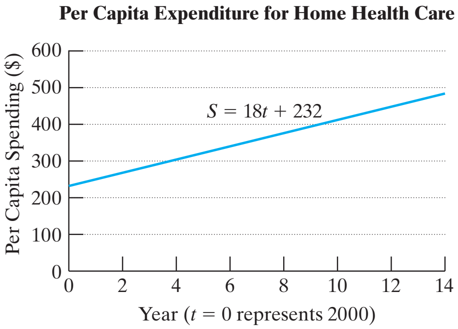
32. For a recent year, the cost C (in \$) for tuition and fees for x credit-hours at a public college was given by $C = 167.95x + 94$.

- **a.** Determine the cost to take 9 credit-hours.
- **b.** If Jenna spent \$2445.30 for her classes, how many credit-hours did she take?

33. The annual per capita spending S (in \$) for prescription drugs can be modeled by $S = 14.2t + 149$, where t is the number of years since 2004. If this model continues, in what year would the average spending for prescription drugs equal \$362 per person? (*Source:* U.S. Centers for Medicare and Medicaid Services, www.census.gov)



34. The annual per capita spending S (in \$) for home health care can be modeled by $S = 18t + 232$, where t is the number of years since 2000. If this model continues, in what year would the average spending for home health care equal \$628 per person? (*Source:* U.S. Centers for Medicare and Medicaid Services, www.census.gov)



35. A motorist drives on State Road 417 to and from work each day and pays \$3.50 in tolls one-way.

a. Write a model for the cost for tolls C (in \$) for x working days.

b. The department of transportation has a prepaid toll program that discounts tolls for high-volume use. The motorist can buy a pass for \$105 per month. How many working days are required for the motorist to save money by buying the pass? (**See Example 3**)

36. A subway ride is \$2.25 per ride.

a. Write a model for the cost C (in \$) for x rides on the subway.

b. A commuter can purchase an unlimited-ride MetroCard for \$89 per month. How many rides are required for a commuter to save money by buying the MetroCard?

37. Helene considers two jobs. One pays \$45,000/yr with an anticipated yearly raise of \$2250. A second job pays \$48,000/yr with yearly raises averaging \$2000.

a. Write a model representing the salary S_1 (in \$) for the first job in x years.

b. Write a model representing the salary S_2 (in \$) for the second job in x years.

c. In how many years will the salary from the first job equal the salary from the second?

38. Tasha considers two sales jobs for different pharmaceutical companies. One pays a base salary of \$25,000 with a 16% commission on sales. The other pays \$30,000 with a 15% commission on sales.

a. Write a model representing the salary S_1 (in \$) for the first job based on x dollars in sales.

b. Write a model representing the salary S_2 (in \$) for the second job based on x dollars in sales.

c. For how much in sales will the two jobs result in equal salaries?

Objective 2: Identify Conditional Equations, Identities, and Contradictions

For Exercises 39–44, identify the equation as a conditional equation, a contradiction, or an identity. Then give the solution set. (See Example 4)

39. $2x - 3 = 4(x - 1) - 1 - 2x$

40. $4(3 - 5n) + 1 = -4n - 8 - 16n$

41. $-(6 - 2w) = 4(w + 1) - 2w - 10$

42. $-5 + 3x = 3(x - 1) - 2$

43. $\frac{1}{2}x + 3 = \frac{1}{4}x + 1$

44. $\frac{2}{3}y - 5 = \frac{1}{6}y - 4$

Objective 3: Solve Rational Equations

For Exercises 45–48, determine the restrictions on x.

45. $\frac{3}{x - 5} + \frac{2}{x + 4} = \frac{5}{7}$

46. $\frac{2}{x + 1} - \frac{5}{x - 7} = \frac{2}{3}$

47. $\frac{5}{2x - 3} - \frac{3}{5x} = \frac{1}{3 - x}$

48. $\frac{1}{2x} - \frac{1}{6 - x} = \frac{2}{4x - 5}$

For Exercises 49–66, solve the equation. (See Examples 5–7)

49. $\frac{1}{2} - \frac{7}{2y} = \frac{5}{y}$

50. $\frac{1}{3} - \frac{4}{3t} = \frac{7}{t}$

51. $\frac{w + 3}{4w} + 1 = \frac{w - 5}{w}$

52. $\frac{x + 2}{6x} + 1 = \frac{x - 7}{x}$

53. $\frac{c}{c - 3} = \frac{3}{c - 3} - \frac{3}{4}$

54. $\frac{7}{d - 7} - \frac{7}{8} = \frac{d}{d - 7}$

55. $\frac{1}{t - 1} = \frac{3}{t^2 - 1}$

56. $\frac{1}{w + 2} = \frac{5}{w^2 - 4}$

57. $\frac{2}{x - 5} - \frac{1}{x + 5} = \frac{11}{x^2 - 25}$

58. $\frac{2}{c + 3} - \frac{1}{c - 3} = \frac{10}{c^2 - 9}$

59. $\frac{-14}{x^2 - x - 12} - \frac{1}{x - 4} = \frac{2}{x + 3}$

60. $\frac{2}{x^2 + 5x + 6} - \frac{2}{x + 2} = \frac{1}{x + 3}$

61. $\frac{5}{x^2 - x - 2} - \frac{2}{x^2 - 4} = \frac{4}{x^2 + 3x + 2}$

62. $\frac{4}{x^2 - 2x - 8} - \frac{1}{x^2 - 16} = \frac{2}{x^2 + 6x + 8}$

63. $\frac{5}{m - 2} = \frac{3m}{m^2 + 2m - 8} - \frac{2}{m + 4}$

64. $\frac{10}{n - 6} = \frac{15n}{n^2 - 2n - 24} - \frac{6}{n + 4}$

65. $\frac{5x}{3x^2 - 5x - 2} - \frac{1}{3x + 1} = \frac{3}{2 - x}$

66. $\frac{3x}{2x^2 + x - 3} - \frac{2}{2x + 3} = \frac{4}{1 - x}$

Objective 4: Solve Literal Equations for a Specified Variable

For Exercises 67–88, solve for the specified variable. (See Examples 8–9)

67. $A = lw$ for l

68. $E = IR$ for R

69. $P = a + b + c$ for c

70. $W = K - T$ for K

71. $\Delta s = s_2 - s_1$ for s_1

72. $\Delta t = t_f - t_i$ for t_i

73. $7x + 2y = 8$ for y

74. $3x + 5y = 15$ for y

75. $5x - 4y = 2$ for y

76. $7x - 2y = 5$ for y

77. $\frac{1}{2}x + \frac{1}{3}y = 1$ for y

78. $\frac{1}{4}x - \frac{2}{3}y = 2$ for y

79. $S = \frac{n}{2}(a + d)$ for d

80. $S = \frac{n}{2}[2a + (n - 1)d]$ for a

81. $V = \frac{1}{3}\pi r^2 h$ for h

82. $V = \frac{1}{3}Bh$ for B

83. $6 = 4x + tx$ for x

84. $8 = 3x + kx$ for x

85. $6x + ay = bx + 5$ for x

86. $3x + 2y = cx + d$ for x

87. $A = P + Prt$ for P

88. $C = A + Ar$ for A

Mixed Exercises

For Exercises 89–102, solve the equation.

89. $\frac{5}{2n + 1} = \frac{-2}{3n - 4}$

90. $\frac{4}{5z - 3} = \frac{-2}{4z + 7}$

91. $5 - 2\{3 - [5v + 3(v - 7)]\} = 8v + 6(3 - 4v) - 61$

92. $6 - \{4 - 2[8u - 2(u - 3)]\} = -4u + 3(2 - u) + 8$

93. $(x - 7)(x + 2) = x^2 + 4x + 13$

94. $(m + 3)(2m - 5) = 2m^2 + 4m - 3$

95. $\frac{3}{c^2 - 4c} - \frac{9}{2c^2 + 3c} = \frac{2}{2c^2 - 5c - 12}$

96. $\frac{4}{d^2 - d} - \frac{5}{2d^2 + 5d} = \frac{2}{2d^2 + 3d - 5}$

97. $\frac{1}{3}x + \frac{1}{2} = \frac{1}{2}(x + 1) - \frac{1}{6}x$

98. $\frac{1}{2}x + \frac{2}{5} = \frac{2}{5}(x + 1) + \frac{1}{10}x$

99. $(t + 2)^2 = (t - 4)^2$

100. $(y - 3)^2 = (y + 1)^2$

101. $\frac{3}{3a + 4} = \frac{5}{5a - 1}$

102. $\frac{8}{8x - 3} = \frac{2}{2x + 5}$

103. Suppose that 40 deer are introduced in a protected wilderness area. The population of the herd P can be approximated by $P = \frac{40 + 20x}{1 + 0.05x}$, where x is the time in years since introducing the deer. Determine the time required for the deer population to reach 200.

104. Starting from rest, an automobile's velocity v (in ft/sec) is given by $v = \frac{180t}{2t + 10}$, where t is the time in seconds after the car begins forward motion. Determine the time required for the car to reach a speed of 60 ft/sec ($\approx$ 41 mph).

105. Brianna's SUV gets 22 mpg in the city and 30 mpg on the highway. The amount of gas she uses A (in gal) is given by $A = \frac{1}{22}c + \frac{1}{30}h$, where c is the number of city miles driven and h is the number of highway miles driven. If Brianna drove 165 mi on the highway and used 7 gal of gas, how many city miles did she drive?

106. Dexter's truck gets 32 mpg on the highway and 24 mpg in the city. The amount of gas he uses A (in gal) is given by $A = \frac{1}{24}c + \frac{1}{32}h$, where c is the number of city miles driven and h is the number of highway miles driven. If Dexter drove 60 mi in the city and used 9 gal of gas, how many highway miles did he drive?

Write About It

107. Explain why the value 5 is not a solution to the equation $\frac{x}{x - 5} + \frac{1}{5} = \frac{5}{x - 5}$.

108. Explain why the value 2 is not the only solution to the equation $2x + 4 = 2(x - 3) + 10$.

109. Explain why $\frac{3}{x} + 12 = 0$ is not a linear equation in one variable.

110. Explain why $2\sqrt{x} + 6 = 0$ is not a linear equation in one variable.

111. Explain why the equation $x + 1 = x + 2$ has no solution.

112. Explain the difference in the process to clear fractions between the two equations.

$$\frac{x}{3} + \frac{1}{2} = 1 \quad \text{and} \quad \frac{3}{x} + \frac{1}{2} = 1$$

Expanding Your Skills

For Exercises 113–116, find the value of a so that the equation has the given solution set.

113. $ax + 6 = 4x + 14 \quad \{4\}$

114. $ax - 3 = 2x + 9 \quad \{3\}$

115. $a(2x - 5) + 6 = 5x + 7 \quad \{16\}$

116. $a(2x + 4) + 12x = 3(2 - x) \quad \{34\}$

SECTION 1.2 Applications with Linear and Rational Equations

OBJECTIVES

1. Solve Applications Involving Simple Interest
2. Solve Applications Involving Mixtures
3. Solve Applications Involving Uniform Motion
4. Solve Applications Involving Rate of Work Done
5. Solve Applications Involving Proportions

1. Solve Applications Involving Simple Interest

In Examples 1–5, we use linear and rational equations to model physical situations to solve applications. While there is no magic formula to apply to all word problems, we do offer the following guidelines to help you organize the given information and to form a useful model.

Problem-Solving Strategy

1. Read the problem carefully. Determine what the problem is asking for, and assign variables to the unknown quantities.
2. Make an appropriate figure or table if applicable. Label the given information and variables in the figure or table.
3. Write an equation that represents the verbal model. The equation may be a known formula or one that you create that is unique to the problem.
4. Solve the equation from step 3.
5. Interpret the solution to the equation and check that it is reasonable in the context of the problem.

Simple interest I for a loan or an investment is based on the principal P (amount of money invested or borrowed), the annual interest rate r, and the time of the loan t in years. The relationship among the variables is given by $I = Prt$.

For example, if \$5000 is invested at 4% simple interest for 18 months (1.5 yr), then the amount of simple interest earned is

$$\begin{aligned} I &= Prt \\ I &= (\$5000)(0.04)(1.5) \\ &= \$300 \end{aligned}$$

The formula for simple interest is used in Example 1.

EXAMPLE 1 Solving an Application Involving Simple Interest

Kent invested a total of \$8000. He invested part of the money for 2 yr in a stock fund that earned the equivalent of 6.5% simple interest. He put the remaining money in an 18-month certificate of deposit (CD) that earned 2.5% simple interest. If the total interest from both investments was \$855, determine the amount invested in each account.

Solution:

We can assign a variable to *either* the amount invested in the stock fund or the amount invested in the CD.

Let x represent the principal invested in the stock fund.

Then, $(8000 - x)$ is the remaining amount in the CD.

The interest from each account is computed from the formula $I = Prt$. Consider organizing this information in a table.

	Stock Fund (6.5% yield)	CD (2.5% yield)	Total
Principal (\$)	x	$8000 - x$	\$8000
Interest (\$)	$x(0.065)(2)$	$(8000 - x)(0.025)(1.5)$	\$855

Avoiding Mistakes

The CD was invested for 18 months. Be sure to convert to years.

18 months = 1.5 yr

To build an equation, note that

$$\begin{pmatrix}\text{Interest from}\\ \text{stock fund}\end{pmatrix} + \begin{pmatrix}\text{Interest}\\ \text{from CD}\end{pmatrix} = \begin{pmatrix}\text{Total}\\ \text{interest}\end{pmatrix}$$

$$\begin{aligned} x(0.065)(2) + (8000 - x)(0.025)(1.5) &= 855 && \text{Second row of table} \\ 0.13x + 0.0375(8000 - x) &= 855 && \text{Simplify.} \\ 0.13x + 300 - 0.0375x &= 855 && \text{Apply the distributive property.} \\ 0.0925x + 300 &= 855 && \text{Combine like terms.} \\ 0.0925x &= 555 && \text{Subtract 300 from both sides.} \\ x &= 6000 \end{aligned}$$

The amount invested in the stock fund is x: \$6000.
The amount invested in the CD is $\$8000 - x = \$8000 - \$6000 = \$2000$.

Avoiding Mistakes

Check that the answer is reasonable.

Amount of interest:
(\$6000)(0.065)(2) = \$780
(\$2000)(0.025)(1.5) = \$75
Total: \$780 + \$75 = \$855 ✓

Skill Practice 1 Franz borrowed a total of \$10,000. Part of the money was borrowed from a lending institution that charged 5.5% simple interest. The rest of the money was borrowed from a friend to whom Franz paid 2.5% simple interest. Franz paid his friend back after 9 months (0.75 yr) and paid the lending institution after 2 yr. If the total amount Franz paid in interest was \$735, how much did he borrow from each source?

Answer

1. Franz borrowed \$4000 from his friend and \$6000 from the lending institution.

2. Solve Applications Involving Mixtures

In Example 1, we "mixed" money between two different investments. We had to find the correct distribution of principal between two accounts to produce the given amount of interest. Example 2 presents a similar type of application that involves mixing different concentrations of a bleach solution to produce a third mixture of a given concentration.

For example, household bleach contains 6% sodium hypochlorite (active ingredient). This means that the remaining 94% of liquid is some other mixing agent such as water. Therefore, given 200 cL of household bleach, 6% would be pure sodium hypochlorite, and 94% would be some other mixing agent.

$$\text{Pure sodium hypochlorite} = (0.06)(200\text{ cL}) = 12\text{ cL}$$
$$\text{Other mixing agent} = (0.94)(200\text{ cL}) = 188\text{ cL}$$

To find the amount of pure sodium hypochlorite, we multiplied the concentration rate by the amount of solution.

EXAMPLE 2 Solving an Application Involving Mixtures

Household bleach contains 6% sodium hypochlorite. How much household bleach should be combined with 70 L of a weaker 1% hypochlorite solution to form a solution that is 2.5% sodium hypochlorite?

Solution:

Let x represent the amount of 6% sodium hypochlorite solution (in liters).
70 L is the amount of 1% sodium hypochlorite solution.
Therefore, $x + 70$ is the amount of the resulting mixture (2.5% solution).

The amount of pure sodium hypochlorite in each mixture is found by multiplying the concentration rate by the amount of solution.

	6% Solution	1% Solution	2.5% Solution
Amount of solution (L)	x	70	$x + 70$
Pure sodium hypochlorite (L)	$0.06x$	$0.01(70)$	$0.025(x + 70)$

To build an equation, note that

$$\left(\begin{array}{c}\text{Amount of sodium}\\ \text{hypochlorite in}\\ \text{6\% solution}\end{array}\right) + \left(\begin{array}{c}\text{Amount of sodium}\\ \text{hypochlorite in}\\ \text{1\% solution}\end{array}\right) = \left(\begin{array}{c}\text{Amount of sodium}\\ \text{hypochlorite in}\\ \text{2.5\% solution}\end{array}\right)$$

$$\begin{aligned} 0.06x + 0.01(70) &= 0.025(x + 70) && \text{Second row in the table} \\ 0.06x + 0.7 &= 0.025x + 1.75 && \text{Solve the equation.} \\ 0.035x &= 1.05 \\ x &= 30 \end{aligned}$$

The amount of household bleach (6% sodium hypochlorite solution) needed is 30 L.

Avoiding Mistakes

Check that the answer is reasonable. The total amount of the resulting solution is 30 L + 70 L, which is 100 L.

Amount of sodium hypochlorite:

$$\begin{aligned} 0.06(30\text{ L}) &= 1.8\text{ L} \\ 0.01(70\text{ L}) &= 0.7\text{ L} \\ \hline 0.025(100\text{ L}) &= 2.5\text{ L}\ \checkmark \end{aligned}$$

Skill Practice 2 How much 4% acid solution should be mixed with 200 mL of a 12% acid solution to make a 9% acid solution?

Answer

2. 120 mL of the 4% acid solution should be used.

3. Solve Applications Involving Uniform Motion

Example 3 involves uniform motion. Recall that the distance that an object travels is given by

$$d = rt \quad \text{Distance} = (\text{Rate})(\text{Time})$$

EXAMPLE 3 Solving an Application Involving Uniform Motion

Donna participated in a 41-mi biathlon that included running and bicycling. She spent 1 hr 45 min on the bike and 45 min running. If her average speed on the bicycle was 12 mph faster than her average speed running, find her average speed running and her average speed riding.

Solution:

There are two unknowns: Donna's average speed on the bike and her average speed running.

Let x represent Donna's average speed running.
Then $x + 12$ represents her speed on the bicycle.

The remaining information can be organized in a table.

	Distance (mi)	Rate (mph)	Time (hr)
Run	$0.75x$	x	0.75
Bike	$1.75(x + 12)$	$x + 12$	1.75

The expressions in this column are found by $d = rt$.

Note that consistency in the units of measurement is important. The speed is given in miles per *hour*. Therefore, we want the time to be in hours.

1 hr 45 min = 1.75 hr
45 min = 0.75 hr

$$\left(\begin{array}{c}\text{Total}\\ \text{distance}\end{array}\right) = \left(\begin{array}{c}\text{Distance}\\ \text{running}\end{array}\right) + \left(\begin{array}{c}\text{Distance}\\ \text{riding}\end{array}\right)$$

To build an equation, note that the total distance equals the sum of the distance running and the distance riding.

$$\begin{aligned} 41 &= 0.75x + 1.75(x + 12) \\ 41 &= 0.75x + 1.75x + 21 \\ 20 &= 2.5x \\ 8 &= x \end{aligned}$$

Solve the equation.

Donna's speed running is 8 mph.
Her speed on the bicycle is 8 + 12 = 20 mph.

Interpret the solution in the context of the problem.

Avoiding Mistakes

Check that the answer is reasonable by verifying that the total distance traveled is 41 mi.

Distance running:
(8 mph)(0.75 hr) = 6 mi

Distance riding:
(20 mph)(1.75 mi) = 35 mi

Total: 6 mi + 35 mi = 41 mi

Skill Practice 3 Rene drove from Miami to Orlando, a total distance of 240 mi. He drove for 1 hr in city traffic and for 3 hr on the highway. If his average speed on the highway was 20 mph faster than his speed in the city, determine his average speed driving in the city and his average speed driving on the highway.

Answer

3. Rene drove 45 mph in the city and 65 mph on the highway.

Point of Interest

The relationship $d = rt$ is a familiar formula indicating that distance equals rate times time, or equivalently that $t = \frac{d}{r}$. However, suppose that a spaceship travels to a distant planet and then returns to Earth. Einstein's theory of special relativity indicates that $t = \frac{d}{r}$ only represents the trip's duration for an observer on Earth. For a person on the spaceship, the time will be shorter by a factor of $\sqrt{1 - \frac{r^2}{c^2}}$, where r is the speed of the spaceship and c is the speed of light.

For example, suppose that a spaceship travels to a planet 10 light-years away (a light-year is the distance that light travels in 1 yr) and then returns. The round trip is 20 light-years. Further suppose that the spaceship travels at half the speed of light, that is, $r = 0.5c$.

To an observer on Earth, the elapsed time of travel (in yr) is

$$t_E = \frac{d}{r} = \frac{20}{0.5} = 40 \text{ yr}$$

To an observer on the spaceship, the elapsed time (in yr) is

$$t_S = \frac{d}{r}\sqrt{1 - \frac{r^2}{c^2}} = \frac{20}{0.5}\sqrt{1 - \frac{(0.5c)^2}{c^2}} \approx 34.6 \text{ yr}$$

The unit of measurement in each case is years. Thus, the observer on Earth perceives the time of travel to be 40 yr, whereas to an observer on the spaceship the time of travel is only 34.6 yr.

4. Solve Applications Involving Rate of Work Done

EXAMPLE 4 Solving an Application Involving "Work" Rates

At a mail-order company, Derrick can process 100 orders in 4 hr. Miguel can process 100 orders in 3 hr.

a. How long would it take them to process 100 orders if they work together?

b. How long would it take them to process 1400 orders if they work together?

Solution:

a. Let t represent the amount of time required to process 100 orders working together.

One method to approach this problem is to add the rates of speed at which each person works.

$$\begin{pmatrix}\text{Derrick's}\\\text{speed}\end{pmatrix} + \begin{pmatrix}\text{Miguel's}\\\text{speed}\end{pmatrix} = \begin{pmatrix}\text{Speed working}\\\text{together}\end{pmatrix}$$

$$\frac{1 \text{ job}}{4 \text{ hr}} + \frac{1 \text{ job}}{3 \text{ hr}} = \frac{1 \text{ job}}{t \text{ hr}} \qquad 1 \text{ job} = 100 \text{ orders.}$$

$$\frac{1}{4} + \frac{1}{3} = \frac{1}{t}$$

$$12t\left(\frac{1}{4} + \frac{1}{3}\right) = 12t\left(\frac{1}{t}\right) \qquad \text{Multiply both sides by the LCD, } 12t.$$

$$3t + 4t = 12 \qquad \text{Apply the distributive properly.}$$

$$7t = 12$$

$$t = \frac{12}{7} \quad \text{or} \quad 1\frac{5}{7}$$

Derrick and Miguel can process 100 orders in $1\frac{5}{7}$ hr working together.

b. The time required to process 1400 orders is 14 times as long as the time to process 100 orders. $\left(\frac{12}{7}\text{ hr}\right)(14) = 24$ hr.

Skill Practice 4 Sheldon and Penny were awarded a contract to paint 16 offices in the new math building at a university. Once all the preparation work is complete, Sheldon can paint an office in 30 min and Penny can paint an office in 45 min.

a. How long would it take them to paint one office working together?

b. How long would it take them to paint all 16 offices?

5. Solve Applications Involving Proportions

An equation that equates two ratios or rates is called a **proportion.** Symbolically, we define a proportion as an equation of the form

$$\frac{a}{b} = \frac{c}{d}, \text{ where } b \neq 0 \text{ and } d \neq 0$$

The method of clearing fractions can be used to solve proportions.

EXAMPLE 5 Solving an Application Involving a Proportion

In a jury pool, there are 8 more men than women. If the ratio of men to women is 8 to 7, determine the number of men and women in the pool.

Solution:

Let x represent the number of women.
Then $x + 8$ represents the number of men. Label the variables.

$$\begin{array}{r} \text{number of men} \rightarrow \\ \text{number of women} \rightarrow \end{array} \frac{x+8}{x} = \frac{8}{7} \begin{array}{l} \leftarrow \text{men} \\ \leftarrow \text{women} \end{array} \qquad \text{Set up a proportion.}$$

$$7\cancel{x}\left(\frac{x+8}{\cancel{x}}\right) = \cancel{7}x\left(\frac{8}{\cancel{7}}\right) \qquad \text{Multiply by the LCD, } 7x.$$

$$7(x + 8) = x(8)$$

$$7x + 56 = 8x \qquad \text{Apply the distributive property.}$$

$$56 = x$$

The number of women is 56. The number of men is $56 + 8 = 64$.

Skill Practice 5 For the 104th Congress, there were 4 more Republicans than Democrats in the U.S. Senate. This resulted in a ratio of 13 Republicans to 12 Democrats. How many senators were Republican and how many were Democrat?

Answers

4. a. It would take 18 min to paint one office working together.
   b. It would take 288 min (4 hr 48 min) to paint 16 offices working together.
5. There were 52 Republicans and 48 Democrats.

SECTION 1.2 Practice Exercises

Prerequisite Review

R.1. At a recent motorcycle rally, the number of men exceeded the number of women by 254. If x represents the number of men, write an expression for the number of women.

R.2. Rebecca downloaded five times as many songs as Nigel. If x represents the number of songs downloaded by Nigel, write an expression for the number of songs downloaded by Rebecca.

R.3. Sidney made \$29 less than eight times Casey's weekly salary. If x represents Casey's weekly salary, write an expression for Sidney's weekly salary.

R.4. David scored 30 points more than six times the number of points Rich scored in a video game. If x represents the number of points scored by Rich, write an expression representing the number of points scored by David.

R.5. Write a proportion for \$8.25 per hour is proportional to \$82.50 per 10 hours.

Concept Connections

1. If \$6000 is borrowed at 7.5% simple interest for 2 yr, then the amount of interest is ________________.

2. Suppose that 8% of a solution is fertilizer by volume and the remaining 92% is water. How much fertilizer is in a 2-L bucket of solution?

3. If $d = rt$, then $t = \frac{\square}{\square}$

4. If $d = rt$, then $r = \frac{\square}{\square}$

5. The formula for the perimeter P of a rectangle with length l and width w is given by ________________.

6. The sum of the measures of the angles inscribed inside a triangle is ________________.

Objective 1: Solve Applications Involving Simple Interest

7. Rocco borrowed a total of \$5000 from two student loans. One loan charged 3% simple interest and the other charged 2.5% simple interest, both payable after graduation. If the interest he owed after 1 yr was \$132.50, determine the amount of principal for each loan. (**See Example 1**)

8. Laura borrowed a total of \$22,000 from two different banks to start a business. One bank charged the equivalent of 4% simple interest, and the other charged 5.5% interest. If the total interest after 1 yr was \$910, determine the amount borrowed from each bank.

9. Fernando invested money in a 3-yr CD (certificate of deposit) that returned the equivalent of 4.4% simple interest. He invested \$2000 less in an 18-month CD that had a 3% return. If the total amount of interest from these investments was \$706.50, determine how much was invested in each CD.

10. Ebony bought a 5-yr Treasury note that paid the equivalent of 2.8% simple interest. She invested \$5000 more in a 10-yr bond earning 3.6% than she did in the Treasury note. If the total amount of interest from these investments was \$5300, determine the amount of principal for each investment.

Objective 2: Solve Applications Involving Mixtures

11. Ethanol fuel mixtures have "E" numbers that indicate the percentage of ethanol in the mixture by volume. For example, E10 is a mixture of 10% ethanol and 90% gasoline. How much E5 should be mixed with 5000 gal of E10 to make an E9 mixture? (**See Example 2**)

12. A nurse mixes 60 cc of a 50% saline solution with a 10% saline solution to produce a 25% saline solution. How much of the 10% solution should he use?

13. The density and strength of concrete are determined by the ratio of cement and aggregate (aggregate is sand, gravel, or crushed stone). Suppose that a contractor has 480 ft^3 of a dry concrete mixture that is 70% sand by volume. How much pure sand must be added to form a new mixture that is 75% sand by volume?

14. Antifreeze is a compound added to water to reduce the freezing point of a mixture. In extreme cold (less than −35°F), one car manufacturer recommends that a mixture of 65% antifreeze be used. How much 50% antifreeze solution should be drained from a 4-gal tank and replaced with pure antifreeze to produce a 65% antifreeze mixture?

Objective 3: Solve Applications Involving Uniform Motion

15. Two passengers leave the airport at Kansas City, Missouri. One flies to Los Angeles, California, in 3.4 hr and the other flies in the opposite direction to New York City in 2.4 hr. With prevailing westerly winds, the speed of the plane to New York City is 60 mph faster than the speed of the plane to Los Angeles. If the total distance traveled by both planes is 2464 mi, determine the average speed of each plane. (**See Example 3**)

17. Darren drives to school in rush hour traffic and averages 32 mph. He returns home in mid-afternoon when there is less traffic and averages 48 mph. What is the distance between his home and school if the total traveling time is 1 hr 15 min?

16. Two planes leave from Atlanta, Georgia. One makes a 5.2-hr flight to Seattle, Washington, and the other makes a 2.5-hr flight to Boston, Massachusetts. The plane to Boston averages 44 mph slower than the plane to Seattle. If the total distance traveled by both planes is 3124 mi, determine the average speed of each plane.

18. Peggy competes in a biathlon by running and bicycling around a large loop through a city. She runs the loop one time and bicycles the loop five times. She can run 8 mph and she can ride 16 mph. If the total time it takes her to complete the race is 1 hr 45 min, determine the distance of the loop.

Objective 4: Solve Applications Involving Rate of Work Done

19. Joel can run around a $\frac{1}{4}$-mi track in 66 sec, and Jason can run around the track in 60 sec. If the runners start at the same point on the track and run in opposite directions, how long will it take the runners to cover $\frac{1}{4}$ mi? (**See Example 4**)

21. One pump can fill a pool in 10 hr. Working with a second slower pump, the two pumps together can fill the pool in 6 hr. How fast can the second pump fill the pool by itself?

20. Marta can vacuum the house in 40 min. It takes her daughter 1 hr to vacuum the house. How long would it take them if they worked together?

22. Brad and Angelina can mow their yard together with two lawn mowers in 30 min. When Brad works alone, it takes him 50 min. How long would it take Angelina to mow the lawn by herself?

Objective 5: Solve Applications Involving Proportions

23. At a construction site, cement, sand, and gravel are mixed to make concrete. The ratio of cement to sand to gravel is 1 to 2.4 to 3.6. If a 150-lb bag of sand is used, how much cement and gravel must be used? (**See Example 5**)

25. In addition to measuring a person's individual HDL and LDL cholesterol levels, doctors also compute the ratio of total cholesterol to HDL cholesterol. Doctors recommend that the ratio of total cholesterol to HDL cholesterol be kept under 4. Suppose that the ratio of a patient's total cholesterol to HDL is 3.4 and her HDL is 60 mg/dL. Determine the patient's LDL level and total cholesterol. (Assume that total cholesterol is the sum of the LDL and HDL levels.)

27. When studying wildlife populations, biologists sometimes use a technique called "mark-recapture." For example, a researcher captured and tagged 30 deer in a wildlife management area. Several months later, the researcher observed a new sample of 80 deer and determined that 5 were tagged. What is the total number of deer in the population?

24. The property tax on a $180,000 house is $1296. At this rate, what is the property tax on a house that is $240,000?

26. For a recent Congress, there were 10 more Democrats than Republicans in the U.S. Senate. This resulted in a ratio of 11 Democrats to 9 Republicans. How many senators were Democrat and how many were Republican?

28. To estimate the number of bass in a lake, a biologist catches and tags 24 bass. Several weeks later, the biologist catches a new sample of 40 bass and finds that 4 are tagged. How many bass are in the lake?

Mixed Exercises

29. Seismographs can record two types of wave energy (P waves and S waves) that travel through the Earth after an earthquake. Traveling through granite, P waves travel approximately 5 km/sec and S waves travel approximately 3 km/sec. If a geologist working at a seismic station measures a time difference of 40 sec between an earthquake's P waves and S waves, how far from the epicenter of the earthquake is the station?

30. Suppose that a shallow earthquake occurs in which the P waves travel 8 km/sec and the S waves travel 4.8 km/sec. If a seismologist measures a time difference of 20 sec between the arrival of the P waves and the S waves, how far is the seismologist from the epicenter of the earthquake?

31. Suppose that a merchant buys a patio set from the wholesaler for \$180. At what price should the merchant mark the patio set so that it may be offered at a discount of 25% but still give the merchant a 40% profit on his \$180 investment?

32. Suppose that a bookstore buys a textbook from the publisher for \$80. At what price should the bookstore mark the textbook so that it may be offered at a discount of 10% but still give the bookstore a 35% profit on the \$80 investment?

33. Henri needs to have a toilet repaired in his house. The cost of the new plumbing fixtures is \$110 and labor is \$60/hr.

a. Write a model that represents the cost of the repair C (in \$) in terms of the number of hours of labor x.

b. After how many hours of labor would the cost of the repair job equal the cost of a new toilet of \$350?

34. After a hurricane, repairs to a roof will cost \$2400 for materials and \$80/hr in labor.

a. Write a model that represents the cost of the repair C (in \$) in terms of the number of hours of labor x.

b. If an estimate for a new roof is \$5520, after how many hours of labor would the cost to repair the roof equal the cost of a new roof?

35. A student 5 ft tall measures the length of the shadow of the Washington Monument to be 444 ft. At the same time, her shadow is 4 ft. Approximate the height of the Washington Monument.

36. A 6-ft man is standing 40 ft from a light post. If the man's shadow is 20 ft, determine the height of the light post.

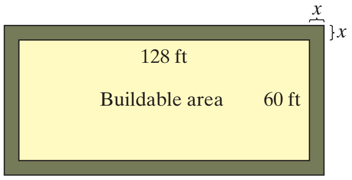
37. A vertical pole is placed in the ground at a campsite outside Salt Lake City, Utah. One winter day, $\frac{1}{8}$ of the pole is in the ground, $\frac{2}{3}$ of the pole is covered in snow, and 1.5 ft is above the snow. How long is the pole, and how deep is the snow?

38. The formula to convert temperature in Fahrenheit F to temperature in Celsius C is given by $C = \frac{5}{9}(F - 32)$. Determine the temperature at which the Celsius and Fahrenheit temperature readings are the same.

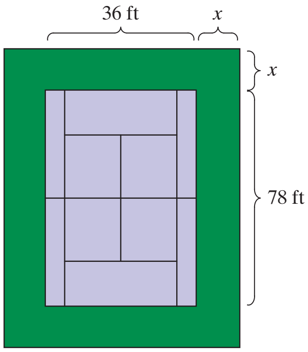
39. A tank contains 40 L of a mixture of plant fertilizer and water in which 20% of the mixture is fertilizer. How much of the mixture should be drained and replaced by an equal amount of water to dilute the mixture to 15% fertilizer?

40. How much water must be evaporated from 200 mL of a 5% salt solution to produce a 25% salt solution?

41. The perimeter of a rectangular lot of land is 440 ft. This includes an easement of x feet of uniform width inside the lot on which no building can be done. If the buildable area is 128 ft by 60 ft, determine the width of the easement.



42. The Arthur Ashe Stadium tennis court is center court to the U.S. Open tennis tournament. The dimensions of the court are 78 ft by 36 ft, with a uniform border of x feet around the outside for additional play area. If the perimeter of the entire play area is 396 ft, determine the value of x.



43. A contractor must tile a rectangular kitchen that is 4 ft longer than it is wide, and the perimeter of the kitchen is 48 ft.

a. Find the dimensions of the kitchen.

b. How many square feet of tile should be ordered if the contractor adds an additional 10% to account for waste?

c. Determine the total cost if the tile costs $12/ft^2 and sales tax is 6%.

45. Aliyah earned an $8000 bonus from her sales job for exceeding her sales goals. After paying taxes at a 28% rate, she invested the remaining money in two stocks. One stock returned the equivalent of 11% simple interest after 1 yr, and the other returned 5% at the end of 1 yr. If her investments returned $453.60 (excluding commissions) how much did she invest in each stock?

44. Max and Molly plan to put down all-weather carpeting on their porch. The length of the porch is 2 ft longer than twice the width, and the perimeter is 64 ft.

a. Find the dimensions of the porch.

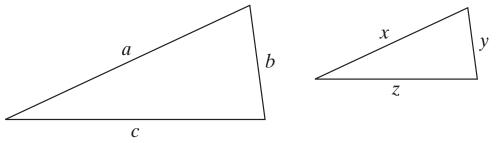
b. How many square feet of carpeting should they buy if they add an additional 10% for waste?

c. Determine the total cost if the carpeting costs $5.85/ft^2 and sales tax is 7.5%.

46. Caitlin invested money in two mutual funds—a stock fund and a balanced fund. She invested twice as much in the stock fund as in the balanced fund. At the end of 1 yr, the stock fund earned the equivalent of 17% simple interest and the balanced fund earned 3.5%. If her total gain was $1125, determine how much she invested in each fund.

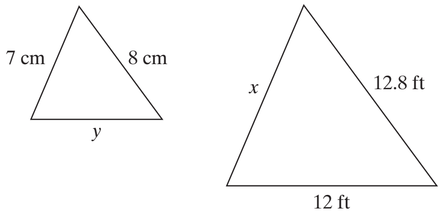
Proportions are used in geometry with similar triangles. If two triangles are similar, then the lengths of the corresponding sides are proportional.

For the similar triangles shown, $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$.

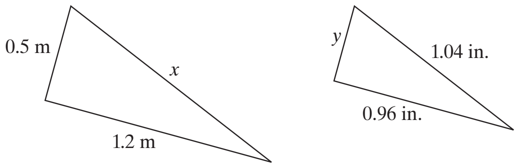


For Exercises 47–48, the triangles are similar with the corresponding sides oriented as shown. Solve for *x* and *y*.

47.



48.



Write About It

49. Is it possible for the measures of the angles in a triangle to be represented by three consecutive odd integers? Explain.

50. Bob wants to change a $100 bill into an equal number of $20 bills, $10 bills, and $5 bills. Is this possible? Explain.

Expanding Your Skills

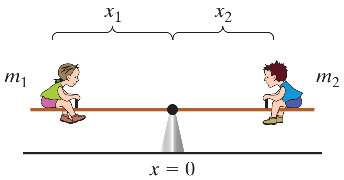
51. One number is 16 more than another number. The quotient of the larger number and smaller number is 3 and the remainder is 2. Find the numbers.

53. The sum of the digits of a two-digit number is 14. If the digits are reversed, the new number is 18 more than the original number. Determine the original number.

52. One number is 25 more than another number. The quotient of the larger number and the smaller number is 4 and the remainder is 1. Find the numbers.

54. The sum of the digits of a two-digit number is 9. If the digits are reversed, the new number is 45 less than the original number. Determine the original number.

Consider a seesaw with two children of masses m_1 and m_2 on either side. Suppose that the position of the fulcrum (pivot point) is labeled as the origin, $x = 0$. Further suppose that the position of each child relative to the origin is x_1 and x_2, respectively. The seesaw will be in equilibrium if $m_1x_1 + m_2x_2 = 0$. Use this equation for Exercises 55–58.



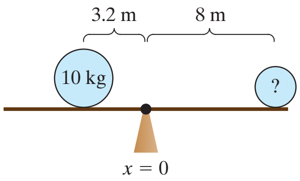
55. Find x_2 so that the system of masses is in equilibrium.

$m_1 = 30$ kg, $x_1 = -1.2$ m and $m_2 = 20$ kg, $x_2 = ?$

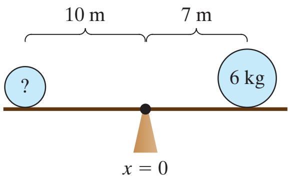
56. Find x_1 so that the system of masses is in equilibrium.

$m_1 = 64$ kg, $x_1 = ?$ and $m_2 = 80$ kg, $x_2 = 2$ m

57. Find the missing mass so that the system is in equilibrium. (*Hint*: Recall that positions to the left of 0 on the number line are negative.)



58. Find the missing mass so that the system is in equilibrium.



SECTION 1.7 Linear, Compound, and Absolute Value Inequalities

OBJECTIVES

1. Solve Linear Inequalities in One Variable
2. Solve Compound Linear Inequalities
3. Solve Absolute Value Inequalities
4. Solve Applications of Inequalities

1. Solve Linear Inequalities in One Variable

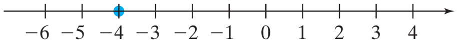
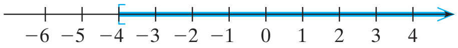
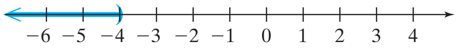
Emily wants to earn an "A" in her College Algebra course and knows that the average of her tests and assignments must be at least 90. She has five test grades of 96, 84, 80, 98, and 88. She also has a score of 100 for online homework, and this carries the same weight as a test grade. She still needs to take the final exam and the final is weighted as two test grades. To determine the scores on the final exam that would result in an average of 90 or more, Emily would solve the following inequality (see Example 9):

$$\frac{96 + 84 + 80 + 98 + 88 + 100 + 2x}{8} \geq 90, \text{ where } x \text{ is Emily's score on the final.}$$

A linear equation in one variable is an equation that can be written as $ax + b = 0$, where a and b are real numbers and $a \neq 0$. A **linear inequality** is any relationship of

TIP For a review of set-builder notation and interval notation, see Section R.1.

the form $ax + b < 0$, $ax + b \le 0$, $ax + b > 0$, or $ax + b \ge 0$. The solution set to a linear equation consists of a single element that can be represented by a point on the number line. The solution set to a linear inequality contains an infinite number of elements and can be expressed in set-builder notation or in interval notation.

Equation/ Inequality	Solution Set	Graph
$x + 4 = 0$	$\{-4\}$	
$x + 4 \ge 0$	$\{x \mid x \ge -4\}$ or $[-4, \infty)$	
$x + 4 < 0$	$\{x \mid x < -4\}$ or $(-\infty, -4)$	

To solve a linear inequality in one variable, we use the following properties of inequality.

Properties of Inequality

Let a, b, and c represent real numbers.

1. If $x < a$, then $a > x$.
2. If $a < b$ and $b < c$, then $a < c$.
3. If $a < b$ and $c < d$, then $a + c < b + d$.
4. If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
5. If c is *positive* and $a < b$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
6. If c is *negative* and $a < b$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

These statements are also true expressed with the symbols $>$, $\le$, and $\ge$.

Property 6 indicates that if both sides of an inequality are multiplied or divided by a negative number, then the direction of the inequality sign must be reversed.

EXAMPLE 1 Solving a Linear Inequality

Solve the inequality. Graph the solution set and write the solution set in set-builder notation and in interval notation.

$$-6x + 4 < 34$$

Solution:

$$-6x + 4 < 34$$

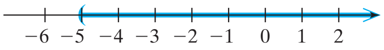
$$-6x + 4 - 4 < 34 - 4 \qquad \text{Subtract 4 from both sides.}$$

$$-6x < 30$$

$$\frac{-6x}{-6} > \frac{30}{-6} \qquad \text{Divide both sides by } -6\text{. Reverse the inequality sign.}$$

$$x > -5$$

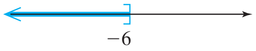
The solution set is $\{x|x > -5\}$.
Interval notation: $(-5, \infty)$



Skill Practice 1 Solve the inequality. Graph the solution set and write the solution set in set-builder notation and in interval notation. $-5t - 6 \ge 24$

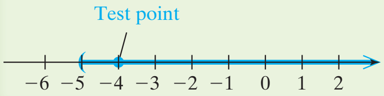
Answer

1.



$\{t \mid t \le -6\}$; $(-\infty, -6]$

TIP In Example 1, the solution set to the inequality $-6x + 4 < 34$ is $\{x \mid x > -5\}$. This means that all numbers greater than -5 make the inequality a true statement. You can check by taking an arbitrary test point from the interval $(-5, \infty)$. For example, the value $x = -4$ makes the original inequality true.



Check: $x = -4$

$$-6(-4) + 4 \overset{?}{<} 34$$

$$24 + 4 \overset{?}{<} 34 \checkmark \quad \text{true}$$

EXAMPLE 2 Solving a Linear Inequality Containing Fractions

Solve the inequality. Graph the solution set and write the solution set in set-builder notation and in interval notation.

$$\frac{x+1}{3} - \frac{2x-4}{6} \le -\frac{x}{2}$$

Solution:

$$\frac{x+1}{3} - \frac{2x-4}{6} \le -\frac{x}{2}$$

$$6\left(\frac{x+1}{3} - \frac{2x-4}{6}\right) \le 6\left(-\frac{x}{2}\right)$$

Multiply both sides by the LCD of 6 to clear fractions.

$$2(x+1) - (2x-4) \le -3x$$

$$2x + 2 - 2x + 4 \le -3x$$

Apply the distributive property.

$$6 \le -3x$$

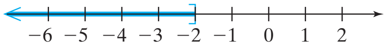
$$\frac{6}{-3} \ge \frac{-3x}{-3}$$

$$-2 \ge x \quad \text{or} \quad x \le -2$$

Divide both sides by -3. Since -3 is a negative number, reverse the inequality sign.

The solution set is $\{x \mid x \le -2\}$.

Interval notation: $(-\infty, -2]$



Skill Practice 2 Solve the inequality. Graph the solution set and write the solution set in set-builder notation and in interval notation.

$$\frac{m-4}{2} - \frac{3m+4}{10} > -\frac{3m}{5}$$

2. Solve Compound Linear Inequalities

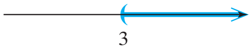
In Examples 3–5, we solve **compound inequalities.** These are statements with two or more inequalities joined by the word "and" or the word "or." For example, suppose that x represents the glucose level measured from a fasting blood sugar test.

- The normal glucose range is given by $x \ge 70$ mg/dL and $x \le 100$ mg/dL.
- An abnormal glucose level is given by $x < 70$ mg/dL or $x > 100$ mg/dL.

To find the solution sets for compound inequalities follow these guidelines.

Answer

2.



$\{m \mid m > 3\}$; $(3, \infty)$

Solving a Compound Inequality

Step 1 To solve a compound inequality, first solve the individual inequalities.

Step 2
- If two inequalities are joined by the word "and," the solutions are the values of the variable that simultaneously satisfy each inequality. That is, we take the *intersection* of the individual solution sets.
- If two inequalities are joined by the word "or," the solutions are the values of the variable that satisfy either inequality. Therefore, we take the *union* of the individual solution sets.

EXAMPLE 3 Solving a Compound Inequality "Or"

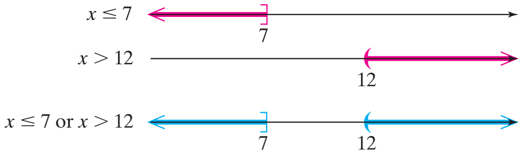
Solve. $x - 2 \leq 5 \quad \text{or} \quad \frac{1}{2}x > 6$

Solution:

$$x - 2 \leq 5 \quad \text{or} \quad \frac{1}{2}x > 6$$

First solve the individual inequalities. Then take the *union* of the individual solution sets.

$$x \leq 7 \quad \text{or} \quad x > 12$$



The solution set is $\{x | x \leq 7 \text{ or } x > 12\}$.

Interval notation: $(-\infty, 7] \cup (12, \infty)$

Skill Practice 3 Solve. $\frac{1}{4}y < -1 \quad \text{or} \quad 3 + y \geq 5$

In Example 4, we solve a compound inequality in which the individual inequalities are joined by the word "and." In this case, we take the intersection of the individual solution sets.

EXAMPLE 4 Solving a Compound Inequality "And"

Solve. $-\frac{1}{4}t < 2 \quad \text{and} \quad 0.52t \geq 1.3$

Solution:

$$-\frac{1}{4}t < 2 \quad \text{and} \quad 0.52t \geq 1.3$$

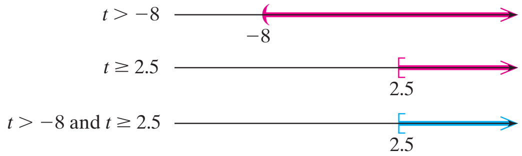
First solve the individual inequalities. Multiply both sides of the first inequality by −4 (reverse the inequality sign). Divide the second inequality by 0.52.

$$t > -8 \quad \text{and} \quad t \geq 2.5$$

Take the *intersection* of the individual solution sets. The intervals overlap for values of t greater than or equal to 2.5.

Answer

3. $\{y | y < -4 \quad \text{or} \quad y \geq 2\}$; $(-\infty, -4) \cup [2, \infty)$



The solution set is $\{t|t \geq 2.5\}$. Interval notation: $[2.5, \infty)$

Skill Practice 4 Solve. $0.36w \leq 0.54$ and $-\frac{1}{2}w > 3$

Sometimes a compound inequality joined by the word "and" is written as a three-part inequality. For example:

$5 < -2x + 7$ and $-2x + 7 \leq 11$

$5 < -2x + 7 \leq 11$

In this example, two simultaneous conditions are imposed on the quantity $-2x + 7$.

To solve a three-part inequality, the goal is to isolate x in the middle region. This is demonstrated in Example 5.

EXAMPLE 5 Solving a Three-Part Compound Inequality

Solve. $5 < -2x + 7 \leq 11$

Solution:

$$5 < -2x + 7 \leq 11$$

$$5 - 7 < -2x + 7 - 7 \leq 11 - 7 \quad \text{Subtract 7 from all three parts of the inequality.}$$

$$-2 < -2x \leq 4$$

$$\frac{-2}{-2} > \frac{-2x}{-2} \geq \frac{4}{-2} \quad \text{Divide all three parts by } -2.$$

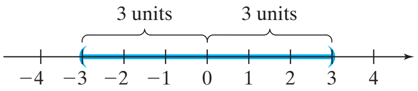
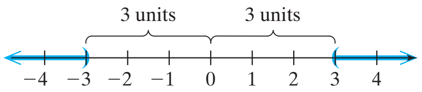
$$1 > x \geq -2 \text{ or equivalently } -2 \leq x < 1.$$

The solution set is $\{x|-2 \leq x < 1\}$.
Interval notation: $[-2, 1)$

Skill Practice 5 Solve. $-16 \leq -3y - 4 < 2$

3. Solve Absolute Value Inequalities

We now investigate the solutions to absolute value inequalities. For example:

Inequality	**Graph**	**Solution Set**
$\|x\| < 3$		$\{x\|-3 < x < 3\}$
$\|x\| > 3$		$\{x\|x < -3 \text{ or } x > 3\}$

We can generalize these observations with the following properties involving absolute value inequalities.

TIP The solution set to $|x| < 3$ is the set of real numbers within 3 units of zero on the number line.

The solution set to $|x| > 3$ is the set of real numbers more than 3 units from zero on the number line.

Answers

4. $\{w\,|\,w < -6\}$; $(-\infty, -6)$
5. $\{y\,|\,-2 < y \leq 4\}$; $(-2, 4]$

Properties Involving Absolute Value Inequalities

For a real number $k > 0$,

1. $|u| < k$ is equivalent to $-k < u < k$. (1)
2. $|u| > k$ is equivalent to $u < -k$ or $u > k$. (2)

Note: The statements also hold true for the inequality symbols $\leq$ and $\geq$, respectively.

Properties (1) and (2) follow directly from the definition: $|u| = \begin{cases} u \text{ if } u \geq 0 \\ -u \text{ if } u < 0 \end{cases}$

By definition, $|u| < k$ is equivalent to

$$u < k \quad \text{and} \quad -u < k$$
$$u < k \quad \text{and} \quad u > -k$$
$$-k < u < k \qquad (1)$$

By definition, $|u| > k$ is equivalent to

$$u > k \quad \text{or} \quad -u > k$$
$$u > k \quad \text{or} \quad u < -k$$
$$u < -k \quad \text{or} \quad u > k \qquad (2)$$

EXAMPLE 6 Solving an Absolute Value Inequality

Solve the inequality and write the solution set in interval notation.

$$2|6 - m| - 3 < 7$$

Solution:

$$2|6 - m| - 3 < 7$$ First isolate the absolute value. Add 3 and divide by 2.

$$|6 - m| < 5$$ The inequality is in the form $|u| < k$, where $u = 6 - m$.

$$-5 < 6 - m < 5$$ Write the equivalent compound inequality $-k < u < k$.

$$-11 < -m < -1$$ Subtract 6 from all three parts.

$$\frac{-11}{-1} > \frac{-m}{-1} > \frac{-1}{-1}$$ Divide by -1 and reverse the inequality signs.

$11 > m > 1$ or equivalently $1 < m < 11$.

The solution set is $\{m \mid 1 < m < 11\}$.
Interval notation: $(1, 11)$

Skill Practice 6 Solve the inequality and write the solution set in interval notation. $3|5 - x| + 2 \leq 14$

EXAMPLE 7 Solving an Absolute Value Inequality

Solve the inequality and write the solution set in interval notation.

$$-4 \geq -2|3x + 1|$$

Answer
6. [1, 9]

Solution:

$-4 \geq -2|3x + 1|$ First isolate the absolute value.

$\frac{-4}{-2} \leq \frac{-2|3x + 1|}{-2}$ Divide both sides by -2 and reverse the inequality sign.

$2 \leq |3x + 1|$

$|3x + 1| \geq 2$ Write the absolute value on the left. Notice that the direction of the inequality sign is also changed. The inequality is now in the form $|u| \geq k$, where $u = 3x + 1$.

$3x + 1 \leq -2$ or $3x + 1 \geq 2$ Write the equivalent form $u \leq -k$ or $u \geq k$.

$x \leq -1$ or $x \geq \frac{1}{3}$ Take the union of the solution sets of the individual inequalities.

The solution set is $\left\{x \,\middle|\, x \leq -1 \quad \text{or} \quad x \geq \frac{1}{3}\right\}$.

Interval notation: $(-\infty, -1] \cup \left[\frac{1}{3}, \infty\right)$

Skill Practice 7 Solve the inequality and write the solution set in interval notation. $-18 > -3|2y - 4|$

An absolute value equation such as $|7x - 3| = -6$ has no solution because an absolute value cannot be equal to a negative number. We must also exercise caution when an absolute value is compared to a negative number or zero within an inequality. This is demonstrated in Example 8.

EXAMPLE 8 Solving Absolute Value Inequalities with Special Case Solution Sets

Solve the inequality and write the solution set in interval notation where appropriate.

a. $|x + 2| < -4$ **b.** $|x + 2| \geq -4$

c. $|x - 5| \leq 0$ **d.** $|x - 5| > 0$

Solution:

a. $|x + 2| < -4$

The solution set is { }.

By definition an absolute value is greater than or equal to zero. Therefore, the absolute value of an expression cannot be less than zero or any negative number. This inequality has no solution.

b. $|x + 2| \geq -4$

The solution set is $\mathbb{R}$.

Interval notation: $(-\infty, \infty)$

An absolute value of any real number is greater than or equal to zero. Therefore, it is also greater than every negative number. This inequality is true for all real numbers, x.

c. $|x - 5| \leq 0$

$|x - 5| = 0$

$x - 5 = 0$

$x = 5$

The solution set is $\{5\}$.

The absolute value of x minus 5 cannot be less than zero, but it can be *equal* to zero.

Answer

7. $(-\infty, -1) \cup (5, \infty)$

d. $|x - 5| > 0$

$|x - 5| > 0$ for all values of x except 5. When $x = 5$, we have $|5 - 5| = 0$, and this is not greater than zero. The solution set is all real numbers excluding 5.

The solution set is $\{x | x < 5 \text{ or } x > 5\}$.
Interval notation: $(-\infty, 5) \cup (5, \infty)$

Skill Practice 8 Solve.

a. $|x - 3| < -2$ **b.** $|x - 3| > -2$

c. $|x + 1| \leq 0$ **d.** $|x + 1| > 0$

4. Solve Applications of Inequalities

In Example 9, we use a linear inequality to solve an application.

EXAMPLE 9 Using a Linear Inequality in an Application of Grades

Emily has test scores of 96, 84, 80, 98, and 88. Her score for online homework is 100 and is weighted as one test grade. Emily still needs to take the final exam, which counts as two test grades. What score does she need on the final exam to have an average of at least 90? (This is the minimum average to earn an "A" in the class.)

Solution:

Let x represent the grade needed on the final exam.

$$\begin{pmatrix}\text{Average of}\\ \text{all scores}\end{pmatrix} \geq 90$$

To earn an "A," Emily's average must be at least 90.

$$\frac{96 + 84 + 80 + 98 + 88 + 100 + 2x}{8} \geq 90$$

Take the sum of all grades. Divide by a total of eight grades.

$$\frac{546 + 2x}{8} \geq 90$$

$$8\left(\frac{546 + 2x}{8}\right) \geq 8(90)$$

Multiply by 8 to clear fractions.

$$546 + 2x \geq 720$$

$$2x \geq 174$$

Subtract 546 from both sides.

$$x \geq 87$$

Divide by 2.

Emily must earn a score of at least 87 to earn an "A" in the class.

Skill Practice 9 For a recent year, the monthly snowfall (in inches) for Chicago, Illinois, for November, December, January, and February was 2, 8.4, 11.2, and 7.9, respectively. How much snow would be necessary in March for Chicago to exceed its monthly average snowfall of 7.28 in. for these five months?

EXAMPLE 10 Applying an Absolute Value Inequality

Suppose that a machine is calibrated to dispense 8 fl oz of orange juice into a plastic bottle, with a measurement error of no more than 0.05 fl oz. Let x represent the actual amount of orange juice poured into the bottle.

Answers

8. a. { } **b.** ℝ; $(-\infty, \infty)$

c. $\{-1\}$ **d.** $(-\infty, -1) \cup (-1, \infty)$

9. Chicago would need more than 6.9 in. of snow in March.

a. Write an absolute value inequality that represents an interval in which to estimate x.

b. Solve the inequality and interpret the answer.

Solution:

a. The measurement error is ±0.05 fl oz. This means that the value of x can deviate from 8 fl oz by as much as ±0.05 fl oz.

$|x - 8| \le 0.05$ The distance between x and 8 is no more than 0.05 unit.

b. $|x - 8| \le 0.05$

$-0.05 \le x - 8 \le 0.05$ The amount of orange juice in the bottle is between 7.95 fl oz and 8.05 fl oz, inclusive.

$7.95 \le x \le 8.05$

Skill Practice 10 A board is to be cut to a length of 24 in. The measurement error is no more than 0.02 in. Let x represent the actual length of the board.

a. Write an absolute value inequality that represents an interval in which to estimate x.

b. Solve the inequality from part (a) and interpret the meaning.

Answers

10. a. $|x - 24| \le 0.02$

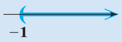
b. $23.98 \le x \le 24.02$; The actual length of the board is between 23.98 in. and 24.02 in., inclusive.

SECTION 1.7 Practice Exercises

Prerequisite Review

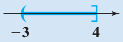
For Exercises R.1–R.3, write the set in set-builder notation (in terms of x) and in interval notation.

R.1.



R.2.

R.3.



For Exercises R.4–R.6, solve the equation.

R.4. $2m + 60 = 84$

R.5. $\frac{4}{5} - \frac{z+5}{20} = \frac{5z+3}{4}$

R.6. $2[3 - (m + 6)] - 18 = 5(5 + m)$

Concept Connections

1. The multiplication and division properties of inequality indicate that if both sides of an inequality are multiplied or divided by a negative real number, the direction of the ___________ sign must be reversed.

2. If a compound inequality consists of two inequalities joined by the word "and," the solution set is the ___________ of the solution sets of the individual inequalities.

3. The compound inequality $a < x$ and $x < b$ can be written as the three-part inequality ___________.

4. If a compound inequality consists of two inequalities joined by the word "or," the solution set is the ___________ of the solution sets of the individual inequalities.

5. If k is a positive real number, then the inequality $|x| < k$ is equivalent to ___________ $< x <$ ___________.

6. If k is a positive real number, then the inequality $|x| > k$ is equivalent to $x <$ ___________ or x ___________ k.

7. If k is a positive real number, then the solution set to the inequality $|x| > -k$ is ___________.

8. If k is a positive real number, then the solution set to the inequality $|x| < -k$ is ___________.

Objective 1: Solve Linear Inequalities in One Variable

For Exercises 9–26, solve the inequality. Graph the solution set, and write the solution set in set-builder notation and interval notation. (See Examples 1–2)

9. $-2x - 5 > 17$

10. $-8t + 1 < 17$

11. $-3 \leq -\frac{4}{3}w + 1$

12. $8 \geq -\frac{5}{2}y - 2$

13. $-1.2 + 0.6a \leq 0.4a + 0.5$

14. $-0.7 + 0.3x \leq 0.9x - 0.4$

15. $-5 > 6(c - 4) + 7$

16. $-14 < 3(m - 7) + 7$

17. $\frac{4 + x}{2} - \frac{x - 3}{5} < -\frac{x}{10}$

18. $\frac{y + 3}{4} - \frac{3y + 1}{6} > -\frac{1}{12}$

19. $\frac{1}{3}(x + 4) - \frac{5}{6}(x - 3) \geq \frac{1}{2}x + 1$

20. $\frac{1}{2}(t - 6) - \frac{4}{3}(t + 2) \geq -\frac{3}{4}t - 2$

21. $5(7 - x) + 2x < 6x - 2 - 9x$

22. $2(3x + 1) - 4x > 2(x + 8) - 5$

23. $5 - 3[2 - 4(x - 2)] \geq 6\{2 - [4 - (x - 3)]\}$

24. $8 - [6 - 10(x - 1)] \geq 2\{1 - 3[2 - (x + 4)]\}$

25. $4 - 3k > -2(k + 3) - k$

26. $2x - 9 < 6(x - 1) - 4x$

Objective 2: Solve Compound Linear Inequalities

For Exercises 27–34, solve the compound inequality. Graph the solution set, and write the solution set in interval notation. (See Examples 3–4)

27. a. $x < 4$ and $x \geq -2$
b. $x < 4$ or $x \geq -2$

28. a. $y \leq -2$ and $y > -5$
b. $y \leq -2$ or $y > -5$

29. a. $m + 1 \leq 6$ or $\frac{1}{3}m < -2$
b. $m + 1 \leq 6$ and $\frac{1}{3}m < -2$

30. a. $n - 6 > 1$ or $\frac{3}{4}n \geq 6$
b. $n - 6 > 1$ and $\frac{3}{4}n \geq 6$

31. a. $-\frac{2}{3}y > -12$ and $2.08 \geq 0.65y$
b. $-\frac{2}{3}y > -12$ or $2.08 \geq 0.65y$

32. a. $-\frac{4}{5}m < 8$ and $0.85 \leq 0.34m$
b. $-\frac{4}{5}m < 8$ or $0.85 \leq 0.34m$

33. a. $3(x - 2) + 2 \leq x - 8$ or $4(x + 1) + 2 > -2x + 4$
b. $3(x - 2) + 2 \leq x - 8$ and $4(x + 1) + 2 > -2x + 4$

34. a. $5(t - 4) + 2 > 3(t + 1) - 3$ or $2t - 6 > 3(t - 4) - 2$
b. $5(t - 4) + 2 > 3(t + 1) - 3$ and $2t - 6 > 3(t - 4) - 2$

35. Write $-2.8 < y \leq 15$ as two separate inequalities joined by "and."

36. Write $-\frac{1}{2} \leq z < 2.4$ as two separate inequalities joined by "and."

For Exercises 37–42, solve the compound inequality. Graph the solution set, and write the solution set in interval notation. (See Example 5)

37. $-3 < -2x + 1 \leq 9$

38. $-6 \leq -3x + 9 < 0$

39. $1 \leq \frac{5x - 4}{2} < 3$

40. $-2 \leq \frac{4x - 1}{3} \leq 5$

41. $-2 \leq \frac{-2x + 1}{-3} \leq 4$

42. $-4 < \frac{-5x - 2}{-2} < 4$

Objective 3: Solve Absolute Value Inequalities

For Exercises 43–46, solve the equation or inequality. Write the solution set to each inequality in interval notation.

43. a. $|x| = 7$
b. $|x| < 7$
c. $|x| > 7$

44. a. $|y| = 8$
b. $|y| < 8$
c. $|y| > 8$

45. a. $|a + 9| + 2 = 6$
b. $|a + 9| + 2 \leq 6$
c. $|a + 9| + 2 \geq 6$

46. a. $|b + 1| - 4 = 1$
b. $|b + 1| - 4 \leq 1$
c. $|b + 1| - 4 \geq 1$

For Exercises 47–60, solve the inequality, and write the solution set in interval notation if possible. (See Examples 6–7)

47. $3|4 - x| - 2 < 16$

48. $2|7 - y| + 1 < 17$

49. $2|x + 3| - 4 \geq 6$

50. $5|x + 1| - 9 \geq -4$

51. $|4w - 5| + 6 \leq 2$

52. $|2x + 7| + 5 < 1$

53. $|5 - p| + 13 > 6$

54. $|12 - 7x| + 5 \geq 4$

55. $-11 \leq 5 - |2p + 4|$

56. $-18 \leq 6 - |3z + 3|$

57. $10 < |-5c - 4| + 2$

58. $15 < |-2d - 3| + 6$

59. $\left|\dfrac{y + 3}{6}\right| < 2$

60. $\left|\dfrac{m - 4}{2}\right| < 14$

For Exercises 61–68, write the solution set. (See Example 8)

61. **a.** $|x| = -9$
- **b.** $|x| < -9$
- **c.** $|x| > -9$

62. **a.** $|y| = -2$
- **b.** $|y| < -2$
- **c.** $|y| > -2$

63. **a.** $18 = 4 - |y + 7|$
- **b.** $18 \leq 4 - |y + 7|$
- **c.** $18 \geq 4 - |y + 7|$

64. **a.** $15 = 2 - |p - 3|$
- **b.** $15 \leq 2 - |p - 3|$
- **c.** $15 \geq 2 - |p - 3|$

65. **a.** $|z| = 0$
- **b.** $|z| < 0$
- **c.** $|z| \leq 0$
- **d.** $|z| > 0$
- **e.** $|z| \geq 0$

66. **a.** $|2w| = 0$
- **b.** $|2w| < 0$
- **c.** $|2w| \leq 0$
- **d.** $|2w| > 0$
- **e.** $|2w| \geq 0$

67. **a.** $|k + 4| = 0$
- **b.** $|k + 4| < 0$
- **c.** $|k + 4| \leq 0$
- **d.** $|k + 4| > 0$
- **e.** $|k + 4| \geq 0$

68. **a.** $|c - 3| = 0$
- **b.** $|c - 3| < 0$
- **c.** $|c - 3| \leq 0$
- **d.** $|c - 3| > 0$
- **e.** $|c - 3| \geq 0$

Objective 4: Solve Applications of Inequalities

For Exercises 69–72, write a three-part inequality to represent the given statement.

69. The normal range for the hemoglobin level x for an adult female is greater than or equal to 12.0 g/dL and less than or equal to 15.2 g/dL.

70. A tennis player must play in the "open" division of a tennis tournament if the player's age a is over 18 yr and under 25 yr.

71. The distance d that Zina hits a 9-iron is at least 90 yd, but no more than 110 yd.

72. A small plane's average speed s is at least 220 mph but not more than 410 mph.

73. Marilee wants to earn an "A" in a class and needs an overall average of at least 92. Her test grades are 88, 92, 100, and 80. The average of her quizzes is 90 and counts as one test grade. The final exam counts as 2.5 test grades. What scores on the final exam would result in Marilee's overall average of 92 or greater? **(See Example 9)**

74. A 10-yr-old competes in gymnastics. For several competitions she received the following "All-Around" scores: 36, 36.9, 37.1, and 37.4. Her coach recommends that gymnasts whose "All-Around" scores average at least 37 move up to the next level. What "All-Around" scores in the next competition would result in the child being eligible to move up?

75. Rita earns scores of 78, 82, 90, 80, and 75 on her five chapter tests for a certain class and a grade of 85 on the class project. The overall average for the course is computed as follows: the average of the five chapter tests makes up 60% of the course grade; the project accounts for 10% of the grade; and the final exam accounts for 30%. What scores can Rita earn on the final exam to earn a "B" in the course if the cut-off for a "B" is an overall score greater than or equal to 80, but less than 90? Assume that 100 is the highest score that can be earned on the final exam and that only whole-number scores are given.

76. Trent earns scores of 66, 84, and 72 on three chapter tests for a certain class. His homework grade is 60 and his grade for a class project is 85. The overall average for the course is computed as follows: the average of the three chapter tests makes up 50% of the course grade; homework accounts for 20% of the grade; the project accounts for 10%; and the final exam accounts for 20%. What scores can Trent earn on the final exam to pass the course if he needs a "C" or better? A "C" or better requires an overall score of 70 or better, and 100 is the highest score that can be earned on the final exam. Assume that only whole-number scores are given.

77. A car travels 50 mph and passes a truck traveling 40 mph. How long will it take the car to be more than 16 mi ahead?

78. A work-study job in the library pays \$10.75/hr and a job in the tutoring center pays \$16.25/hr. How long would it take for a tutor to make over \$500 more than a student working in the library? Round to the nearest hour.

79. A rectangular garden is to be constructed so that the width is 100 ft. What are the possible values for the length of the garden if at most 800 ft of fencing is to be used?

81. For a certain bowling league, a beginning bowler computes her handicap by taking 90% of the difference between 220 and her average score in league play. Determine the average scores that would produce a handicap of 72 or less. Also assume that a negative handicap is not possible in this league.

83. Donovan has offers for two sales jobs. Job A pays a base salary of $25,000 plus a 10% commission on sales. Job B pays a base salary of $30,000 plus 8% commission on sales.

a. How much would Donovan have to sell for the salary from Job A to exceed the salary from Job B?

b. If Donovan routinely sells more than $500,000 in merchandise, which job would result in a higher salary?

80. The lengths of the sides of a triangle are given by three consecutive integers greater than 1. What are the possible values for the shortest side if the perimeter is not to exceed 24 ft?

82. Body temperature is usually maintained between 36.5°C and 37.5°C, inclusive. Determine the corresponding range of temperatures in Fahrenheit. Use the relationship between degrees Celsius C and degrees Fahrenheit F: $C = \frac{5}{9}(F - 32)$.

84. Nancy wants to vacation in Austin, Texas. Hotel A charges $179 per night with a 14% nightly room tax and free parking. Hotel B charges $169 per night with an 18% nightly room tax plus a one-time $40 parking fee. After how many nights will Hotel B be less expensive?

For Exercises 85–88,

a. Write an absolute value inequality to represent each statement.

b. Solve the inequality. Write the solution set in interval notation.

85. The variation between the measured value v and 16 oz is less than 0.01 oz.

87. The value of x differs from 4 by more than 1 unit.

89. A refrigerator manufacturer recommends that the temperature t (in °F) inside a refrigerator be 36.5°F. If the thermostat has a margin of error of no more than 1.5°F,

a. Write an absolute value inequality that represents an interval in which to estimate t. **(See Example 10)**

b. Solve the inequality and interpret the answer.

91. The results of a political poll indicate that the leading candidate will receive 51% of the votes with a margin of error of no more than 3%. Let x represent the true percentage of votes received by this candidate.

a. Write an absolute value inequality that represents an interval in which to estimate x.

b. Solve the inequality and interpret the answer.

86. The variation between the measured value t and 60 min is less than 0.2 min.

88. The value of y differs from 10 by more than 2 units.

90. A box of cereal is labeled to contain 16 oz. A consumer group takes a sample of 50 boxes and measures the contents of each box. The individual content of each box differs slightly from 16 oz, but by no more than 0.5 oz.

a. If x represents the exact weight of the contents of a box of cereal, write an absolute value inequality that represents an interval in which to estimate x.

b. Solve the inequality and interpret the answer.

92. A police officer uses a radar detector to determine that a motorist is traveling 34 mph in a 25-mph school zone. The driver goes to court and argues that the radar detector is not accurate. The manufacturer claims that the radar detector is calibrated to be in error by no more than 3 mph.

a. If x represents the motorist's actual speed, write an inequality that represents an interval in which to estimate x.

b. Solve the inequality and interpret the answer. Should the motorist receive a ticket?

Mixed Exercises

For Exercises 93–98, determine the set of values for x for which the radical expression would produce a real number. For example, the expression $\sqrt{x-1}$ is a real number if $x - 1 \geq 0$ or equivalently, $x \geq 1$.

93. a. $\sqrt{x-2}$ **b.** $\sqrt{2-x}$ **94. a.** $\sqrt{x-6}$ **b.** $\sqrt{6-x}$

95. a. $\sqrt{x+4}$ **b.** $\sqrt[3]{x+4}$ **96. a.** $\sqrt{x+7}$ **b.** $\sqrt[3]{x+7}$

97. a. $\sqrt{2x-9}$ **b.** $\sqrt[4]{2x-9}$ **98. a.** $\sqrt{3x-7}$ **b.** $\sqrt[4]{3x-7}$

For Exercises 99–102, answer true or false given that $a > 0$, $b < 0$, $c > 0$, and $d < 0$.

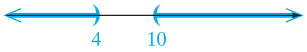
99. $cd > a$ **100.** $ab < c$ **101.** If $a > c$, then $ad < cd$. **102.** If $a < c$, then $ab < bc$.

For Exercises 103–106, write an absolute value inequality whose solution set is shown in the graph.

103.

104.

105.



106.

Write About It

107. How is the process to solve a linear inequality different from the process to solve a linear equation?

109. Explain the difference between the solution sets for the following inequalities:

$$|x - 3| \leq 0 \quad \text{and} \quad |x - 3| > 0$$

108. Explain why $8 < x < 2$ has no solution.

110. Explain why $x^2 = 4$ is equivalent to the equation $|x| = 2$.

Expanding Your Skills

For Exercises 111–116, solve the inequality and write the solution set in interval notation.

111. $|x| + x < 11$ (*Hint*: Use the definition of $|x|$ to consider two cases.)
Case 1: $x + x < 11$ if $x \geq 0$.
Case 2: $-x + x < 11$ if $x < 0$.

112. $|x| - x > 10$

113. $1 < |x| < 9$

114. $2 < |y| < 11$

115. $5 \leq |2x + 1| \leq 7$

116. $7 \leq |3x - 5| \leq 13$

117. Solve the inequality for p: $|p - \hat{p}| < z\sqrt{\dfrac{\hat{p}\hat{q}}{n}}$. (Do not rationalize the denominator.)

118. Solve the inequality for μ: $|\mu - \bar{x}| < \dfrac{z\sigma}{\sqrt{n}}$. (Do not rationalize the denominator.)

SECTION 1.4 Quadratic Equations

OBJECTIVES

1. Solve Quadratic Equations by Using the Zero Product Property
2. Solve Quadratic Equations by Using the Square Root Property
3. Complete the Square
4. Solve Quadratic Equations by Using the Quadratic Formula
5. Use the Discriminant
6. Solve an Equation for a Specified Variable

1. Solve Quadratic Equations by Using the Zero Product Property

A linear equation in one variable is an equation of the form $ax + b = 0$, where $a \neq 0$. A linear equation is also called a first-degree equation. We now turn our attention to a quadratic equation. This is identified as a second-degree equation.

Definition of a Quadratic Equation

Let a, b, and c represent real numbers, where $a \neq 0$. A **quadratic equation** in the variable x is an equation of the form

$$ax^2 + bx + c = 0.$$

To solve a quadratic equation, we make use of the zero product property.

Zero Product Property

If $mn = 0$, then $m = 0$ or $n = 0$.

To solve a quadratic equation using the zero product property, set one side of the equation equal to zero and factor the other side.

EXAMPLE 1 Applying the Zero Product Property

Solve the equations. **a.** $x^2 - 8x = 0$ **b.** $2x(2x - 7) = -12$

Solution:

a.

$$\begin{aligned} x^2 - 8x &= 0 && \text{One side of the equation is already zero.}\\ x(x - 8) &= 0 && \text{Factor the other side.}\\ x = 0 \text{ or } x - 8 &= 0 && \text{Set each factor equal to zero.}\\ x &= 8 \end{aligned}$$

Check: $x^2 - 8x = 0$

$(0)^2 - 8(0) \stackrel{?}{=} 0$ $\qquad$ $(8)^2 - 8(8) \stackrel{?}{=} 0$

$0 - 0 = 0$ ✓ $\qquad$ $64 - 64 = 0$ ✓

The solution set is $\{0, 8\}$.

b.

$$\begin{aligned} 2x(2x - 7) &= -12 \\ 4x^2 - 14x &= -12 && \text{Apply the distributive property.}\\ 4x^2 - 14x + 12 &= 0 && \text{Set one side to zero.}\\ 2(2x^2 - 7x + 6) &= 0 && \text{Factor.}\\ 2(x - 2)(2x - 3) &= 0 \\ 2 = 0 \quad \text{or} \quad x - 2 = 0 \quad &\text{or} \quad 2x - 3 = 0 && \text{Set each factor equal to zero.}\\ \uparrow \qquad \qquad & \\ \text{contradiction} \qquad x = 2 \qquad & \qquad x = \frac{3}{2} && \text{Both solutions check.} \end{aligned}$$

The solution set is $\left\{2, \frac{3}{2}\right\}$.

Skill Practice 1 Solve the equations.

a. $y^2 + 3y = 0$ **b.** $10(3x^2 - 13x) = -40$

TIP After applying the zero product property in Example 1(b), we have three equations. The first equation does not contain the variable *x*. It is a contradiction and does not yield a solution for *x*.

2. Solve Quadratic Equations by Using the Square Root Property

The zero product property can be used to solve equations of the form $x^2 = k$.

$$\begin{aligned} x^2 &= k \\ x^2 - k &= 0 \\ (x - \sqrt{k})(x + \sqrt{k}) &= 0 \\ x - \sqrt{k} = 0 \quad \text{or} \quad x + \sqrt{k} &= 0 \\ x = \sqrt{k} \qquad \qquad x &= -\sqrt{k} \end{aligned}$$

The solutions can also be written as $x = \pm\sqrt{k}$. This is read as "*x* equals plus or minus the square root of *k*."

The solution set is $\{\pm\sqrt{k}\}$.

This result is formalized as the square root property.

Square Root Property

If $x^2 = k$, then $x = \pm\sqrt{k}$.

The solution set is $\{\sqrt{k}, -\sqrt{k}\}$ or more concisely $\{\pm\sqrt{k}\}$.

To apply the square root property to solve a quadratic equation, first isolate the square term on one side and the constant term on the other side.

Answers

1. a. $\{-3, 0\}$ **b.** $\left\{\frac{1}{3}, 4\right\}$

EXAMPLE 2 Applying the Square Root Property

Solve the equations by using the square root property.

a. $x^2 = 64$ **b.** $2y^2 + 36 = 0$ **c.** $(w + 3)^2 = 8$

Solution:

a. $x^2 = 64$

$x = \pm\sqrt{64}$ Apply the square root property.

$x = \pm 8$ Both solutions check in the original equation.

The solution set is $\{\pm 8\}$.

b. $2y^2 + 36 = 0$

$2y^2 = -36$ Isolate the square term.

$y^2 = -18$ Write the equation in the form $y^2 = k$.

$y = \pm\sqrt{-18}$ Apply the square root property.

$y = \pm 3i\sqrt{2}$ Simplify the radical.

$\sqrt{-18} = i\sqrt{18} = i\sqrt{3^2 \cdot 2} = 3i\sqrt{2}$

The solution set is $\{\pm 3i\sqrt{2}\}$. Both solutions check in the original equation.

TIP The solutions to the equation in Example 2(b) are written concisely as $\pm 3i\sqrt{2}$. Do not forget that this actually represents two solutions:

$y = 3i\sqrt{2}$ and
$y = -3i\sqrt{2}$

c. $(w + 3)^2 = 8$

$w + 3 = \pm\sqrt{8}$ Apply the square root property.

$w = -3 \pm \sqrt{8}$ Subtract 3 from both sides to isolate w.

$w = -3 \pm 2\sqrt{2}$ Simplify the radical. $\sqrt{8} = \sqrt{2^3} = 2\sqrt{2}$

The solution set is $\{-3 \pm 2\sqrt{2}\}$. Both solutions check.

Skill Practice 2 Solve the equations by using the square root property.

a. $a^2 = 49$ **b.** $2c^2 + 80 = 0$ **c.** $(t + 4)^2 = 24$

Point of Interest

Unfortunate names? In the long history of mathematics, number systems have been expanded to accommodate meaningful solutions to equations. But the negative connotation of their names may suggest a reluctance by early mathematicians to accept these new concepts. Negative numbers for example are not unpleasant or disagreeable. Irrational numbers are not illogical or absurd, and imaginary numbers are not "fake." Instead these sets of numbers are necessary to render solutions to such equations as

$$2x + 10 = 0 \qquad x^2 - 5 = 0 \qquad x^2 + 4 = 0$$

3. Complete the Square

In Example 2(c), the left side of the equation is the square of a binomial and the right side is a constant. We can manipulate a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ to write it as the square of a binomial equal to a constant. First look at the relationship between a perfect square trinomial and its factored form.

Perfect Square Trinomial		Factored Form
$x^2 + 10x + 25$	$\longrightarrow$	$(x + 5)^2$
$t^2 - 6t + 9$	$\longrightarrow$	$(t - 3)^2$
$p^2 - 14p + 49$	$\longrightarrow$	$(p - 7)^2$

Answers

2. a. $\{\pm 7\}$ **b.** $\{\pm 2i\sqrt{10}\}$
c. $\{-4 \pm 2\sqrt{6}\}$

For a perfect square trinomial with a leading coefficient of 1, the constant term is the square of one-half the linear term coefficient. For example:

$$x^2 + 10x + \overset{\uparrow}{25}$$
$$\left[\tfrac{1}{2}(10)\right]^2$$

In general, an expression of the form $x^2 + bx + n$ is a perfect square trinomial if $n = \left(\frac{1}{2}b\right)^2$. The process to create a perfect square trinomial is called **completing the square.**

EXAMPLE 3 Completing the Square

Determine the value of n that makes the polynomial a perfect square trinomial. Then factor the result.

Solution:

Expression	Value of n	Complete the square
a. $x^2 + 18x + n$	$\left[\frac{1}{2}(18)\right]^2 = (9)^2 = 81$	$x^2 + 18x + 81 = (x + 9)^2$
b. $x^2 - 13x + n$	$\left[\frac{1}{2}(-13)\right]^2 = \left(-\frac{13}{2}\right)^2 = \frac{169}{4}$	$x^2 - 13x + \frac{169}{4} = \left(x - \frac{13}{2}\right)^2$
c. $x^2 + \frac{4}{7}x + n$	$\left[\frac{1}{2}\left(\frac{4}{7}\right)\right]^2 = \left(\frac{2}{7}\right)^2 = \frac{4}{49}$	$x^2 + \frac{4}{7}x + \frac{4}{49} = \left(x + \frac{2}{7}\right)^2$

Skill Practice 3 Determine the value of n that makes the polynomial a perfect square trinomial. Then factor the result.

a. $x^2 + 12x + n$ **b.** $x^2 + 5x + n$ **c.** $x^2 - \frac{2}{3}x + n$

TIP When factoring a perfect square trinomial, the constant term in the binomial will always be one-half the x term coefficient.

$$x^2 + 18x + 81$$
$$= (x + 9)^2$$

Note: $9 = \frac{1}{2}(18)$

We can solve a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ by completing the square and then applying the square root property.

Solving a Quadratic Equation $ax^2 + bx + c = 0$ by Completing the Square and Applying the Square Root Property

Step 1 Divide both sides by a to make the leading coefficient 1.

Step 2 Isolate the variable terms on one side of the equation.

Step 3 Complete the square.
- Add the square of one-half the linear term coefficient to both sides.
- Factor the resulting perfect square trinomial.

Step 4 Apply the square root property and solve for x.

Answers

3. a. $n = 36$; $(x + 6)^2$
b. $n = \frac{25}{4}$; $\left(x + \frac{5}{2}\right)^2$
c. $n = \frac{1}{9}$; $\left(x - \frac{1}{3}\right)^2$

EXAMPLE 4 Completing the Square and Solving a Quadratic Equation

Solve the equation by completing the square and applying the square root property. $x^2 - 3 = -10x$

Solution:

$x^2 - 3 = -10x$	Write the equation in the form $ax^2 + bx + c = 0$.
$x^2 + 10x - 3 = 0$	**Step 1:** Notice that the leading coefficient is already 1.
$x^2 + 10x + \underline{\quad} = 3 + \underline{\quad}$	**Step 2:** Add 3 to both sides to isolate the variable terms.
$x^2 + 10x + 25 = 3 + 25$	**Step 3:** Add $\left[\frac{1}{2}(10)\right]^2 = [5]^2 = 25$ to both sides.
$(x + 5)^2 = 28$	Factor.
$x + 5 = \pm\sqrt{28}$	**Step 4:** Apply the square root property and solve for x.
$x = -5 \pm 2\sqrt{7}$	Both solutions check in the original equation.
$\{-5 \pm 2\sqrt{7}\}$	Write the solution set.

Skill Practice 4 Solve the equation by completing the square and applying the square root property. $x^2 - 2 = 8x$

In Example 5, we encounter a quadratic equation in which the leading coefficient is not 1. The first step is to divide both sides by the leading coefficient.

EXAMPLE 5 Completing the Square and Solving a Quadratic Equation

Solve the equation by completing the square and applying the square root property. $-2x^2 - 3x - 5 = 0$

Solution:

$-2x^2 - 3x - 5 = 0$	The equation is in the form $ax^2 + bx + c = 0$.
$\frac{-2x^2}{-2} - \frac{3x}{-2} - \frac{5}{-2} = \frac{0}{-2}$	**Step 1:** Divide by the leading coefficient, -2.
$x^2 + \frac{3}{2}x + \frac{5}{2} = 0$	The new leading coefficient is 1.
$x^2 + \frac{3}{2}x + \underline{\quad} = -\frac{5}{2} + \underline{\quad}$	**Step 2:** Subtract $\frac{5}{2}$ from both sides to isolate the variable terms.
$x^2 + \frac{3}{2}x + \frac{9}{16} = -\frac{5}{2} + \frac{9}{16}$	**Step 3:** Add $\left[\frac{1}{2}\left(\frac{3}{2}\right)\right]^2 = \left[\frac{3}{4}\right]^2 = \frac{9}{16}$ to both sides.
$\left(x + \frac{3}{4}\right)^2 = -\frac{40}{16} + \frac{9}{16}$	Factor.
$\left(x + \frac{3}{4}\right)^2 = -\frac{31}{16}$	

Answer

4. $\{4 \pm 3\sqrt{2}\}$

$$x + \frac{3}{4} = \pm\sqrt{-\frac{31}{16}}$$

$$x = -\frac{3}{4} \pm \frac{\sqrt{31}}{4}i$$

$$\left\{-\frac{3}{4} \pm \frac{\sqrt{31}}{4}i\right\}$$

Step 4: Apply the square property and solve for x.

$$\sqrt{-\frac{31}{16}} = i\sqrt{\frac{31}{16}} = i\frac{\sqrt{31}}{\sqrt{16}} = \frac{\sqrt{31}}{4}i$$

The solutions both check in the original equation.

Skill Practice 5 Solve the equation by completing the square and applying the square root property. $-3x^2 + 5x - 7 = 0$

4. Solve Quadratic Equations by Using the Quadratic Formula

If we solve a general quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ by completing the square and using the square root property, the result is a formula that gives the solutions for x in terms of a, b, and c.

$$ax^2 + bx + c = 0$$

Begin with a quadratic equation in standard form with $a > 0$.

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

Divide by the leading coefficient.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Isolate the terms containing x.

$$x^2 + \frac{b}{a}x + \left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 = \left(\frac{1}{2} \cdot \frac{b}{a}\right)^2 - \frac{c}{a}$$

Add the square of $\frac{1}{2}$ the linear term coefficient to both sides of the equation.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Factor the left side as a perfect square.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Combine fractions on the right side by finding a common denominator.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Apply the square root property.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Simplify the denominator.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides.

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Combine fractions.

The result is called the quadratic formula.

The Quadratic Formula

For a quadratic equation of the form $ax^2 + bx + c = 0$ $(a \neq 0)$, the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Answer

5. $\left\{\frac{5}{6} \pm \frac{\sqrt{59}}{6}i\right\}$

TIP When applying the quadratic formula, note that *a*, *b*, and *c* are constants. The variable is *x*.

EXAMPLE 6 Using the Quadratic Formula

Solve the equation by applying the quadratic formula. $x(x - 6) = 3$

Solution:

$$x(x - 6) = 3$$
$$x^2 - 6x - 3 = 0$$

Write the equation in the form $ax^2 + bx + c = 0$.

$$a = 1,\ b = -6,\ c = -3$$

Identify the values of a, b, and c.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-3)}}{2(1)}$$

Apply the quadratic formula. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{6 \pm \sqrt{48}}{2}$$

Simplify.

$$= \frac{6 \pm 4\sqrt{3}}{2}$$

Simplify the radical. $\sqrt{48} = \sqrt{2^4 \cdot 3} = 2^2\sqrt{3} = 4\sqrt{3}$

$$= \frac{\cancel{2}(3 \pm 2\sqrt{3})}{\cancel{2}}$$

Factor the numerator.

$$= 3 \pm 2\sqrt{3}$$

Simplify the fraction.

The solution set is $\{3 \pm 2\sqrt{3}\}$.

The solutions both check in the original equation.

Skill Practice 6 Solve the equation by applying the quadratic formula. $x(x - 8) = 3$

If a quadratic equation has fractional or decimal coefficients, we have the option of clearing fractions or decimals to create integer coefficients. This makes the application of the quadratic formula easier, as demonstrated in Example 7.

EXAMPLE 7 Using the Quadratic Formula

Solve the equation by applying the quadratic formula. $\frac{3}{10}x^2 - \frac{2}{5}x + \frac{7}{10} = 0$

Solution:

$$\frac{3}{10}x^2 - \frac{2}{5}x + \frac{7}{10} = 0$$

The equation is in the form $ax^2 + bx + c = 0$.

$$10\left(\frac{3}{10}x^2 - \frac{2}{5}x + \frac{7}{10}\right) = 10(0)$$

Multiply by 10 to clear fractions.

$$3x^2 - 4x + 7 = 0$$

$$a = 3,\ b = -4,\ c = 7$$

Identify the values of a, b, and c.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(7)}}{2(3)}$$

Apply the quadratic formula. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{4 \pm \sqrt{-68}}{6}$$

Simplify.

$$= \frac{4 \pm 2i\sqrt{17}}{6}$$

Simplify the radical.

Answer

6. $\{4 \pm \sqrt{19}\}$

$$= \frac{\cancel{2}(2 \pm i\sqrt{17})}{\cancel{2} \cdot 3}$$ Factor the numerator and denominator.

$$= \frac{2 \pm i\sqrt{17}}{3}$$ Simplify the fraction.

$$= \frac{2}{3} \pm \frac{\sqrt{17}}{3}i$$ Write the solutions in standard form, $a + bi$.

The solution set is $\left\{\frac{2}{3} \pm \frac{\sqrt{17}}{3}i\right\}$. The solutions both check in the original equation.

Skill Practice 7 Solve the equation by applying the quadratic formula.

$$\frac{5}{12}x^2 - \frac{1}{2}x + \frac{1}{4} = 0$$

Three methods have been presented to solve a quadratic equation. We offer these guidelines to choose an appropriate and efficient method to solve a given quadratic equation.

Methods to Solve a Quadratic Equation

Method/Notes	Examples
Apply the Zero Product Property • Set one side of the equation equal to zero and factor the other side. Then apply the zero product property.	Solve. $x^2 - x = 12$ $x^2 - x - 12 = 0$ $(x - 4)(x + 3) = 0$ $x = 4$ or $x = -3$
Complete the Square and Apply the Square Root Property • Good choice if the equation is in the form $x^2 = k$. • Good choice if the equation is in the form $ax^2 + bx + c = 0$, where $a = 1$ and b is an even real number.	Solve. $c^2 = -6$ $c = \pm\sqrt{-6}$ $c = \pm i\sqrt{6}$ Solve. $x^2 + 6x + 2 = 0$ $x^2 + 6x + 9 = -2 + 9$ $(x + 3)^2 = 7$ $x + 3 = \pm\sqrt{7}$ $x = -3 \pm \sqrt{7}$
Apply the Quadratic Formula • Applies in all situations. • Consider clearing fractions or decimals if the coefficients are not integer values.	Solve. $0.2x^2 + 0.5x + 0.1 = 0$ $10(0.2x^2 + 0.5x + 0.1) = 10(0)$ $2x^2 + 5x + 1 = 0$ $x = \frac{-(5) \pm \sqrt{(5)^2 - 4(2)(1)}}{2(2)}$ $x = \frac{-5 \pm \sqrt{17}}{4}$

5. Use the Discriminant

The solutions to a quadratic equation are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The radicand, $b^2 - 4ac$, is called the *discriminant*. The value of the discriminant tells us the number and type of solutions to the equation. We examine three different cases.

Answer

7. $\left\{\frac{3}{5} \pm \frac{\sqrt{6}}{5}i\right\}$

Using the Discriminant to Determine the Number and Type of Solutions to a Quadratic Equation

Given a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$, the quantity $b^2 - 4ac$ is called the **discriminant.**

Discriminant $b^2 - 4ac$	**Number and Type of Solutions**	**Examples**	**Result of Quadratic Formula**
$b^2 - 4ac < 0$	2 nonreal solutions	$2x^2 - 3x + 5 = 0$ $b^2 - 4ac = (-3)^2 - 4(2)(5)$ $= -31$	$x = \dfrac{3 \pm \sqrt{-31}}{4}$
$b^2 - 4ac = 0$	1 real solution	$x^2 + 6x + 9 = 0$ $b^2 - 4ac = (6)^2 - 4(1)(9)$ $= 0$	$x = \dfrac{-6 \pm \sqrt{0}}{2} = -3$
$b^2 - 4ac > 0$	2 real solutions	$2x^2 + 7x - 1 = 0$ $b^2 - 4ac = (7)^2 - 4(2)(-1)$ $= 57$	$x = \dfrac{-7 \pm \sqrt{57}}{4}$

EXAMPLE 8 Using the Discriminant

Use the discriminant to determine the number and type of solutions for each equation.

a. $5x^2 - 3x + 1 = 0$ **b.** $2x^2 = 3 - 6x$ **c.** $4x^2 + 12x = -9$

Solution:

Equation	**$b^2 - 4ac$**	**Solution Type and Number**
a. $5x^2 - 3x + 1 = 0$	$(-3)^2 - 4(5)(1)$ $= -11$	Because $-11 < 0$, there are two nonreal solutions.
b. $2x^2 = 3 - 6x$		
$2x^2 + 6x - 3 = 0$	$(6)^2 - 4(2)(-3)$ $= 60$	Because $60 > 0$, there are two real solutions.
c. $4x^2 + 12x = -9$		
$4x^2 + 12x + 9 = 0$	$(12)^2 - 4(4)(9)$ $= 0$	Because the discriminant is 0, there is one real solution.

Skill Practice 8 Use the discriminant to determine the number and type of solutions for each equation.

a. $2x^2 - 4x + 5 = 0$ **b.** $25x^2 = 10x - 1$ **c.** $x^2 + 10x = -9$

Answers

8. a. Discriminant: -24 (2 nonreal solutions)
b. Discriminant: 0 (1 real solution)
c. Discriminant: 64 (2 real solutions)

6. Solve an Equation for a Specified Variable

In Examples 9 and 10, we manipulate literal equations to solve for a specified variable.

EXAMPLE 9 Solving an Equation for a Specified Variable

Solve for r. $V = \frac{1}{3}\pi r^2 h \quad (r > 0)$

Solution:

$$V = \frac{1}{3}\pi r^2 h$$

This equation is quadratic in the variable r. The strategy in this example is to isolate r^2 and then apply the square root property.

$$3(V) = 3\left(\frac{1}{3}\pi r^2 h\right)$$

Multiply both sides by 3 to clear fractions.

$$3V = \pi r^2 h$$

$$\frac{3V}{\pi h} = \frac{\pi r^2 h}{\pi h}$$

Divide both sides by πh to isolate r^2.

$$\frac{3V}{\pi h} = r^2$$

$$r = \sqrt{\frac{3V}{\pi h}} \text{ or } r = \frac{\sqrt{3V\pi h}}{\pi h}$$

Apply the square root property. Since $r > 0$, we take the positive square root only.

TIP The equation $V = \frac{1}{3}\pi r^2 h$ is linear in the variables V and h, and quadratic in the variable r.

TIP The formula $V = \frac{1}{3}\pi r^2 h$ gives the volume of a right circular cone with radius r. Therefore, $r > 0$.

Skill Practice 9 Solve for v. $E = \frac{1}{2}mv^2 \quad (v > 0)$

EXAMPLE 10 Solving an Equation for a Specified Variable

Solve for t. $mt^2 + nt = z$

Solution:

This equation is quadratic in the variable t. The strategy is to write the polynomial in descending order by powers of t. Then since there are two t terms with different exponents, we cannot isolate t directly. Instead we apply the quadratic formula.

$$mt^2 + nt = z$$

$$mt^2 + nt - z = 0$$

Write the polynomial in descending order by t.

$$a = m, b = n, c = -z$$

Identify the coefficients of each term.

$$t = \frac{-(n) \pm \sqrt{(n)^2 - 4(m)(-z)}}{2m}$$

Apply the quadratic formula.

$$t = \frac{-n \pm \sqrt{n^2 + 4mz}}{2m}$$

Simplify.

Avoiding Mistakes

In the equation $mt^2 + nt - z = 0$, t is the variable, and m, n, and z are the coefficients.

Skill Practice 10 Solve for p. $cp^2 - dp = k$

Answers

9. $v = \sqrt{\frac{2E}{m}}$ or $v = \frac{\sqrt{2Em}}{m}$

10. $p = \frac{d \pm \sqrt{d^2 + 4ck}}{2c}$

SECTION 1.4 Practice Exercises

Prerequisite Review

R.1. a. Multiply the binomials. $(x - 4)(x - 4)$

b. Factor $x^2 - 8x + 16$.

For Exercises R.2–R.4, factor completely.

R.2. $5x^2 + 17xy - 12y^2$

R.3. $4p^3 - 48p^2q + 144pq^2$

R.4. $16x^2 + 40x + 25$

R.5. Simplify the expression. $\dfrac{12 - 4\sqrt{5}}{8}$

Concept Connections

1. A _______________ equation is a second-degree equation of the form $ax^2 + bx + c = 0$ where $a \neq 0$.

2. A _______________ equation is a first-degree equation of the form $ax + b = 0$ where $a \neq 0$.

3. The square root property indicates that if $x^2 = k$, then $x =$ _______________.

4. The value of n that would make the trinomial $x^2 + 20x + n$ a perfect square trinomial is _______________.

5. Given $ax^2 + bx + c = 0$ $(a \neq 0)$, write the quadratic formula.

6. For a quadratic equation $ax^2 + bx + c = 0$, the discriminant is given by the expression _______________.

Objective 1: Solve Quadratic Equations by Using the Zero Product Property

For Exercises 7–18, solve by applying the zero product property. (See Example 1)

7. $(x - 3)(x + 7) = 0$

8. $(t + 4)(t - 1) = 0$

9. $n^2 + 5n = 24$

10. $y^2 = 18 - 7y$

11. $8t(t + 3) = 2t - 5$

12. $6m(m + 4) = m - 15$

13. $40p^2 - 90 = 0$

14. $32n^2 - 162 = 0$

15. $3x^2 = 12x$

16. $z^2 = 25z$

17. $(m + 4)(m - 5) = -8$

18. $(n + 2)(n - 4) = 27$

Objective 2: Solve Quadratic Equations by Using the Square Root Property

For Exercises 19–30, solve by using the square root property. (See Example 2)

19. $x^2 = 81$

20. $w^2 = 121$

21. $5y^2 - 35 = 0$

22. $6v^2 - 30 = 0$

23. $4u^2 + 64 = 0$

24. $8p^2 + 72 = 0$

25. $(k + 2)^2 = 28$

26. $3(z + 11)^2 - 10 = 110$

27. $2(w - 5)^2 + 5 = 23$

28. $(c - 3)^2 = 49$

29. $\left(t - \dfrac{1}{2}\right)^2 = -\dfrac{17}{4}$

30. $\left(a - \dfrac{1}{3}\right)^2 = -\dfrac{47}{9}$

Objective 3: Complete the Square

For Exercises 31–38, determine the value of n that makes the polynomial a perfect square trinomial. Then factor as the square of a binomial. (See Example 3)

31. $x^2 + 14x + n$

32. $y^2 + 22y + n$

33. $p^2 - 26p + n$

34. $u^2 - 4u + n$

35. $w^2 - 3w + n$

36. $v^2 - 11v + n$

37. $m^2 + \dfrac{2}{9}m + n$

38. $k^2 + \dfrac{2}{5}k + n$

For Exercises 39–50, solve by completing the square and applying the square root property. (See Examples 4–5)

39. $y^2 + 22y - 4 = 0$

40. $x^2 + 14x - 3 = 0$

41. $t^2 - 8t = -24$

42. $p^2 - 24p = -156$

43. $4z^2 + 24z = -160$

44. $2m^2 + 20m = -70$

45. $2x(x - 3) = 4 + x$

46. $5c(c - 2) = 6 + 3c$

47. $-4y^2 - 12y + 5 = 0$

48. $-2x^2 - 14x + 5 = 0$

49. $3x^2 + 5x - 6 = 0$

50. $4x^2 + 3x - 8 = 0$

Objective 4: Solve Quadratic Equations by Using the Quadratic Formula

For Exercises 51–54, identify values of *a*, *b*, and *c* that could be used with the quadratic formula to solve the equation.

51. $x^2 = 7x - 4$

52. $x^2 = 3(x - 2)$

53. $5x^2 + 3x = 0$

54. $2x^2 - 18 = 0$

For Exercises 55–70, solve by using the quadratic formula. (See Examples 6–7)

55. $x^2 - 3x - 7 = 0$

56. $x^2 - 5x - 9 = 0$

57. $y^2 = -4y - 6$

58. $z^2 = -8z - 19$

59. $t(t - 6) = -10$

60. $m(m + 10) = -34$

61. $-7c + 3 = -5c^2$

62. $-5d + 2 = -6d^2$

63. $(6x + 5)(x - 3) = -2x(7x + 5) + x - 12$

64. $(5c + 7)(2c - 3) = -2c(c + 15) - 35$

65. $9x^2 + 49 = 0$

66. $121x^2 + 4 = 0$

67. $\frac{1}{2}x^2 - \frac{2}{7} = \frac{5}{14}x$

68. $\frac{1}{3}x^2 - \frac{7}{6} = \frac{3}{2}x$

69. $0.4y^2 = 2y - 2.5$

70. $0.09n^2 = 0.42n - 0.49$

Mixed Exercises

For Exercises 71–78, determine if the equation is linear, quadratic, or neither. If the equation is linear or quadratic, find the solution set.

71. $2y + 4 = 0$

72. $3z - 9 = 0$

73. $2y^2 + 4y = 0$

74. $3z^2 - 9z = 0$

75. $5x(x + 6) = 5x^2 + 27x + 3$

76. $3x(x - 4) = 3x^2 - 11x + 4$

77. $2x^2(x + 7) = x^2 + 3x + 1$

78. $-x(x^2 - 5) + 4 = x^2 + 5$

For Exercises 79–96, solve the equation by using any method.

79. $(3x - 4)^2 = 0$

80. $(2x + 1)^2 = 0$

81. $m^2 + 4m = -2$

82. $n^2 + 8n = -3$

83. $\frac{x^2 - 4x}{6} - \frac{5x}{3} = 1$

84. $\frac{m^2 + 2m}{7} - \frac{9m}{14} = \frac{3}{2}$

85. $2(x + 4) + x^2 = x(x + 2) + 8$

86. $3(y - 5) + y^2 = y(y + 3) - 15$

87. $\frac{3}{5}x^2 - \frac{1}{10}x = \frac{1}{2}$

88. $\frac{1}{12}x^2 - \frac{11}{24}x = -\frac{1}{2}$

89. $x^2 - 5x = 5x(x - 1) - 4x^2 + 1$

90. $p^2 - 4p = 4p(p - 1) - 3p^2 + 2$

91. $(2y + 7)(y + 1) = 2y^2 - 11$

92. $(3z - 8)(z + 2) = 3z^2 + 10$

93. $7d^2 + 5 = 0$

94. $11t^2 + 3 = 0$

95. $x^2 - \sqrt{5} = 0$

96. $y^2 - \sqrt{11} = 0$

Objective 5: Use the Discriminant

For Exercises 97–104, (a) evaluate the discriminant and (b) determine the number and type of solutions to each equation. (See Example 8)

97. $3x^2 - 4x + 6 = 0$

98. $5x^2 - 2x + 4 = 0$

99. $-2w^2 + 8w = 3$

100. $-6d^2 + 9d = 2$

101. $3x(x - 4) = x - 4$

102. $2x(x - 2) = x + 3$

103. $-1.4m + 0.1 = -4.9m^2$

104. $3.6n + 0.4 = -8.1n^2$

Objective 6: Solve an Equation for a Specified Variable

For Exercises 105–118, solve for the indicated variable. (See Examples 9–10)

105. $A = \pi r^2$ for $r > 0$

106. $V = \pi r^2 h$ for $r > 0$

107. $s = \frac{1}{2}gt^2$ for $t > 0$

108. $c = \frac{d^2 t}{2}$ for $d > 0$

109. $a^2 + b^2 = c^2$ for $a > 0$

110. $a^2 + b^2 + c^2 = d^2$ for $c > 0$

111. $L = c^2 I^2 Rt$ for $I > 0$

112. $I = cN^2 r^2 s$ for $N > 0$

113. $kw^2 - cw = r$ for w

114. $dy^2 + my = p$ for y

115. $s = v_0 t + \frac{1}{2}at^2$ for t

116. $S = 2\pi rh + \pi r^2 h$ for r

117. $LI^2 + RI + \frac{1}{C} = 0$ for I

118. $A = \pi r^2 + \pi rs$ for r

Write About It

119. Explain why the zero product property cannot be applied directly to solve the equation $(2x - 3)(x + 1) = 6$.

120. Given a quadratic equation, what is the discriminant and what information does it provide about the given quadratic equation?

Expanding Your Skills

For Exercises 121–122, solve for the indicated variable.

121. $x^2 - xy - 2y^2 = 0$ for x

122. $3a^2 + 2ab - b^2 = 0$ for a

For Exercises 123–132, write an equation with integer coefficients and the variable x that has the given solution set. [*Hint*: Apply the zero product property in reverse. For example, to build an equation whose solution set is $\left\{2, -\frac{5}{2}\right\}$ we have $(x - 2)(2x + 5) = 0$, or simply $2x^2 + x - 10 = 0$.]

123. $\{4, -2\}$

124. $\{7, -1\}$

125. $\left\{\frac{2}{3}, \frac{1}{4}\right\}$

126. $\left\{\frac{3}{5}, \frac{1}{7}\right\}$

127. $\{\sqrt{5}, -\sqrt{5}\}$

128. $\{\sqrt{2}, -\sqrt{2}\}$

129. $\{2i, -2i\}$

130. $\{9i, -9i\}$

131. $\{1 \pm 2i\}$

132. $\{2 \pm 9i\}$

The solutions to the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are $x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. For Exercises 133–134, prove the given statements.

133. Prove that $x_1 + x_2 = -\frac{b}{a}$.

134. Prove that $x_1 x_2 = \frac{c}{a}$.

SECTION 1.5 Applications of Quadratic Equations

OBJECTIVES

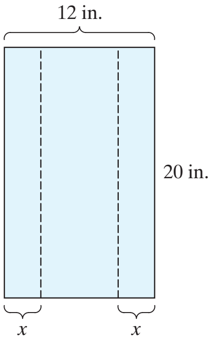
1. Solve Applications Involving Quadratic Equations and Geometry
2. Solve Applications Involving Quadratic Models

1. Solve Applications Involving Quadratic Equations and Geometry

In this section, we solve applications that involve quadratic equations. Examples 1 and 2 involve applications with geometric figures.

EXAMPLE 1 Solving an Application Involving Volume

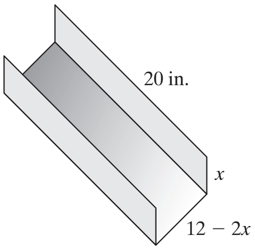
A trough at the end of a gutter spout is meant to direct water away from a house. The homeowner makes the trough from a rectangular piece of aluminum that is 20 in. long and 12 in. wide. He makes a fold along the two long sides a distance of x inches from the edge. If he wants the trough to hold 360 in.^{3} of water, how far from the edge should he make the fold?



Solution:

Let x represent the distance between the edge of the sheet and the fold.

Information is given about the volume of the trough. When the fold is made, the trough will be in the shape of a rectangular solid with the ends missing. The volume is given by the product of length, width, and height.



$$V = lwh$$

$$360 = (20)(12 - 2x)(x)$$ The length is 20 in., the width is 12 − 2x, and the height is x.

$$360 = 240x - 40x^2$$ Apply the distributive property.

$$40x^2 - 240x + 360 = 0$$ Set one side equal to zero.

$$40(x^2 - 6x + 9) = 0$$ Factor.

$$40(x - 3)^2 = 0$$

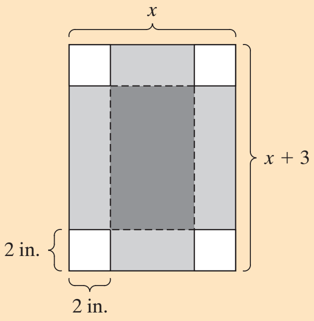
$$\cancel{40 = 0} \quad \text{or} \quad x - 3 = 0$$ Apply the zero product property. Set each factor equal to zero.

$$x = 3$$ The first equation is a contradiction. The only solution is 3.

The sheet of aluminum should be folded 3 in. from the edges.

TIP In Example 1, we choose to move the terms to the left side of the equation so that the leading coefficient is positive. This makes the polynomial easier to factor.

Skill Practice 1 A box is to be formed by taking a sheet of cardboard and cutting away four 2-in. by 2-in. squares from each corner. Then the sides are turned up to form a box that holds 56 in.^{3}. If the length of the original piece of cardboard is 3 in. more than the width, find the dimensions of the original sheet of cardboard.



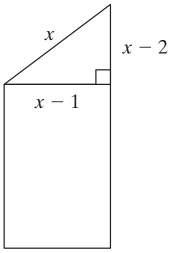
Answer

1. The sheet of cardboard is 8 in. by 11 in.

In Example 2, we use the Pythagorean theorem and a quadratic equation to find the lengths of the sides of a right triangle.

EXAMPLE 2 Solving an Application Involving the Pythagorean Theorem

A window is in the shape of a rectangle with an adjacent right triangle above (see figure). The length of one leg of the right triangle is 2 ft less than the length of the hypotenuse. The length of the other leg is 1 ft less than the length of the hypotenuse. Find the lengths of the sides.



Solution:

Let x represent the length of the hypotenuse.
$x - 1$ represents the length of the longer leg.
$x - 2$ represents the length of the shorter leg.

Use the Pythagorean theorem to relate the lengths of the sides.

$a^2 + b^2 = c^2$	Pythagorean theorem
$(x - 1)^2 + (x - 2)^2 = (x)^2$	Substitute $x - 1$, $x - 2$, and x for the lengths of the sides.
$x^2 - 2x + 1 + x^2 - 4x + 4 = x^2$	
$x^2 - 6x + 5 = 0$	Set one side of the equation equal to zero.
$(x - 1)(x - 5) = 0$	Factor.
$x - 1 = 0$ or $x - 5 = 0$	Apply the zero product property.
$\cancel{x = 1}$ or $x = 5$	Reject $x = 1$ because if x were 1, the lengths of the legs would be 0 ft and -1 ft, which is impossible.

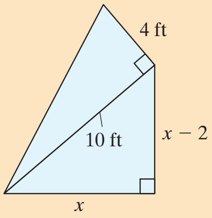
Avoiding Mistakes

In Example 2, be sure to square the binomials correctly. Recall that $(a - b)^2 = a^2 - 2ab + b^2$.

Therefore,
$(x - 1)^2 = x^2 - 2x + 1$
$(x - 2)^2 = x^2 - 4x + 4$

The hypotenuse is 5 ft.
The length of the longer leg is given by $x - 1$: 5 ft $-$ 1 ft = 4 ft.
The length of the shorter leg is given by $x - 2$: 5 ft $-$ 2 ft = 3 ft.

Skill Practice 2 A sail on a sailboat is in the shape of two adjacent right triangles. The hypotenuse of the lower triangle is 10 ft, and one leg is 2 ft shorter than the other leg. Find the lengths of the legs of the lower triangle.



2. Solve Applications Involving Quadratic Models

In the study of physical science, a common model used to represent the vertical position of an object moving vertically under the influence of gravity is given in Table 1-1.

Answer

2. The longer leg is 8 ft and the shorter leg is 6 ft.

Table 1-1 Vertical Position of an Object

Suppose that an object has an initial vertical position of s_0 and initial velocity v_0 straight upward. The vertical position s of the object is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0, \text{ where}$$

g	is the acceleration due to gravity. On Earth, $g = 32$ ft/sec^2 or $g = 9.8$ m/sec^2.
t	is the time of travel.
v_0	is the initial velocity.
s_0	is the initial vertical position.
s	is the vertical position of the object at time t.

TIP The value of g is chosen to be consistent with the units for position and velocity. In this case, the initial height is given in ft. The initial velocity is given in ft/sec. Therefore, we choose g in ft/sec^2 rather than m/sec^2.

For example, suppose that a child tosses a ball straight upward from a height of 1.5 ft, with an initial velocity of 48 ft/sec.

The initial height is $s_0 = 1.5$ ft.
The initial velocity is $v_0 = 48$ ft/sec.
The acceleration due to gravity is $g = 32$ ft/sec^2.

The vertical position of the ball (in feet) is given by

$$s = -\frac{1}{2}gt^2 + v_0t + s_0$$

$$s = -\frac{1}{2}(32)t^2 + (48)t + (1.5)$$

$$= -16t^2 + 48t + 1.5$$

EXAMPLE 3 Analyzing an Object Moving Vertically

A toy rocket is shot straight upward from a launch pad of 1 m above ground level with an initial velocity of 24 m/sec.

a. Write a model to express the height of the rocket s (in meters) above ground level.

b. Find the time(s) at which the rocket is at a height of 20 m. Round to 1 decimal place.

c. Find the time(s) at which the rocket is at a height of 40 m.

Solution:

a. $s = -\frac{1}{2}gt^2 + v_0t + s_0$

$$s = -\frac{1}{2}(9.8)t^2 + (24)t + (1)$$

$$= -4.9t^2 + 24t + 1$$

In this example,
$s_0 = 1$ m
$v_0 = 24$ m/sec
$g = 9.8$ m/sec^2

TIP Choose $g = 9.8$ m/sec^2 because the height is given in meters and velocity is given in meters per second.

b. $20 = -4.9t^2 + 24t + 1$ Substitute 20 for s.

$4.9t^2 - 24t + 19 = 0$ Set one side equal to zero.

$$t = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4.9)(19)}}{2(4.9)}$$ Apply the quadratic formula.

$$t = \frac{24 \pm \sqrt{203.6}}{9.8} \begin{array}{l} \nearrow \; t = \dfrac{24 + \sqrt{203.6}}{9.8} \approx 3.9 \\ \searrow \; t = \dfrac{24 - \sqrt{203.6}}{9.8} \approx 1.0 \end{array}$$

The rocket will be at a height of 20 m at 1 sec and 3.9 sec after launch.

c. $40 = -4.9t^2 + 24t + 1$ Substitute 40 for s.

$4.9t^2 - 24t + 39 = 0$

$$t = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(4.9)(39)}}{2(4.9)}$$ Apply the quadratic formula.

$$t = \frac{24 \pm \sqrt{-188.4}}{9.8}$$ The solutions are not real numbers.

There is no real number t for which the height of the rocket will be 40 m. The rocket will not reach a height of 40 m.

Skill Practice 3 A fireworks mortar is launched straight upward from a pool deck 2 m off the ground at an initial velocity of 40 m/sec.

a. Write a model to express the height of the mortar s (in meters) above ground level.

b. Find the time(s) at which the mortar is at a height of 60 m. Round to 1 decimal place.

c. Find the time(s) at which the rocket is at a height of 100 m.

Answers

3. a. $s = -4.9t^2 + 40t + 2$

b. The mortar will be at a height of 60 m at 1.9 sec and 6.3 sec after launch.

c. The mortar will not reach a height of 100 m.

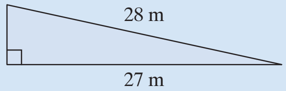
SECTION 1.5 Practice Exercises

Prerequisite Review

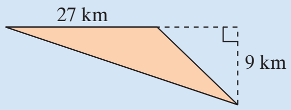
R.1. Find the area of a rectangle that measures 28 ft by $9\frac{2}{7}$ ft.

R.2. Find the volume of a rectangular solid that measures 6 in. by 8 ft by 14 ft.

R.3. Find the length of the unknown side.



R.4. Find the area of the triangle.



Concept Connections

1. Write a formula for the area of a triangle of base b and height h.

2. Write a formula for the area of a circle of radius r.

3. Write a formula for the volume of a rectangular solid of length l, width w, and height h.

4. Write the Pythagorean theorem for a right triangle with the lengths of the legs given by a and b and the length of the hypotenuse given by c.

Objective 1: Solve Applications Involving Quadratic Equations and Geometry

For Exercises 5–12, refer to the geometry formulas in the inside back cover.

a. Write an equation in terms of x that represents the given relationship.

b. Solve the equation to find the dimensions of the given shape.

5. The length of a rectangle is 3 yd more than twice the width x. The area is 629 yd^2.

7. The height of a triangle is 2 ft less than the base x. The area is 40 ft^2.

9. The width of a rectangular box is 8 in. The height is one-fifth the length x. The volume is 640 in.^{3}.

11. The length of the longer leg of a right triangle is 2 ft longer than the length of the shorter leg x. The hypotenuse is 2 ft shorter than twice the length of the shorter leg.

13. A rectangular garden covers 40 yd^2. The length is 2 yd longer than the width. Find the length and width. Round to the nearest tenth of a yard.

15. The height of a triangular truss is 8 ft less than the base. The amount of drywall needed to cover the triangular area is 86 ft^2. Find the base and height of the triangle to the nearest tenth of a foot.

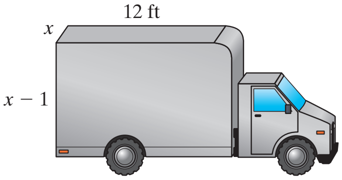
17. a. Write an equation representing the fact that the product of two consecutive even integers is 120.

b. Solve the equation from part (a) to find the two integers.

19. a. Write an equation representing the fact that the sum of the squares of two consecutive integers is 113.

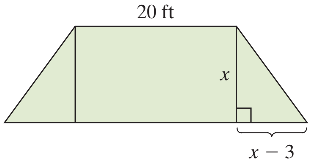
b. Solve the equation from part (a) to find the two integers.

21. On moving day, Guyton needs to rent a truck. The length of the cargo space is 12 ft, and the height is 1 ft less than the width. The brochure indicates that the truck can hold 504 ft^3. What are the dimensions of the cargo space? Assume that the cargo space is in the shape of a rectangular solid. (**See Example 1**)



23. A sprinkler rotates 360° to water a circular region. If the total area watered is approximately 2000 yd^2, determine the radius of the region (the radius is length of the stream of water). Round the answer to the nearest yard.

25. A patio is configured from a rectangle with two right triangles of equal size attached at the two ends. The length of the rectangle is 20 ft. The base of the right triangle is 3 ft less than the height of the triangle. If the total area of the patio is 348 ft^2, determine the base and height of the triangular portions.



6. The width of a rectangle is 2 m less than one-quarter of the length x. The area is 252 m^2.

8. The height of a triangle is 4 yd longer than the base x. The area is 70 yd^2.

10. The height of a rectangular box is 4 ft. The length is 1 ft longer than twice the width x. The volume is 312 ft^3.

12. The longer leg of a right triangle is 7 cm longer than the length of the shorter leg x. The hypotenuse is 17 cm.

14. A rectangular piece of carpet covers 200 yd^2. The width is 9 yd less than the length. Find the length and width. Round to the nearest tenth of a yard.

16. The base of a triangular piece of fabric is 6 in. more than the height. The area is 600 in.^{2}. Find the base and height of the triangle to the nearest tenth of an inch.

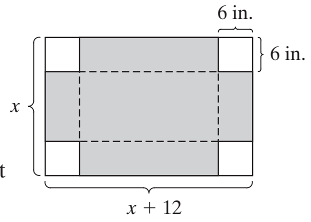
18. a. Write an equation representing the fact that the product of two consecutive odd integers is 35.

b. Solve the equation from part (a) to find the two integers.

20. a. Write an equation representing the fact that the sum of the squares of two consecutive integers is 181.

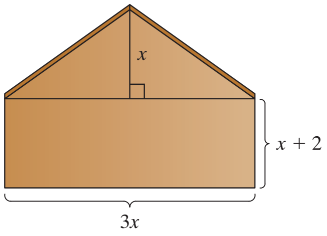
b. Solve the equation from part (a) to find the two integers.

22. Lorene plans to make several open-topped boxes in which to carry plants. She makes the boxes from rectangular sheets of cardboard from which she cuts out 6-in. squares from each corner. The length of the original piece of cardboard is 12 in. more than the width. If the volume of the box is 1728 in.^{3}, determine the dimensions of the original piece of cardboard.

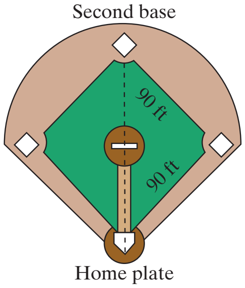


24. An earthquake could be felt over a 46,000-mi^2 area. Up to how many miles from the epicenter could the earthquake be felt? Round to the nearest mile.

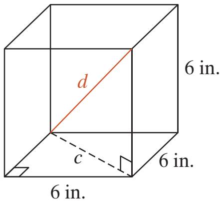
26. The front face of a house is in the shape of a rectangle with a Queen post roof truss above. The length of the rectangular region is 3 times the height of the truss. The height of the rectangle is 2 ft more than the height of the truss. If the total area of the front face of the house is 336 ft^2, determine the length and width of the rectangular region.



27. A baseball diamond is in the shape of a square with 90-ft sides. How far is it from home plate to second base? Give the exact value and give an approximation to the nearest tenth of a foot.



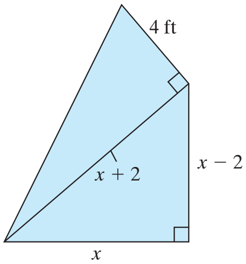
28. The figure shown is a cube with 6-in. sides. Find the exact length of the diagonal through the interior of the cube d by following these steps.



a. Apply the Pythagorean theorem using the sides on the base of the cube to find the length of diagonal c.

b. Apply the Pythagorean theorem using c and the height of the cube as the legs of the right triangle through the interior of the cube.

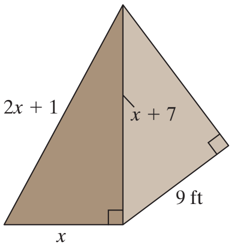
29. The sail on a sailboat is in the shape of two adjacent right triangles. In the lower triangle, the shorter leg is 2 ft less than the longer leg. The hypotenuse is 2 ft more than the longer leg. (**See Example 2**)



a. Find the lengths of the sides of the lower triangle.

b. Find the total sail area.

30. A portion of a roof truss is given in the figure. The triangle on the left is configured such that the longer leg is 7 ft longer than the shorter leg, and the hypotenuse is 1 ft more than twice the shorter leg.



a. Find the lengths of the sides of the triangle on the left.

b. Find the lengths of the sides of the triangle on the right.

31. The display area on a cell phone has a 3.5-in. diagonal.

a. If the aspect ratio of length to width is 1.5 to 1, determine the length and width of the display area. Round the values to the nearest hundredth of an inch.

b. If the phone has 326 pixels per inch, approximate the dimensions in pixels.

32. The display area on a computer has a 15-in. diagonal. If the aspect ratio of length to width is 1.6 to 1, determine the length and width of the display area. Round the values to the nearest hundredth of an inch.

Objective 2: Solve Applications Involving Quadratic Models

33. In a round-robin tennis tournament, each player plays every other player exactly one time. The number of matches N is given by $N = \frac{1}{2}n(n - 1)$, where n is the number of players in the tournament. If 28 matches were played, how many players were in the tournament?

34. The sum of the first n natural numbers, $S = 1 + 2 + 3 + \cdots + n$, is given by $S = \frac{1}{2}n(n + 1)$. If the sum of the first n natural numbers is 171, determine the value of n.

35. The population P of a culture of *Pseudomonas aeruginosa* bacteria is given by $P = -1718t^2 + 82{,}000t + 10{,}000$, where t is the time in hours since the culture was started. Determine the time(s) at which the population was 600,000. Round to the nearest hour.

36. The gas mileage for a certain vehicle can be approximated by $m = -0.04x^2 + 3.6x - 49$, where x is the speed of the vehicle in mph. Determine the speed(s) at which the car gets 30 mpg. Round to the nearest mph.

37. The distance d (in ft) required to stop a car that was traveling at speed v (in mph) before the brakes were applied depends on the amount of friction between the tires and the road and the driver's reaction time. After an accident, a legal team hired an engineering firm to collect data for the stretch of road where the accident occurred. Based on the data, the stopping distance is given by $d = 0.05v^2 + 2.2v$.

a. Determine the distance required to stop a car going 50 mph.

b. Up to what speed (to the nearest mph) could a motorist be traveling and still have adequate stopping distance to avoid hitting a deer 330 ft away?

38. Leptin is a hormone that has a central role in fat metabolism. One study published in the *New England Journal of Medicine* measured serum leptin concentrations versus the percentage of body fat for 275 individuals. The concentration of leptin c (in ng/mL) is approximated by $c = 219x^2 - 26.7x + 1.64$, where x is percentage of body fat.

a. Determine the concentration of leptin in an individual with 22% body fat ($x = 0.22$). Round to 1 decimal place.

b. If an individual has 3 ng/mL of leptin, determine the percentage of body fat. Round to the nearest whole percent.

(*Source*: "Serum Immunoreactive-Leptin Concentrations in Normal-Weight and Obese Humans," *New England Journal of Medicine*, Feb., 1996)

For Exercises 39–42, use the model $s = -\frac{1}{2}gt^2 + v_0t + s_0$ with $g = 32$ ft/sec^2 or $g = 9.8$ m/sec^2. (See Example 3)

39. NBA basketball legend Michael Jordan had a 48-in. vertical leap. Suppose that Michael jumped from ground level with an initial velocity of 16 ft/sec.

a. Write a model to express Michael's height (in ft) above ground level t seconds after leaving the ground.

b. Use the model from part (a) to determine how long it would take Michael to reach his maximum height of 48 in. (4 ft).

41. A bad punter on a football team kicks a football approximately straight upward with an initial velocity of 75 ft/sec.

a. If the ball leaves his foot from a height of 4 ft, write an equation for the vertical height s (in ft) of the ball t seconds after being kicked.

b. Find the time(s) at which the ball is at a height of 80 ft. Round to 1 decimal place.

40. At the time of this printing, the highest vertical leap on record is 60 in., held by Kadour Ziani. For this record-setting jump, Kadour left the ground with an initial velocity of $8\sqrt{5}$ ft/sec.

a. Write a model to express Kadour's height (in ft) above ground level t seconds after leaving the ground.

b. Use the model from part (a) to determine how long it would take Kadour to reach his maximum height of 60 in. (5 ft). Round to the nearest hundredth of a second.

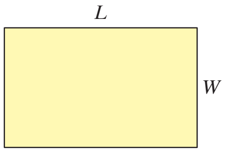
42. In a classic *Seinfeld* episode, Jerry tosses a loaf of bread (a marble rye) straight upward to his friend George who is leaning out of a third-story window.

a. If the loaf of bread leaves Jerry's hand at a height of 1 m with an initial velocity of 18 m/sec, write an equation for the vertical position of the bread s (in meters) t seconds after release.

b. How long will it take the bread to reach George if he catches the bread on the way up at a height of 16 m? Round to the nearest tenth of a second.

Expanding Your Skills

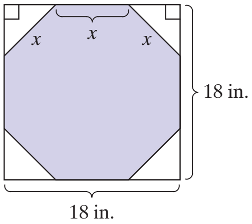
43. A **golden rectangle** is a rectangle in which the ratio of its length to its width is equal to the ratio of the sum of its length and width to its length: $\frac{L}{W} = \frac{L + W}{L}$ (values of L and W that meet this condition are said to be in the **golden ratio**).



a. Suppose that a golden rectangle has a width of 1 unit. Solve the equation to find the exact value for the length. Then give a decimal approximation to 2 decimal places.

b. To create a golden rectangle with a width of 9 ft, what should be the length? Round to 1 decimal place.

44. An artist has been commissioned to make a stained glass window in the shape of a regular octagon. The octagon must fit inside an 18-in. square space. Determine the length of each side of the octagon. Round to the nearest hundredth of an inch.

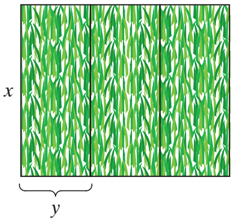


45. A farmer has 160 yd of fencing material and wants to enclose three rectangular pens. Suppose that x represents the length of each pen and y represents the width as shown in the figure.

a. Assuming that the farmer uses all 160 yd of fencing, write an expression for y in terms of x.

b. Write an expression in terms of x for the area of one individual pen.

c. If the farmer wants to design the structure so that each pen encloses 250 yd^2, determine the dimensions of each pen.



46. At noon, a ship leaves a harbor and sails south at 10 knots. Two hours later, a second ship leaves the harbor and sails east at 15 knots. When will the ships be 100 nautical miles apart? Round to the nearest minute.

Chapter: Polynomial and Rational Functions

SECTION 3.1 Quadratic Functions and Applications

OBJECTIVES

1. Graph a Quadratic Function Written in Vertex Form
2. Write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in Vertex Form
3. Find the Vertex of a Parabola by Using the Vertex Formula
4. Solve Applications Involving Quadratic Functions
5. Create Quadratic Models Using Regression

1. Graph a Quadratic Function Written in Vertex Form

In Chapter 2, we defined a function of the form $f(x) = mx + b$ $(m \neq 0)$ as a linear function. The function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a *quadratic function.* Notice that a quadratic function has a leading term of second degree. We are already familiar with the graph of $f(x) = x^2$ (Figure 3-1). The graph is a parabola opening upward with vertex at the origin. Also note that the graph is symmetric with respect to the vertical line through the vertex called the **axis of symmetry.**

We can write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in the form $f(x) = a(x - h)^2 + k$ by completing the square. Furthermore, from Section 2.6 we know that the graph of $f(x) = a(x - h)^2 + k$ is related to the graph of $y = x^2$ by a vertical shrink or stretch determined by a, a horizontal shift determined by h, and a vertical shift determined by k. Therefore, the graph of a quadratic function is a parabola with vertex at (h, k).

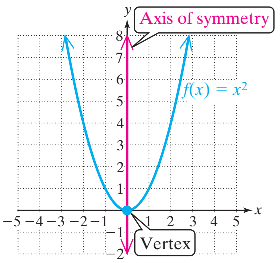
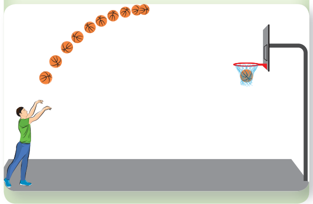


Figure 3-1

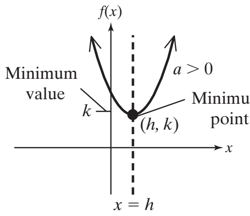
TIP A quadratic function is often used as a model for projectile motion. This is motion followed by an object influenced by an initial force and by the force of gravity.

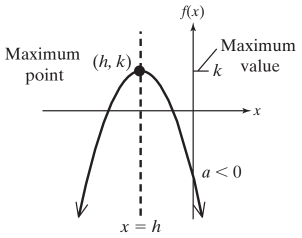


Quadratic Function

A function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a **quadratic function.** By completing the square, $f(x)$ can be expressed in **vertex form** as $f(x) = a(x - h)^2 + k$.

- The graph of f is a parabola with vertex (h, k).
- If $a > 0$, the parabola opens upward, and the vertex is the minimum point. The minimum *value* of f is k.
- If $a < 0$, the parabola opens downward, and the vertex is the maximum point. The maximum *value* of f is k.
- The axis of symmetry is $x = h$. This is the vertical line that passes through the vertex.





In Example 1, we analyze and graph a quadratic function by identifying the vertex, axis of symmetry, and x- and y-intercepts. From the graph, the minimum or maximum value of the function is readily apparent.

EXAMPLE 1 Analyzing and Graphing a Quadratic Function

Given $f(x) = -2(x - 1)^2 + 8$,

a. Determine whether the graph of the parabola opens upward or downward.
b. Identify the vertex.
c. Determine the x-intercept(s).
d. Determine the y-intercept.
e. Sketch the function.
f. Determine the axis of symmetry.
g. Determine the maximum or minimum value of f.
h. Write the domain and range in interval notation.

Solution:

a. $f(x) = -2(x - 1)^2 + 8$

The parabola opens downward.

The function is written as $f(x) = a(x - h)^2 + k$, where $a = -2$, $h = 1$, and $k = 8$. Since $a < 0$, the parabola opens downward.

b. The vertex is (1, 8).

The vertex is (h, k), which is (1, 8).

c.

$$\begin{aligned} f(x) &= -2(x - 1)^2 + 8 \\ 0 &= -2(x - 1)^2 + 8 \\ -8 &= -2(x - 1)^2 \\ 4 &= (x - 1)^2 \\ \pm\sqrt{4} &= x - 1 \\ 1 \pm 2 &= x \\ x = 3 \quad &\text{or} \quad x = -1 \end{aligned}$$

To find the x-intercept(s), find all real solutions to the equation $f(x) = 0$.

The x-intercepts are (3, 0) and (−1, 0).

d. $f(0) = -2(0 - 1)^2 + 8$
$= 6$

To find the y-intercept, evaluate $f(0)$.

The y-intercept is (0, 6).

e. The graph of f is shown in Figure 3-2.

f. The axis of symmetry is the vertical line through the vertex: $x = 1$.

g. The maximum value is 8.

h. The domain is $(-\infty, \infty)$.
The range is $(-\infty, 8]$.

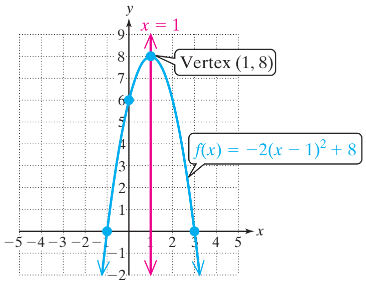


Figure 3-2

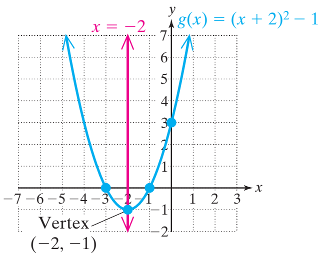
Skill Practice 1 Repeat Example 1 with $g(x) = (x + 2)^2 - 1$.

2. Write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in Vertex Form

In Section 2.2, we learned how to complete the square to write an equation of a circle $x^2 + y^2 + Ax + By + C = 0$ in standard form $(x - h)^2 + (y - k)^2 = r^2$. We use the same process to write a quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in vertex form $f(x) = a(x - h)^2 + k$. However, we will work on the right side of the equation only. This is demonstrated in Example 2.

Answers

1. a. Upward **b.** (−2, −1)
c. (−3, 0) and (−1, 0) **d.** (0, 3)
e.



f. $x = -2$
g. The minimum value is −1.
h. The domain is $(-\infty, \infty)$.
The range is $[-1, \infty)$.

EXAMPLE 2 Writing a Quadratic Function in Vertex Form

Given $f(x) = 3x^2 + 12x + 5$,

a. Write the function in vertex form: $f(x) = a(x - h)^2 + k$.
b. Identify the vertex.
c. Identify the x-intercept(s).
d. Identify the y-intercept.
e. Sketch the function.
f. Determine the axis of symmetry.
g. Determine the minimum or maximum value of f.
h. Write the domain and range in interval notation.

Solution:

a.
$$\begin{aligned} f(x) &= 3x^2 + 12x + 5 \\ &= 3(x^2 + 4x \qquad\qquad) + 5 \end{aligned}$$

Factor out the leading coefficient of the x^2 term from the two terms containing x. The leading term within parentheses now has a coefficient of 1.

$$= 3(x^2 + 4x + 4 - 4) + 5$$

Complete the square within parentheses. Add and subtract $\left[\frac{1}{2}(4)\right]^2 = 4$ within parentheses.

$$= 3(x^2 + 4x + 4) + 3(-4) + 5$$

Remove -4 from within parentheses, along with a factor of 3.

$$= 3(x + 2)^2 - 7 \text{ (vertex form)}$$

b. The vertex is $(-2, -7)$.

c.
$$\begin{aligned} f(x) &= 3x^2 + 12x + 5 \\ 0 &= 3x^2 + 12x + 5 \end{aligned}$$

To find the x-intercept(s), find the real solutions to the equation $f(x) = 0$.

$$x = \frac{-12 \pm \sqrt{(12)^2 - 4(3)(5)}}{2(3)}$$

The right side is not factorable. Apply the quadratic formula.

$$= \frac{-12 \pm \sqrt{84}}{6}$$

$$= \frac{-12 \pm 2\sqrt{21}}{6}$$

$$= \frac{-6 \pm \sqrt{21}}{3} \begin{array}{l} \nearrow x \approx -0.47 \\ \searrow x \approx -3.53 \end{array}$$

The x-intercepts are $\left(\frac{-6 + \sqrt{21}}{3}, 0\right)$ and $\left(\frac{-6 - \sqrt{21}}{3}, 0\right)$ or approximately $(-0.47, 0)$ and $(-3.53, 0)$.

d.
$$\begin{aligned} f(0) &= 3(0)^2 + 12(0) + 5 \\ &= 5 \end{aligned}$$

To find the y-intercept, evaluate $f(0)$. The y-intercept is $(0, 5)$.

e. The graph of f is shown in Figure 3-3.

f. The axis of symmetry is $x = -2$.

g. The minimum value is -7.

h. The domain is $(-\infty, \infty)$.

The range is $[-7, \infty)$.

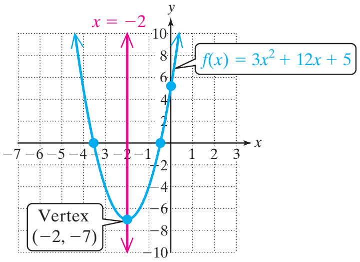


Figure 3-3

Skill Practice 2 Repeat Example 2 with $f(x) = 3x^2 - 6x + 1$.

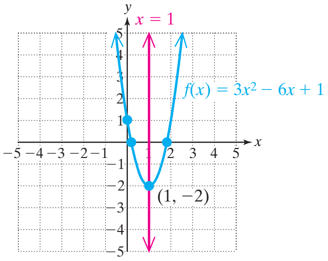
Answers

2. a. $f(x) = 3(x - 1)^2 - 2$ **b.** $(1, -2)$

c. $\left(\frac{3 \pm \sqrt{6}}{3}, 0\right)$

d. $(0, 1)$

e.



f. $x = 1$
g. The minimum value is -2.
h. The domain is $(-\infty, \infty)$.
The range is $[-2, \infty)$.

3. Find the Vertex of a Parabola by Using the Vertex Formula

Completing the square and writing a quadratic function in the form $f(x) = a(x - h)^2 + k$ is one method to find the vertex of a parabola. Another method is to use the vertex formula. The vertex formula can be derived by completing the square on $f(x) = ax^2 + bx + c$.

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$

Factor out a from the x terms, and complete the square within parentheses. $\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \frac{b^2}{4a^2}$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c$$

Remove the term $-\frac{b^2}{4a^2}$ from within parentheses along with a factor of a.

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

Factor the trinomial.

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

Obtain a common denominator and add the terms outside parentheses.

$$= a\left[x - \left(\frac{-b}{2a}\right)\right]^2 + \frac{4ac - b^2}{4a}$$

$f(x)$ is now written in vertex form.

$$f(x) = a(x - \underset{\uparrow}{h})^2 + \underset{\uparrow}{k}$$

$h = \frac{-b}{2a}$ and $k = \frac{4ac - b^2}{4a}$

The vertex is $\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right)$.

The y-coordinate of the vertex is given by $\frac{4ac - b^2}{4a}$ and is often hard to remember. Therefore, it is usually easier to evaluate the x-coordinate first from $\frac{-b}{2a}$, and then evaluate $f\left(\frac{-b}{2a}\right)$.

Vertex Formula to Find the Vertex of a Parabola

For $f(x) = ax^2 + bx + c$ $(a \neq 0)$, the vertex is given by $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

EXAMPLE 3 Using the Vertex Formula

Given $f(x) = -x^2 + 4x - 5$,

a. State whether the graph of the parabola opens upward or downward.
b. Determine the vertex of the parabola by using the vertex formula.
c. Determine the x-intercept(s).
d. Determine the y-intercept.
e. Sketch the graph.
f. Determine the axis of symmetry.
g. Determine the minimum or maximum value of f.
h. Write the domain and range in interval notation.

Solution:

a. $f(x) = -x^2 + 4x - 5$

The parabola opens downward.

The function is written as $f(x) = ax^2 + bx + c$ where $a = -1$. Since $a < 0$, the parabola opens downward.

b. x-coordinate: $\dfrac{-b}{2a} = \dfrac{-(4)}{2(-1)} = 2$

y-coordinate: $f(2) = -(2)^2 + 4(2) - 5$

$= -1$

The vertex is $(2, -1)$.

c. Since the vertex of the parabola is below the x-axis and the parabola opens downward, the parabola cannot cross or touch the x-axis.

Therefore, there are no x-intercepts.

Solving the equation $f(x) = 0$ to find the x-intercepts results in imaginary solutions:

$0 = -x^2 + 4x - 5$

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(-1)(-5)}}{2(-1)}$$

$x = 2 \pm i$

d. To find the y-intercept, evaluate $f(0)$.

$f(0) = -(0)^2 + 4(0) - 5 = -5$

The y-intercept is $(0, -5)$.

e. The graph of f is shown in Figure 3-4.

f. The axis of symmetry is $x = 2$.

g. The maximum value of f is -1.

h. The domain is $(-\infty, \infty)$.

The range is $(-\infty, -1]$.

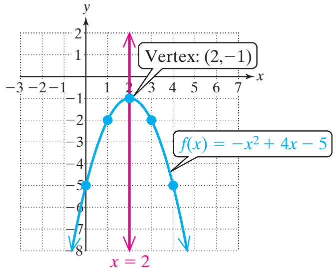


Figure 3-4

TIP For more accuracy in the graph, plot one or two points near the vertex. Then use the symmetry of the curve to find additional points on the graph.

For example, the points (1, −2) and (0, −5) are on the left branch of the parabola. The corresponding points to the right of the axis of symmetry are (3, −2) and (4, −5).

Skill Practice 3 Repeat Example 3 with $f(x) = -x^2 - 4x - 7$.

The x-intercepts of a quadratic function defined by $f(x) = ax^2 + bx + c$ are the real solutions to the equation $f(x) = 0$. The discriminant $b^2 - 4ac$ enables us to determine the number of real solutions to the equation and thus, the number of x-intercepts of the graph of the function.

Using the Discriminant to Determine the Number of *x*-Intercepts

Given a quadratic function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$,

- If $b^2 - 4ac = 0$, the graph of $y = f(x)$ has one x-intercept.
- If $b^2 - 4ac > 0$, the graph of $y = f(x)$ has two x-intercepts.
- If $b^2 - 4ac < 0$, the graph of $y = f(x)$ has no x-intercept.

From Example 2, the discriminant of $3x^2 + 12x + 5 = 0$ is $(12)^2 - 4(3)(5) = 84 > 0$. Therefore, the graph of $f(x) = 3x^2 + 12x + 5$ has two x-intercepts (Figure 3-3).

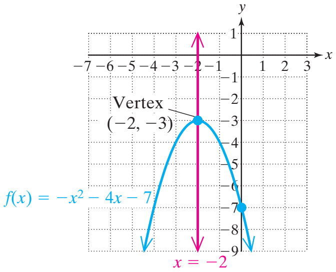
From Example 3, the discriminant of $-x^2 + 4x - 5 = 0$ is $(4)^2 - 4(-1)(-5) = -4 < 0$. Therefore, the graph of $f(x) = -x^2 + 4x - 5$ has no x-intercept (Figure 3-4).

Answers

3. a. Downward **b.** $(-2, -3)$

c. No x-intercepts **d.** $(0, -7)$

e.



f. $x = -2$

g. The maximum value is −3.

h. The domain is $(-\infty, \infty)$.

The range is $(-\infty, -3]$.

4. Solve Applications Involving Quadratic Functions

Quadratic functions can be used in a variety of applications in which a variable is optimized. That is, the vertex of a parabola gives the maximum or minimum value of the dependent variable. We show three such applications in Examples 4–6.

EXAMPLE 4 Using a Quadratic Function for Projectile Motion

A stone is thrown from a 100-m cliff at an initial speed of 20 m/sec at an angle of 30° from the horizontal. The height of the stone can be modeled by $h(t) = -4.9t^2 + 10t + 100$, where $h(t)$ is the height in meters and t is the time in seconds after the stone is released.

a. Determine the time at which the stone will be at its maximum height. Round to 2 decimal places.

b. Determine the maximum height. Round to the nearest meter.

c. Determine the time at which the stone will hit the ground.

Solution:

a. The time at which the stone will be at its maximum height is the t-coordinate of the vertex.

$$t = \frac{-b}{2a} = \frac{-10}{2(-4.9)} \approx 1.02$$

The stone will be at its maximum height approximately 1.02 sec after release.

Given $h(t) = -4.9t^2 + 10t + 100$, the coefficients are $a = -4.9$, $b = 10$, and $c = 100$.

The vertex is given by $\left(\frac{-b}{2a}, h\left(\frac{-b}{2a}\right)\right)$.

b. The maximum height is the value of $h(t)$ at the vertex.

$$h(1.02) = -4.9(1.02)^2 + 10(1.02) + 100$$

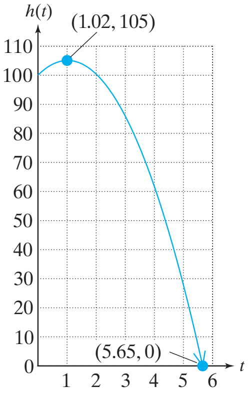
$$\approx 105$$ The maximum height is 105 m.

c. The stone will hit the ground when $h(t) = 0$.

$$h(t) = -4.9t^2 + 10t + 100$$
$$0 = -4.9t^2 + 10t + 100$$
$$t = \frac{-10 \pm \sqrt{(10)^2 - 4(-4.9)(100)}}{2(-4.9)}$$

$t \approx 5.65$ or $t \approx -\cancel{3.61}$ Reject the negative solution.

The stone will hit the ground in approximately 5.65 sec.



Skill Practice 4 A quarterback throws a football with an initial velocity of 72 ft/sec at an angle of 25°. The height of the ball can be modeled by $h(t) = -16t^2 + 30.4t + 5$, where $h(t)$ is the height (in ft) and t is the time in seconds after release.

a. Determine the time at which the ball will be at its maximum height.

b. Determine the maximum height of the ball.

c. Determine the amount of time required for the ball to reach the receiver's hands if the receiver catches the ball at a point 3 ft off the ground.

Point of Interest

The movie *Apollo 13* starring Tom Hanks was filmed in part in a "Vomit Comet," an aircraft that uses a parabolic flight trajectory to produce weightlessness. As the plane climbs toward the top of the parabolic path, occupants experience a force of nearly 2 Gs (twice their body weight). Once the plane goes over the vertex of the parabola, flyers free fall inside the plane. Such motion often produces motion sickness, thus earning the aircraft its name.

Answers

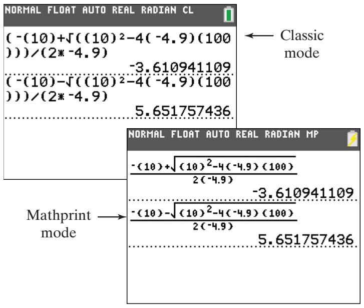
4. a. 0.95 sec **b.** 19.44 ft

c. Approximately 1.96 sec

TECHNOLOGY CONNECTIONS

Compute Solutions to a Quadratic Equation

The syntax to compute the expressions from Example 4(c) is shown for a calculator in Classic mode and in Mathprint mode. In Classic mode, parentheses are required around the numerator and denominator of the fraction and around the radicand within the square root. In Mathprint mode, select the ALPHA key followed by F1 to access the fraction template.



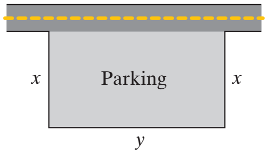
```
NORMAL FLOAT AUTO REAL RADIAN CL
(-(10)+√((10)²-4(-4.9)(100
)))/(2*-4.9)
                   -3.610941109
(-(10)-√((10)²-4(-4.9)(100
)))/(2*-4.9)
                    5.651757436
```

```
NORMAL FLOAT AUTO REAL RADIAN MP
-(10)+√(10)²-4(-4.9)(100)
-------------------------
         2(-4.9)
                   -3.610941109
-(10)-√(10)²-4(-4.9)(100)
-------------------------
         2(-4.9)
                    5.651757436
```

In Example 5, we present a type of application called an optimization problem. The goal is to maximize or minimize the value of the dependent variable by finding an optimal value of the independent variable.

EXAMPLE 5 Applying a Quadratic Function to Geometry

A parking area is to be constructed adjacent to a road. The developer has purchased 340 ft of fencing. Determine dimensions for the parking lot that would maximize the area. Then find the maximum area.



Solution:

Let x represent the width of the parking area.
Let y represent the length.
Let A represent the area.

Read the problem carefully, draw a representative diagram, and label the unknowns.

We need to find the values of x and y that maximize the area A of the rectangular region. The area is given by $A = \text{(length)(width)} = yx = xy$.

To write the area as a function of one variable only, we need an equation that relates x and y. We know that the parking area is limited by a fixed amount of fencing. That is, the sum of the lengths of the three sides to be fenced can be at most 340 ft.

$2x + y = 340$

The equation $2x + y = 340$ is called a **constraint equation.** This equation gives an implied restriction on x and y due to the limited amount of fencing.

Solve for y.

$y = 340 - 2x$

Solve the constraint equation, $2x + y = 340$ for either x or y. In this case, we have solved for y.

$A = xy$

$A(x) = x(340 - 2x)$

Substitute $340 – 2x$ for y in the equation $A = xy$.

$= -2x^2 + 340x$

Function A is a quadratic function with a negative leading coefficient. The graph of the parabola opens downward, so the vertex is the maximum point on the function.

x-coordinate of vertex:

$$x = \frac{-b}{2a} = \frac{-340}{2(-2)} = 85$$

$$y = 340 - 2(85) = 170$$

The x-coordinate of the vertex $\frac{-b}{2a}$ is the value of x that will maximize the area.

The second dimension of the parking lot can be determined from the constraint equation.

The values of x and y that would maximize the area are $x = 85$ ft and $y = 170$ ft.

$$A(85) = -2(85)^2 + 340(85) = 14{,}450$$

The value of the function at $x = 85$ gives the maximum area.

The maximum area is 14,450 ft^2.

Avoiding Mistakes

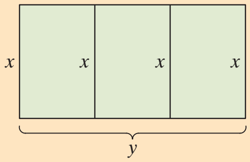
To check, verify that the value $A(85)$ is the same as the product of length and width, xy.

$$A(85) = 14{,}450 \text{ ft}^2$$

$$xy = (85 \text{ ft})(170 \text{ ft})$$

$$= 14{,}450 \text{ ft}^2 \checkmark$$

Skill Practice 5 A farmer has 200 ft of fencing and wants to build three adjacent rectangular corrals. Determine the dimensions that should be used to maximize the area, and find the area of each individual corral.



5. Create Quadratic Models Using Regression

In Section 2.5, we introduced linear regression. A regression line is a linear model based on all observed data points. In a similar fashion, we can create a quadratic function using regression. For example, suppose that a scientist growing bacteria measures the population of bacteria as a function of time. A scatter plot reveals that the data follow a curve that is approximately parabolic (Figure 3-5). In Example 6, we use a graphing calculator to find a quadratic function that models the population of the bacteria as a function of time.

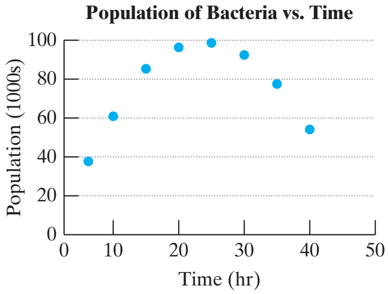


Figure 3-5

EXAMPLE 6 Creating a Quadratic Function Using Regression

The data in the table represent the population of bacteria $P(t)$ (in 1000s) versus the number of hours t since the culture was started.

a. Use regression to find a quadratic function to model the data. Round the coefficients to 3 decimal places.

b. Use the model to determine the time at which the population is the greatest. Round to the nearest hour.

c. What is the maximum population? Round to the nearest hundred.

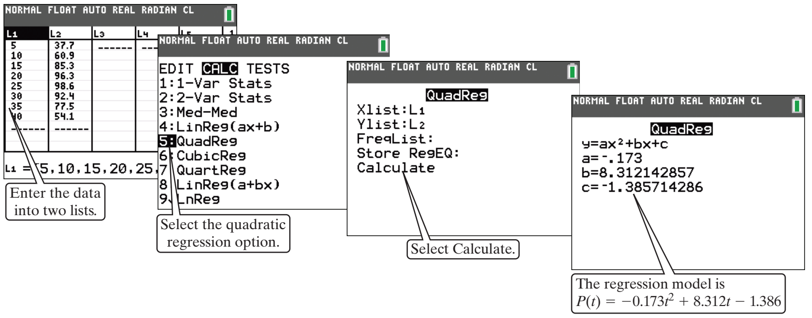
Time (hr) t	Population (1000s) $P(t)$
5	37.7
10	60.9
15	85.3
20	96.3
25	98.6
30	92.4
35	77.5
40	54.1

Answer

5. The dimensions should be $x = 25$ ft and $y = 50$ ft. The area of each individual corral is $\frac{1250}{3} = 416.\overline{6}$ ft^2.

Solution:

a. From the graph in Figure 3-5, it appears that the data follow a parabolic curve. Therefore, a quadratic model would be reasonable.



```
NORMAL FLOAT AUTO REAL RADIAN CL
L1    L2     L3      L4
5     37.7   ------  ---
10    60.9
15    85.3
20    96.3
25    98.6
30    92.4
35    77.5
40    54.1
------ ------
L1 = {5,10,15,20,25,
```

Enter the data into two lists.

```
NORMAL FLOAT AUTO REAL RADIAN CL
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7:QuartReg
8:LinReg(a+bx)
9↓LnReg
```

Select the quadratic regression option.

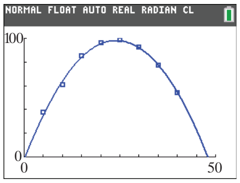
```
NORMAL FLOAT AUTO REAL RADIAN CL
QuadReg
Xlist:L1
Ylist:L2
FreqList:
Store RegEQ:
Calculate
```

Select Calculate.

```
NORMAL FLOAT AUTO REAL RADIAN CL
QuadReg
y=ax²+bx+c
a=-.173
b=8.312142857
c=-1.385714286
```

The regression model is $P(t) = -0.173t^2 + 8.312t - 1.386$

b. From the graph, the time when the population is greatest is the t-coordinate of the vertex.



$$t = \frac{-b}{2a} = \frac{-(8.312)}{2(-0.173)} \approx 24$$

The population is greatest 24 hr after the culture is started.

c. The maximum population of the bacteria is the $P(t)$ value at the vertex.

$$P(24) = -0.173(24)^2 + 8.312(24) - 1.386$$
$$\approx 98.5 \text{ The maximum number of bacteria is approximately 98,500.}$$

Skill Practice 6 The funding $f(t)$ (in $ millions) for a drug rehabilitation center is given in the table for selected years t.

t	0	3	6	9	12	15
$f(t)$	3.5	2.2	2.1	3	4.9	8

a. Use regression to find a quadratic function to model the data.

b. During what year is the funding the least? Round to the nearest year.

c. What is the minimum yearly amount of funding received? Round to the nearest million.

Answers

6. a. $f(t) = 0.060t^2 - 0.593t + 3.486$
b. Year 5 **c.** $2 million

SECTION 3.1 Practice Exercises

Prerequisite Review

R.1. Solve the equation. $x^2 + 3x - 18 = 0$

R.2. a. Find the values of x for which $f(x) = 0$.

b. Find $f(0)$.

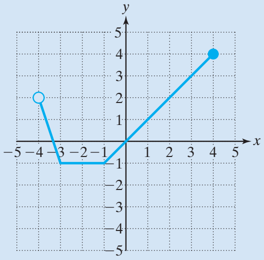
$$f(x) = 3x^2 - 7x - 20$$

R.3. Solve the equation by completing the square and applying the square root property.

$$x^2 + 8x + 12 = 0$$

R.4. Find $g\left(-\frac{1}{2}\right)$ for $g(x) = -x^2 + 2x - 4$.

R.5. Write the domain and range in interval notation.



Concept Connections

1. A function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$ is called a ______________ function.

2. The vertical line drawn through the vertex of a quadratic function is called the ______________ of symmetry.

3. Given $f(x) = a(x - h)^2 + k$ $(a \neq 0)$, the vertex of the parabola is the point ______________.

4. Given $f(x) = a(x - h)^2 + k$, if $a < 0$, the parabola opens (upward/downward) and the (minimum/maximum) value is ______________.

5. Given $f(x) = a(x - h)^2 + k$, if $a > 0$, the parabola opens (upward/downward) and the (minimum/maximum) value is ______________.

6. The graph of $f(x) = a(x - h)^2 + k$, $a \neq 0$, is a parabola and the axis of symmetry is the line given by ______________.

Objective 1: Graph a Quadratic Function Written in Vertex Form

For Exercises 7–14,

a. Determine whether the graph of the parabola opens upward or downward.

b. Identify the vertex.

c. Determine the x-intercept(s).

d. Determine the y-intercept.

e. Sketch the function.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of the function.

h. Write the domain and range in interval notation. (**See Example 1**)

7. $f(x) = -(x - 4)^2 + 1$

8. $g(x) = -(x + 2)^2 + 4$

9. $h(x) = 2(x + 1)^2 - 8$

10. $k(x) = 2(x - 3)^2 - 2$

11. $m(x) = 3(x - 1)^2$

12. $n(x) = \frac{1}{2}(x + 2)^2$

13. $p(x) = -\frac{1}{5}(x + 4)^2 + 1$

14. $q(x) = -\frac{1}{3}(x - 1)^2 + 1$

Objective 2: Write $f(x) = ax^2 + bx + c$ $(a \neq 0)$ in Vertex Form

For Exercises 15–24,

a. Write the function in vertex form.

b. Identify the vertex.

c. Determine the x-intercept(s).

d. Determine the y-intercept.

e. Sketch the function.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of the function.

h. Write the domain and range in interval notation. (**See Example 2**)

15. $f(x) = x^2 + 6x + 5$

16. $g(x) = x^2 + 8x + 7$

17. $p(x) = 3x^2 - 12x - 7$

18. $q(x) = 2x^2 - 4x - 3$

19. $c(x) = -2x^2 - 10x + 4$

20. $d(x) = -3x^2 - 9x + 8$

21. $h(x) = -2x^2 + 7x$

22. $k(x) = 3x^2 - 8x$

23. $p(x) = x^2 + 9x + 17$

24. $q(x) = x^2 + 11x + 26$

Objective 3: Find the Vertex of a Parabola by Using the Vertex Formula

For Exercises 25–32, find the vertex of the parabola by applying the vertex formula.

25. $f(x) = 3x^2 - 42x - 91$

26. $g(x) = 4x^2 - 64x + 107$

27. $k(a) = -\frac{1}{3}a^2 + 6a + 1$

28. $j(t) = -\frac{1}{4}t^2 + 10t - 5$

29. $f(c) = 4c^2 - 5$

30. $h(a) = 2a^2 + 14$

31. $P(x) = 1.2x^2 + 1.8x - 3.6$
(Write the coordinates of the vertex as decimals.)

32. $Q(x) = 7.5x^2 - 2.25x + 4.75$
(Write the coordinates of the vertex as decimals.)

For Exercises 33–42,

a. State whether the graph of the parabola opens upward or downward.

b. Identify the vertex.

c. Determine the x-intercept(s).

d. Determine the y-intercept.

e. Sketch the graph.

f. Determine the axis of symmetry.

g. Determine the minimum or maximum value of the function.

h. Write the domain and range in interval notation. (**See Example 3**)

33. $g(x) = -x^2 + 2x - 4$

34. $h(x) = -x^2 - 6x - 10$

35. $f(x) = 5x^2 - 15x + 3$

36. $k(x) = 2x^2 - 10x - 5$

37. $f(x) = 2x^2 + 3$

38. $g(x) = -x^2 - 1$

39. $f(x) = -2x^2 - 20x - 50$

40. $m(x) = 2x^2 - 8x + 8$

41. $n(x) = x^2 - x + 3$

42. $r(x) = x^2 - 5x + 7$

Objective 4: Solve Applications Involving Quadratic Functions

43. The monthly profit for a small company that makes long-sleeve T-shirts depends on the price per shirt. If the price is too high, sales will drop. If the price is too low, the revenue brought in may not cover the cost to produce the shirts. After months of data collection, the sales team determines that the monthly profit is approximated by $f(p) = -50p^2 + 1700p - 12{,}000$, where p is the price per shirt and $f(p)$ is the monthly profit based on that price. (**See Example 4**)

a. Find the price that generates the maximum profit.

b. Find the maximum profit.

c. Find the price(s) that would enable the company to break even.

44. The monthly profit for a company that makes decorative picture frames depends on the price per frame. The company determines that the profit is approximated by $f(p) = -80p^2 + 3440p - 36{,}000$, where p is the price per frame and $f(p)$ is the monthly profit based on that price.

a. Find the price that generates the maximum profit.

b. Find the maximum profit.

c. Find the price(s) that would enable the company to break even.

45. A long jumper leaves the ground at an angle of 20° above the horizontal, at a speed of 11 m/sec. The height of the jumper can be modeled by $h(x) = -0.046x^2 + 0.364x$, where h is the jumper's height in meters and x is the horizontal distance from the point of launch.

a. At what horizontal distance from the point of launch does the maximum height occur? Round to 2 decimal places.

b. What is the maximum height of the long jumper? Round to 2 decimal places.

c. What is the length of the jump? Round to 1 decimal place.

46. A firefighter holds a hose 3 m off the ground and directs a stream of water toward a burning building. The water leaves the hose at an initial speed of 16 m/sec at an angle of 30°. The height of the water can be approximated by $h(x) = -0.026x^2 + 0.577x + 3$, where $h(x)$ is the height of the water in meters at a point x meters horizontally from the firefighter to the building.

a. Determine the horizontal distance from the firefighter at which the maximum height of the water occurs. Round to 1 decimal place.

b. What is the maximum height of the water? Round to 1 decimal place.

c. The flow of water hits the house on the downward branch of the parabola at a height of 6 m. How far is the firefighter from the house? Round to the nearest meter.

47. The population $P(t)$ of a culture of the bacterium *Pseudomonas aeruginosa* is given by $P(t) = -1718t^2 + 82{,}000t + 10{,}000$, where t is the time in hours since the culture was started.

a. Determine the time at which the population is at a maximum. Round to the nearest hour.

b. Determine the maximum population. Round to the nearest thousand.

48. The gas mileage $m(x)$ (in mpg) for a certain vehicle can be approximated by $m(x) = -0.028x^2 + 2.688x - 35.012$, where x is the speed of the vehicle in mph.

a. Determine the speed at which the car gets its maximum gas mileage.

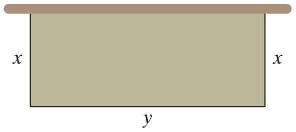
b. Determine the maximum gas mileage.

49. The sum of two positive numbers is 24. What two numbers will maximize the product? (**See Example 5**)

50. The sum of two positive numbers is 1. What two numbers will maximize the product?

51. The difference of two numbers is 10. What two numbers will minimize the product?

53. Suppose that a family wants to fence in an area of their yard for a vegetable garden to keep out deer. One side is already fenced from the neighbor's property. (**See Example 5**)

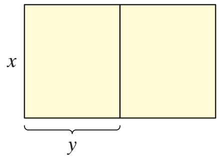


a. If the family has enough money to buy 160 ft of fencing, what dimensions would produce the maximum area for the garden?

b. What is the maximum area?

52. The difference of two numbers is 30. What two numbers will minimize the product?

54. Two chicken coops are to be built adjacent to one another using 120 ft of fencing.



a. What dimensions should be used to maximize the area of an individual coop?

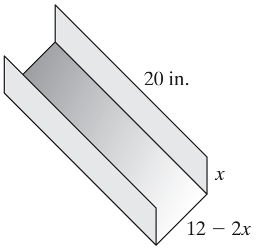
b. What is the maximum area of an individual coop?

55. A trough at the end of a gutter spout is meant to direct water away from a house. The homeowner makes the trough from a rectangular piece of aluminum that is 20 in. long and 12 in. wide. He makes a fold along the two long sides a distance of x inches from the edge.

a. Write a function to represent the volume in terms of x.

b. What value of x will maximize the volume of water that can be carried by the gutter?

c. What is the maximum volume?

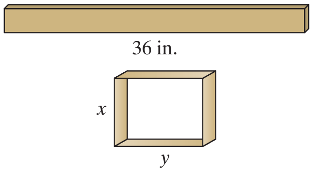


56. A rectangular frame of uniform depth for a shadow box is to be made from a 36-in. piece of wood.

a. Write a function to represent the display area in terms of x.

b. What dimensions should be used to maximize the display area?

c. What is the maximum area?



Objective 5: Create Quadratic Models Using Regression

57. *Tetanus bacillus* bacteria are cultured to produce tetanus toxin used in an inactive form for the tetanus vaccine. The amount of toxin produced per batch increases with time and then decreases as the culture becomes unstable. The variable t is the time in hours after the culture has started, and $y(t)$ is the yield of toxin in grams. (**See Example 6**)

t	8	16	24	32	40	48	56	64	72	80	88	96
$y(t)$	0.60	1.12	1.60	1.78	1.90	2.00	1.94	1.80	1.48	1.30	0.66	0.10

a. Use regression to find a quadratic function to model the data.

b. At what time is the yield the greatest? Round to the nearest hour.

c. What is the maximum yield? Round to the nearest gram.

58. Gas mileage is tested for a car under different driving conditions. At lower speeds, the car is driven in stop-and-go traffic. At higher speeds, the car must overcome more wind resistance. The variable x given in the table represents the speed (in mph) for a compact car, and $m(x)$ represents the gas mileage (in mpg).

x	25	30	35	40	45	50	55	60	65
$m(x)$	22.7	25.1	27.9	30.8	31.9	30.9	28.4	24.2	21.9

a. Use regression to find a quadratic function to model the data.

b. At what speed is the gas mileage the greatest? Round to the nearest mile per hour.

c. What is the maximum gas mileage? Round to the nearest mile per gallon.

59. Fluid runs through a drainage pipe with a 10-cm radius and a length of 30 m (3000 cm). The velocity of the fluid gradually decreases from the center of the pipe toward the edges as a result of friction with the walls of the pipe. For the data shown, $v(x)$ is the velocity of the fluid (in cm/sec) and x represents the distance (in cm) from the center of the pipe toward the edge.

x	0	1	2	3	4
$v(x)$	195.6	195.2	194.2	193.0	191.5

x	5	6	7	8	9
$v(x)$	189.8	188.0	185.5	183.0	180.0

a. The pipe is 30 m long (3000 cm). Determine how long it will take fluid to run the length of the pipe through the center of the pipe. Round to 1 decimal place.

b. Determine how long it will take fluid at a point 9 cm from the center of the pipe to run the length of the pipe. Round to 1 decimal place.

c. Use regression to find a quadratic function to model the data.

d. Use the model from part (c) to predict the velocity of the fluid at a distance 5.5 cm from the center of the pipe. Round to 1 decimal place.

60. The braking distance required for a car to stop depends on numerous variables such as the speed of the car, the weight of the car, reaction time of the driver, and the coefficient of friction between the tires and the road. For a certain vehicle on one stretch of highway, the braking distances $d(s)$ (in ft) are given for several different speeds s (in mph).

s	30	35	40	45	50
$d(s)$	109	134	162	191	223

s	55	60	65	70	75
$d(s)$	256	291	328	368	409

a. Use regression to find a quadratic function to model the data.

b. Use the model from part (a) to predict the stopping distance for the car if it is traveling 62 mph before the brakes are applied. Round to the nearest foot.

c. Suppose that the car is traveling 53 mph before the brakes are applied. If a deer is standing in the road at a distance of 245 ft from the point where the brakes are applied, will the car hit the deer?

Mixed Exercises

For Exercises 61–64, given a quadratic function defined by $f(x) = ax^2 + bx + c$ $(a \neq 0)$, answer true or false. If an answer is false, explain why.

61. The graph of f can have two y-intercepts.

62. The graph of f can have two x-intercepts.

63. If $a < 0$, then the vertex of the parabola is the maximum point on the graph of f.

64. The axis of symmetry of the graph of f is the line defined by $y = c$.

For Exercises 65–70, determine the number of x-intercepts of the graph of $f(x) = ax^2 + bx + c$ $(a \neq 0)$, based on the discriminant of the related equation $f(x) = 0$. (*Hint*: Recall that the discriminant is $b^2 - 4ac$.)

65. $f(x) = 4x^2 + 12x + 9$

66. $f(x) = 25x^2 - 20x + 4$

67. $f(x) = -x^2 - 5x + 8$

68. $f(x) = -3x^2 + 4x + 9$

69. $f(x) = -3x^2 + 6x - 11$

70. $f(x) = -2x^2 + 5x - 10$

For Exercises 71–78, given a quadratic function defined by $f(x) = a(x - h)^2 + k$ $(a \neq 0)$, match the graph with the function based on the conditions given.

71. $a > 0$, $h < 0$, $k > 0$

72. $a > 0$, $h < 0$, $k < 0$

73. $a < 0$, $h < 0$, $k < 0$

74. $a < 0$, $h < 0$, $k > 0$

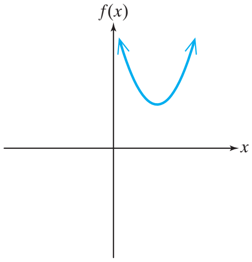
75. $a > 0$, axis of symmetry $x = 2$, $k < 0$

76. $a < 0$, axis of symmetry $x = 2$, $k > 0$

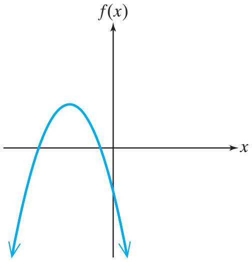
77. $a < 0$, $h = 2$, maximum value equals -2

78. $a > 0$, $h = 2$, minimum value equals 2

a.



b.



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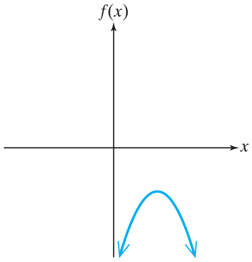

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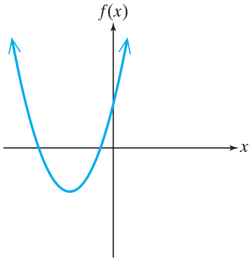
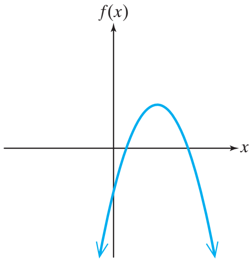
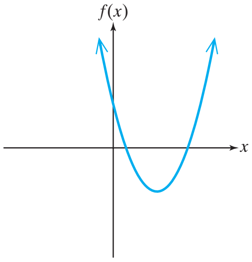


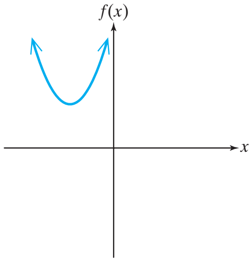
e.



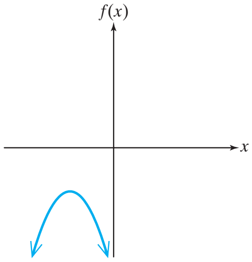
f.



g.



h.



Write About It

79. Explain why a parabola opening upward has a minimum value but no maximum value. Use the graph of $f(x) = x^2$ to explain.

80. Explain why a quadratic function whose graph opens downward with vertex $(4, -3)$ has no x-intercept.

81. Explain why a quadratic function given by $f(x) = ax^2 + bx + c$ cannot have two y-intercepts.

82. Explain how to use the discriminant to determine the number of x-intercepts for the graph of $f(x) = ax^2 + bx + c$.

83. If a quadratic function given by $y = f(x)$ has x-intercepts of $(2, 0)$ and $(6, 0)$, explain why the vertex must be $(4, f(4))$.

84. Given an equation of a parabola in the form $y = a(x - h)^2 + k$, explain how to determine by inspection if the parabola has no x-intercepts.

Expanding Your Skills

For Exercises 85–88, define a quadratic function $y = f(x)$ that satisfies the given conditions.

85. Vertex $(2, -3)$ and passes through $(0, 5)$

86. Vertex $(-3, 1)$ and passes through $(0, -17)$

87. Axis of symmetry $x = 4$, maximum value 6, passes through $(1, 3)$

88. Axis of symmetry $x = -2$, minimum value 5, passes through $(2, 13)$

For Exercises 89–92, find the value of *b* or *c* that gives the function the given minimum or maximum value.

89. $f(x) = 2x^2 + 12x + c$; minimum value -9

90. $f(x) = 3x^2 + 12x + c$; minimum value -4

91. $f(x) = -x^2 + bx + 4$; maximum value 8

92. $f(x) = -x^2 + bx - 2$; maximum value 7

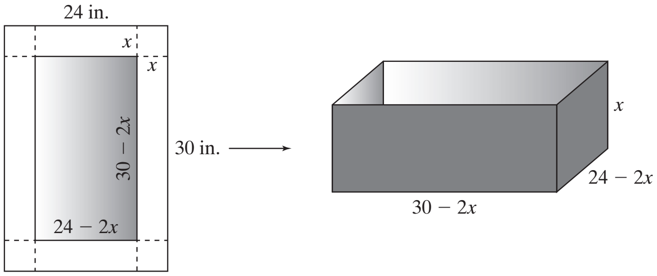
SECTION 3.2 Introduction to Polynomial Functions

OBJECTIVES

1. Determine the End Behavior of a Polynomial Function
2. Identify Zeros and Multiplicities of Zeros
3. Apply the Intermediate Value Theorem
4. Sketch a Polynomial Function

1. Determine the End Behavior of a Polynomial Function

A solar oven is to be made from an open box with reflective sides. Each box is made from a 30-in. by 24-in. rectangular sheet of aluminum with squares of length x (in inches) removed from each corner. Then the flaps are folded up to form an open box.



The volume $V(x)$ (in cubic inches) of the box is given by

$$V(x) = 4x^3 - 108x^2 + 720x, \text{ where } 0 < x < 12.$$

From the graph of $y = V(x)$ (Figure 3-6), the maximum volume appears to occur when squares of approximately 4 inches in length are cut from the corners of the sheet of aluminum. See Exercise 99.

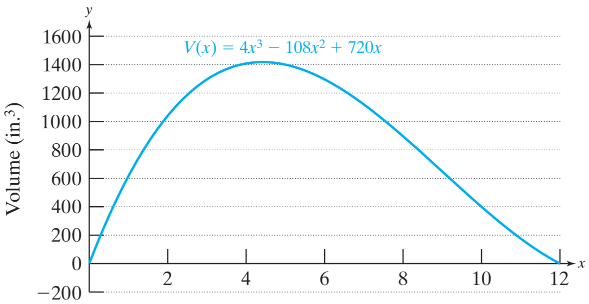


Figure 3-6

The function defined by $V(x) = 4x^3 - 108x^2 + 720x$ is an example of a polynomial function of degree 3.

Definition of a Polynomial Function

Let n be a whole number and $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ be real numbers, where $a_n \neq 0$. Then a function defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

is called a **polynomial function of degree *n*.**

The coefficients of each term of a polynomial function are real numbers, and the exponents on x must be whole numbers.

Polynomial Function	Not a Polynomial Function
$f(x) = 4x^5 - 3x^4 + 2x^2$	$f(x) = 4\sqrt{x} - \frac{3}{x} + (3 + 2i)x^2$

$\sqrt{x} = x^{1/2}$
Exponent not a whole number

$3/x = 3x^{-1}$
Exponent not a whole number

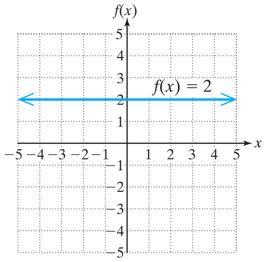
$(3 + 2i)$
Coefficient not a real number

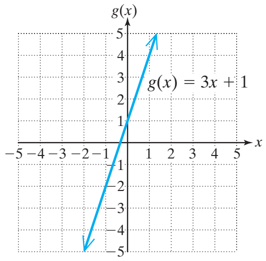
TIP A third-degree polynomial function is referred to as a *cubic* polynomial function.

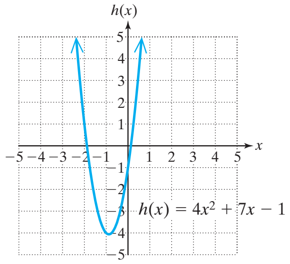
A fourth-degree polynomial function is referred to as a *quartic* polynomial function.

We have already studied several special cases of polynomial functions. For example:

$f(x) = 2$	constant function	(polynomial function, degree 0)
$g(x) = 3x + 1$	linear function	(polynomial function, degree 1)
$h(x) = 4x^2 + 7x - 1$	quadratic function	(polynomial function, degree 2)

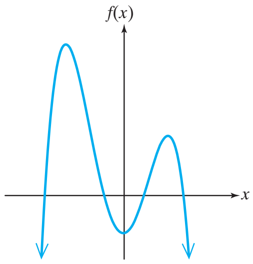




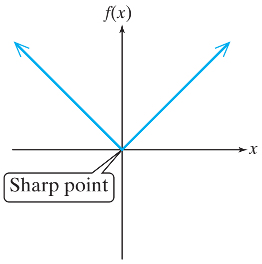


The domain of a polynomial function is all real numbers. Furthermore, the graph of a polynomial function is both continuous and smooth. Informally, a continuous function can be drawn without lifting the pencil from the paper. A smooth function has no sharp corners or points. For example, the first curve shown here could be a polynomial function, but the last three are not polynomial functions.

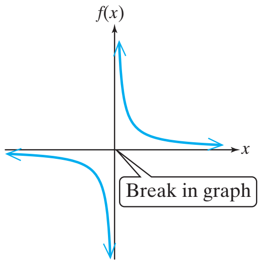
Smooth and Continuous



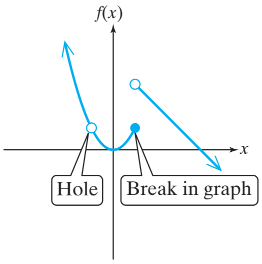
Not Smooth



Not Continuous



Not Continuous



To begin our analysis of polynomial functions, we first consider the graphs of functions of the form $f(x) = ax^n$, where a is a real number and n is a positive integer. These fall into a category of functions called **power functions.** The graphs of three power functions with even degrees and positive coefficients are shown in Figure 3-7. The graphs of three power functions with odd degrees and positive coefficients are shown in Figure 3-8.

TIP For a positive integer n, the graph of the power function $y = x^n$ becomes "flatter" near the x-intercept for higher powers of n.

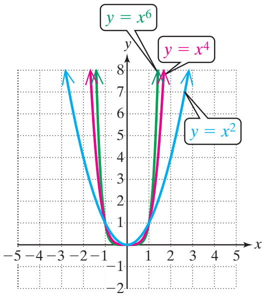


Figure 3-7

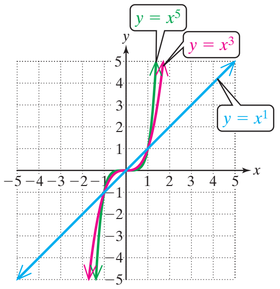


Figure 3-8

From Figure 3-7, notice that for even powers of n, the behavior of $y = x^n$ is similar to the graph of $y = x^2$ with variations on the "steepness" of the curve. Figure 3-8 shows that for odd powers, the behavior of $y = x^n$ with $n \geq 3$ is similar to the graph of $y = x^3$. For any power function $y = ax^n$, the coefficient a will impose a vertical shrink or stretch on the graph of $y = x^n$ by a factor of $|a|$. If $a < 0$, then the graph is reflected across the x-axis.

Power functions are helpful to analyze the "end behavior" of a polynomial function with multiple terms. The end behavior is the general direction that the fuction follows as x approaches ∞ or $-\infty$. To describe end behavior, we have the following notation.

Notation for Infinite Behavior of $y = f(x)$

$x \to \infty$	is read as "x approaches infinity." This means that x becomes infinitely large in the positive direction.
$x \to -\infty$	is read as "x approaches negative infinity." This means that x becomes infinitely "large" in the negative direction.
$f(x) \to \infty$	is read as "$f(x)$ approaches infinity." This means that the y value becomes infinitely large in the positive direction.
$f(x) \to -\infty$	is read as "$f(x)$ approaches negative infinity." This means that the y value becomes infinitely "large" in the negative direction.

Consider the function defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

The leading term has the greatest exponent on x.

The leading term has the greatest exponent on x. Therefore, as $|x|$ gets large (that is, as $x \to \infty$ or as $x \to -\infty$), the leading term will be relatively larger in absolute value than all other terms. In fact, x^n will eventually be greater in absolute value than the *sum* of all other terms. Therefore, the end behavior of the function is dictated only by the leading term, and the graph of the function far to the left and far to the right will follow the general behavior of the power function $y = ax^n$.

The Leading Term Test

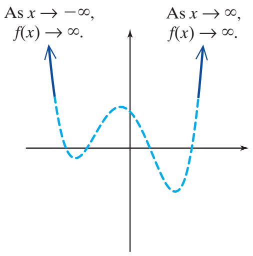
Consider a polynomial function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$

As $x \to \infty$ or as $x \to -\infty$, f eventually becomes forever increasing or forever decreasing and will follow the general behavior of $y = a_n x^n$.

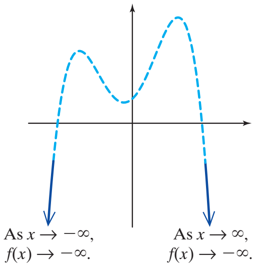
n is even

a_n positive



End behavior:
up left/up right

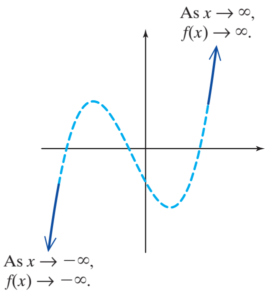
a_n negative



End behavior:
down left/down right

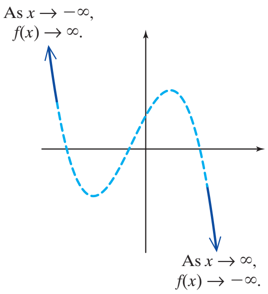
n is odd

a_n positive



End behavior:
down left/up right

a_n negative



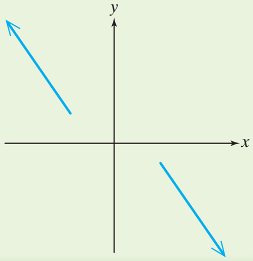
End behavior:
up left/down right

EXAMPLE 1 Determining End Behavior

Use the leading term to determine the end behavior of the graph of the function.

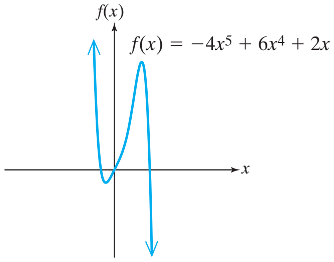
a. $f(x) = -4x^5 + 6x^4 + 2x$ **b.** $g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2$

TIP The graph of $y = f(x)$ from Example 1(a) will exhibit the same behavior as the graph of the power function $y = -4x^5$ for values of x far to the right and far to the left. This is similar to the graph of $y = x^5$ reflected across the x-axis.



Solution:

a. $f(x) = \underbrace{-4}_{\text{negative}}x^{\underbrace{5}_{\text{odd}}} + 6x^4 + 2x$



The leading coefficient is negative and the degree is odd. By the leading term test, the end behavior is up to the left and down to the right.

As $x \to -\infty$, $f(x) \to \infty$.
As $x \to \infty$, $f(x) \to -\infty$.

b. $g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2$

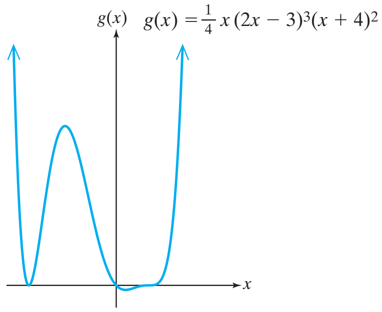
$g(x) = \frac{1}{4}x(2x - 3)^3(x + 4)^2 = \overbrace{2}^{\text{positive}}x^{\overbrace{6}^{\text{even}}} + \cdots$

To determine the leading term, multiply the leading terms from each factor. That is,

$\frac{1}{4}x(2x)^3(x)^2 = 2x^6$.

The leading coefficient is positive and the degree is even. By the leading term test, the end behavior is up to the left and up to the right.

As $x \to -\infty$, $f(x) \to \infty$.
As $x \to \infty$, $f(x) \to \infty$.



Skill Practice 1 Use the leading term to determine the end behavior of the graph of the function.

a. $f(x) = -0.3x^4 - 5x^2 - 3x + 4$ **b.** $g(x) = \frac{6}{7}(x - 9)^4(x + 4)^2(3x - 5)$

2. Identify Zeros and Multiplicities of Zeros

Consider a polynomial function defined by $y = f(x)$. The values of x in the domain of f for which $f(x) = 0$ are called the **zeros** of the function. These are the real solutions (or **roots**) of the equation $f(x) = 0$ and correspond to the x-intercepts of the graph of $y = f(x)$.

Answers

1. **a.** Down to the left, down to the right.
   As $x \to -\infty$, $f(x) \to -\infty$.
   As $x \to \infty$, $f(x) \to -\infty$.
   b. Down to the left, up to the right.
   As $x \to -\infty$, $f(x) \to -\infty$.
   As $x \to \infty$, $f(x) \to \infty$.

EXAMPLE 2 Determining the Zeros of a Polynomial Function

Find the zeros of the function defined by $f(x) = x^3 + x^2 - 9x - 9$.

Solution:

$f(x) = x^3 + x^2 - 9x - 9$

$0 = x^3 + x^2 - 9x - 9$ To find the zeros of f, set $f(x) = 0$ and solve for x.

$0 = x^2(x + 1) - 9(x + 1)$ Factor by grouping.

$0 = (x + 1)(x^2 - 9)$

$0 = (x + 1)(x - 3)(x + 3)$ Factor the difference of squares.

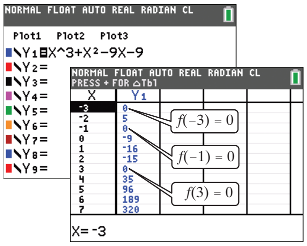
$x = -1, x = 3, x = -3$ Set each factor equal to zero and solve for x.

The zeros of f are -1, 3, and -3.

The graph of f is shown in Figure 3-9. The zeros of the function are real numbers and correspond to the x-intercepts of the graph. By inspection, we can evaluate $f(0) = -9$, indicating that the y-intercept is $(0, -9)$.

Check:

A table of points can be used to check that $f(-1)$, $f(3)$, and $f(-3)$ all equal 0.



```
NORMAL FLOAT AUTO REAL RADIAN CL
Plot1  Plot2  Plot3
\Y1■X^3+X²-9X-9
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
\Y8=
\Y9=
```

```
NORMAL FLOAT AUTO REAL RADIAN CL
PRESS + FOR △Tbl
  X    Y1
 -3    0
 -2    5
 -1    0
  0   -9
  1  -16
  2  -15
  3    0
  4   35
  5   96
  6  189
  7  320
X=-3
```

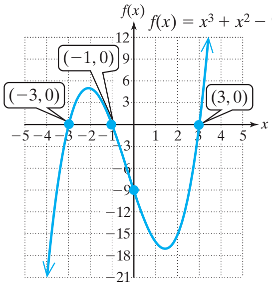


Figure 3-9

Skill Practice 2 Find the zeros of the function defined by $f(x) = 4x^3 - 4x^2 - 25x + 25$.

EXAMPLE 3 Determining the Zeros of a Polynomial Function

Find the zeros of the function defined by $f(x) = -x^3 + 8x^2 - 16x$.

Solution:

$f(x) = -x^3 + 8x^2 - 16x$ To find the zeros of f, set $f(x) = 0$ and solve for x.

$0 = -x(x^2 - 8x + 16)$ Factor out the GCF.

$0 = -x(x - 4)^2$ Factor the perfect square trinomial.

$x = 0, x = 4$ Set each factor equal to zero and solve for x.

The zeros of f are 0 and 4.

The graph of f is shown in Figure 3-10. The zeros of the function are real numbers and correspond to the x-intercepts $(0, 0)$ and $(4, 0)$.

The leading term of $f(x)$ is $-x^3$. The coefficient is negative and the exponent is odd. The graph shows the end behavior up to the left and down to the right as expected.

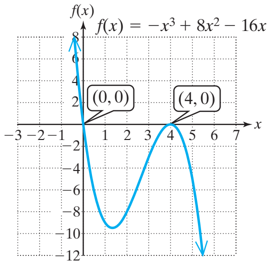


Figure 3-10

Skill Practice 3 Find the zeros of the function defined by $f(x) = x^3 + 10x^2 + 25x$.

Answers

2. $1, \frac{5}{2}, -\frac{5}{2}$

3. 0, −5

From Example 3, $f(x) = -x^3 + 8x^2 - 16x$ can be written as a product of linear factors:

$$f(x) = -x(x - 4)^2$$

Notice that the factor $(x - 4)$ appears to the second power. Therefore, we say that the corresponding zero, 4, has a multiplicity of 2. In general, we say that if a polynomial function has a factor $(x - c)$ that appears exactly k times, then c is a **zero of multiplicity *k*.** For example, consider:

$g(x) = x^2(x - 2)^3(x + 4)^7$
0 is a zero of multiplicity 2.
2 is a zero of multiplicity 3.
-4 is a zero of multiplicity 7.

The graph of a polynomial function behaves in the following manner based on the multiplicity of the zeros.

Touch Points and Cross Points

Let f be a polynomial function and let c be a real zero of f. Then the point $(c, 0)$ is an x-intercept of the graph of f. Furthermore,

- If c is a zero of odd multiplicity, then the graph *crosses* the x-axis at c. The point $(c, 0)$ is called a **cross point.**
- If c is a zero of even multiplicity, then the graph *touches* the x-axis at c and turns back around (does not cross the x-axis). The point $(c, 0)$ is called a **touch point.**

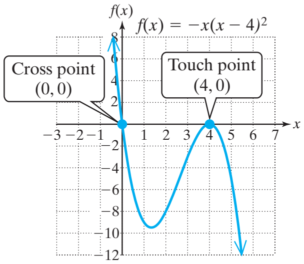


Figure 3-11

To illustrate the behavior of a polynomial function at its real zeros, consider the graph of $f(x) = -x(x - 4)^2$ from Example 3 (Figure 3-11).

- 0 has a multiplicity of 1 (odd multiplicity). The graph *crosses* the x-axis at (0, 0).
- 4 has a multiplicity of 2 (even multiplicity). The graph *touches* the x-axis at (4, 0) and turns back around.

EXAMPLE 4 Determining Zeros and Multiplicities

Determine the zeros and their multiplicities for the given functions.

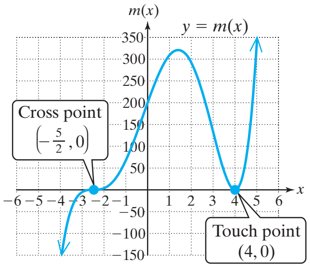
a. $m(x) = \frac{1}{10}(x - 4)^2(2x + 5)^3$ **b.** $n(x) = x^4 - 2x^2$

Solution:

a. $m(x) = \frac{1}{10}(x - 4)^{\overset{\text{even}}{2}}(2x + 5)^{\overset{\text{odd}}{3}}$ The function is factored into linear factors. The zeros are 4 and $-\frac{5}{2}$.

The function has a zero of 4 with multiplicity 2 (even). The graph has a touch point at (4, 0).

The function has a zero of $-\frac{5}{2}$ with multiplicity 3 (odd). The graph has a cross point at $\left(-\frac{5}{2}, 0\right)$.

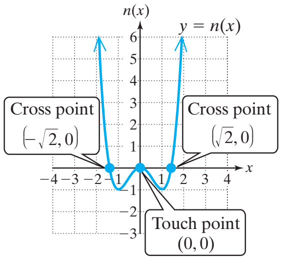


b. $n(x) = x^4 - 2x^2$
$= x^2(x^2 - 2)$
$= x^2(x - \sqrt{2})^1(x + \sqrt{2})^1$

The function has a zero of 0 with multiplicity 2 (even). The graph has a touch point at (0, 0).

The function has a zero of $\sqrt{2}$ with multiplicity 1 (odd). The graph has a cross point at $(\sqrt{2}, 0) \approx (1.41, 0)$.

The function has a zero of $-\sqrt{2}$ with multiplicity 1 (odd). The graph has a cross point at $(-\sqrt{2}, 0) \approx (-1.41, 0)$.



Skill Practice 4 Determine the zeros and their multiplicities for the given functions.

a. $p(x) = -\frac{3}{5}(x + 3)^4(5x - 1)^5$ **b.** $q(x) = 2x^6 - 14x^4$

3. Apply the Intermediate Value Theorem

In Examples 2–4, the zeros of the functions were easily identified by first factoring the polynomial. However, in most cases, the real zeros of a polynomial are difficult or impossible to determine algebraically. For example, the function given by $f(x) = x^4 + 6x^3 - 26x + 15$ has zeros of $-1 \pm \sqrt{6}$ and $-2 \pm \sqrt{7}$. At this point, we do not have the tools to find the zeros of this function analytically. However, we can use the intermediate value theorem to help us search for zeros of a polynomial function and approximate their values.

Intermediate Value Theorem

Let f be a polynomial function. For $a < b$, if $f(a)$ and $f(b)$ have opposite signs, then f has at least one zero on the interval $[a, b]$.

EXAMPLE 5 Applying the Intermediate Value Theorem

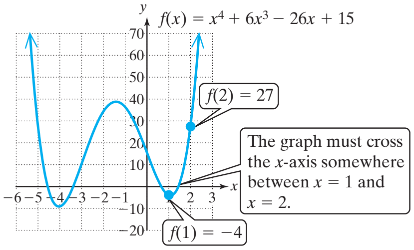
Show that $f(x) = x^4 + 6x^3 - 26x + 15$ has a zero on the interval [1, 2].

Solution:

$f(x) = x^4 + 6x^3 - 26x + 15$
$f(1) = (1)^4 + 6(1)^3 - 26(1) + 15 = -4$
$f(2) = (2)^4 + 6(2)^3 - 26(2) + 15 = 27$

Since $f(1)$ and $f(2)$ have opposite signs, then by the intermediate value theorem, we know that the function must have at least one zero on the interval [1, 2].

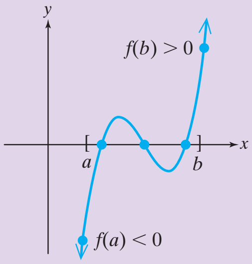
The actual value of the zero on the interval [1, 2] is $-1 + \sqrt{6} \approx 1.45$.



Skill Practice 5 Show that $f(x) = x^4 + 6x^3 - 26x + 15$ has a zero on the interval [−4, −3].

Avoiding Mistakes

For a polynomial function f, if $f(a)$ and $f(b)$ have opposite signs, then f must have at least one zero on the interval $[a, b]$. This includes the possibility that f may have more than one zero on $[a, b]$.



TIP It is important to note that if the signs of $f(a)$ and $f(b)$ are the same, then the intermediate value theorem is inconclusive.

Answers

4. a. −3 (multiplicity 4) and $\frac{1}{5}$ (multiplicity 5)
b. 0 (multiplicity 4), $\sqrt{7}$ (multiplicity 1), and $-\sqrt{7}$ (multiplicity 1)

5. $f(-4) = -9$ and $f(-3) = 12$. Since $f(-4)$ and $f(-3)$ have opposite signs, then the intermediate value theorem guarantees the existence of at least one zero on the interval [−4, −3].

The intermediate value theorem can be used repeatedly in a technique called the bisection method to approximate the value of a zero. See the online group activity "Investigating the Bisection Method for Finding Zeros."

Point of Interest

The modern definition of a computer is a programmable device designed to carry out a sequence of arithmetic or logical operations. However, the word "computer" originally referred to a person who did such calculations using paper and pencil. "Human computers" were notably used in the eighteenth century to predict the path of Halley's comet and to produce astronomical tables critical to surveying and navigation. Later, during World Wars I and II, human computers developed ballistic firing tables that would describe the trajectory of a shell.

Computing tables of values was very time consuming, and the "computers" would often interpolate to find intermediate values within a table. Interpolation is a method by which intermediate values between two numbers are estimated. Often the interpolated values were based on a polynomial function.

4. Sketch a Polynomial Function

The graph of a polynomial function may also have "turning points." These correspond to relative maxima and minima. For example, consider $f(x) = x(x + 2)(x - 2)^2$. See Figure 3-12.

Multiplying the leading terms within the factors, we have a leading term of $(x)(x)(x)^2 = x^4$. Therefore, the end behavior of the graph is up to the left and up to the right.

TIP Even with advanced techniques from calculus or the use of a graphing utility, it is often difficult or impossible to find the exact location of the turning points of a polynomial function.

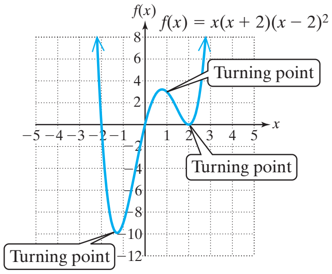


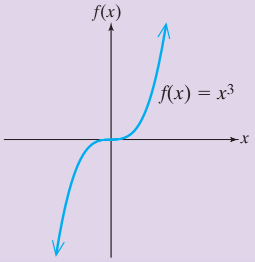
Figure 3-12

Starting from the far left, the graph of f decreases to the x-intercept of -2. Since -2 is a zero with an odd multiplicity, the graph must cross the x-axis at -2. For the same reason, the graph must cross the x-axis again at the origin. Therefore, somewhere between $x = -2$ and $x = 0$, the graph must "turn around." This point is called a "turning point."

The turning points of a polynomial function are the points where the function changes from increasing to decreasing or vice versa.

Avoiding Mistakes

A polynomial of degree n may have fewer than $n - 1$ turning points. For example, $f(x) = x^3$ is a degree 3 polynomial function (indicating that it could have a maximum of two turning points), yet the graph has no turning points.



Number of Turning Points of a Polynomial Function

Let f represent a polynomial function of degree n. Then the graph of f has at most $n - 1$ turning points.

At this point we are ready to outline a strategy for sketching a polynomial function.

Graphing a Polynomial Function

To graph a polynomial function defined by $y = f(x)$,

1. Use the leading term to determine the end behavior of the graph.
2. Determine the y-intercept by evaluating $f(0)$.
3. Determine the real zeros of f and their multiplicities (these are the x-intercepts of the graph of f).
4. Plot the x- and y-intercepts and sketch the end behavior.
5. Draw a sketch starting from the left-end behavior. Connect the x- and y-intercepts in the order that they appear from left to right using these rules:
   - The curve will cross the x-axis at an x-intercept if the corresponding zero has an odd multiplicity.
   - The curve will touch but not cross the x-axis at an x-intercept if the corresponding zero has an even multiplicity.
6. If a test for symmetry is easy to apply, use symmetry to plot additional points. Recall that
   - f is an even function (symmetric to the y-axis) if $f(-x) = f(x)$.
   - f is an odd function (symmetric to the origin) if $f(-x) = -f(x)$.
7. Plot more points if a greater level of accuracy is desired. In particular, to estimate the location of turning points, find several points between two consecutive x-intercepts.

In Examples 6 and 7, we demonstrate the process of graphing a polynomial function.

EXAMPLE 6 Graphing a Polynomial Function

Graph $f(x) = x^3 - 9x$.

Solution:

$f(x) = x^3 - 9x$

1. The leading term is x^3. The end behavior is down to the left and up to the right.

The exponent on the leading term is odd and the leading coefficient is positive.

2. $f(0) = (0)^3 - 9(0) = 0$

The y-intercept is $(0, 0)$.

Determine the y-intercept by evaluating $f(0)$.

3. $0 = x^3 - 9x$

$0 = x(x^2 - 9)$

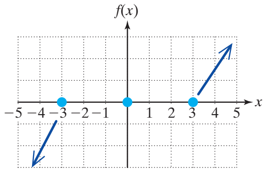
$0 = x(x - 3)(x + 3)$

The zeros of the function are 0, 3, and -3, and each has a multiplicity of 1.

Find the real zeros of f by solving for the real solutions to the equation $f(x) = 0$.

The zeros are real numbers and correspond to x-intercepts on the graph. Since the multiplicity of each zero is an odd number, the graph will cross the x-axis at the zeros.

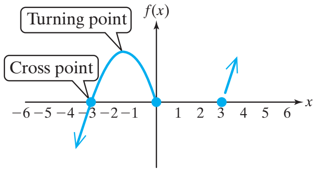
4.

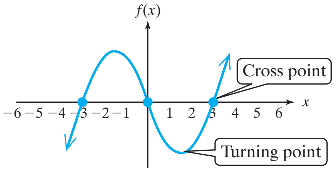


Plot the x- and y-intercepts and sketch the end behavior.

5. Moving from left to right, the curve increases from the far left and then crosses the x-axis at -3. The graph must have a turning point between $x = -3$ and $x = 0$ so that the curve can pass through the next x-intercept of (0, 0).

The graph crosses the x-axis at $x = 0$. The graph must then have another turning point between $x = 0$ and $x = 3$ so that the curve can pass through the next x-intercept of (3, 0). Finally, the graph crosses the x-axis at $x = 3$ and continues to increase to the far right.





6. $f(x) = x^3 - 9x$

$$f(-x) = (-x)^3 - 9(-x) \qquad -f(x) = -(x^3 - 9x)$$
$$= -x^3 + 9x \xleftrightarrow[\text{(same)}]{f(-x) = -f(x)} = -x^3 + 9x$$

Testing for symmetry, we see that $f(-x) = -f(x)$. Therefore, f is an odd function and is symmetric with respect to the origin.

7. If more accuracy is desired, plot additional points. In this case, since f is symmetric to the origin, if a point (x, y) is on the graph, then so is $(-x, -y)$. The graph of f is shown in Figure 3-13.

x	$f(x)$
1	−8
2	−10
4	28

Use symmetry. $\longrightarrow$

x	$f(x)$
−1	8
−2	10
−4	−28

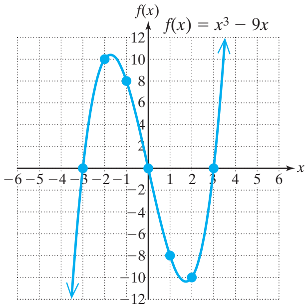


Figure 3-13

TIP Techniques of calculus can be used to find the exact coordinates of the turning points of the polynomial function in Example 6.

Skill Practice 6 Graph $g(x) = -x^3 + 4x$.

EXAMPLE 7 Graphing a Polynomial Function

Graph $g(x) = -0.1(x - 1)(x + 2)(x - 4)^2$.

Solution:

$g(x) = -0.1(x - 1)(x + 2)(x - 4)^2$

1. Multiplying the leading terms within the factors, we have a leading term of $-0.1(x)(x)(x)^2 = -0.1x^4$. The end behavior is down to the left and down to the right.

The exponent on the leading term is even and the leading coefficient is negative.

2. $g(0) = -0.1(0 - 1)(0 + 2)(0 - 4)^2 = 3.2$
The y-intercept is (0, 3.2).

Determine the y-intercept by evaluating $g(0)$.

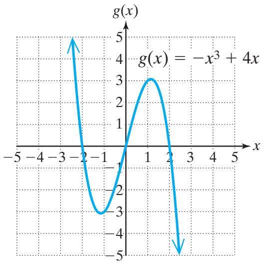
3. $0 = -0.1(x - 1)(x + 2)(x - 4)^2$
The zeros of the function are 1, −2, and 4.
The multiplicity of 1 is 1.
The multiplicity of −2 is 1.
The multiplicity of 4 is 2.

Find the real zeros of g by solving for the real solutions of the equation $g(x) = 0$.

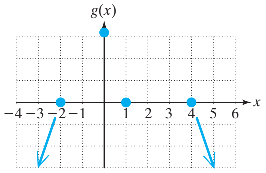
The zeros are real numbers and correspond to x-intercepts on the graph: (1, 0), (−2, 0), and (4, 0).

Answer

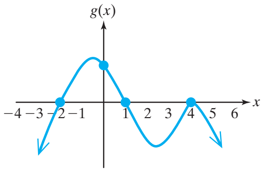
6.



4.



Plot the x- and y-intercepts and sketch the end behavior.



5. Moving from left to right, the curve increases from the far left. It then crosses the x-axis at $x = -2$ and turns back around to pass through the next x-intercept at $x = 1$.

The curve has another turning point between $x = 1$ and $x = 4$ so that it can touch the x-axis at 4. From there it turns back downward and continues to decrease to the far right.

6. From our preliminary sketch in step 5, we see that the function is not symmetric with respect to either the y-axis or origin.

7. If more accuracy is desired, plot additional points. The graph is shown in Figure 3-14.

x	$g(x)$
−3	−19.6
−1	5
2	−1.6
3	−1
5	−2.8

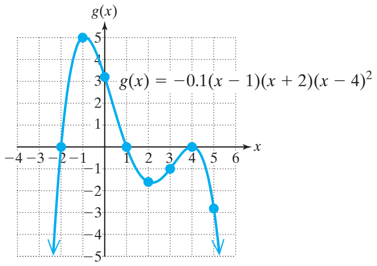
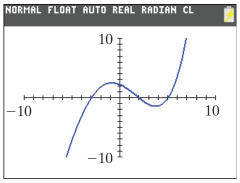


Figure 3-14

Skill Practice 7 Graph $h(x) = 0.5x(x - 1)(x + 3)^2$.

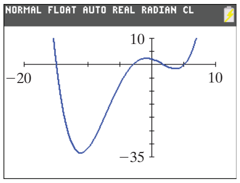
TECHNOLOGY CONNECTIONS

Using a Graphing Utility to Graph a Polynomial Function



It is important to have a strong knowledge of algebra to use a graphing utility effectively. For example, consider the graph of $f(x) = 0.005(x - 2)(x + 3)(x - 5)(x + 15)$ on the standard viewing window.

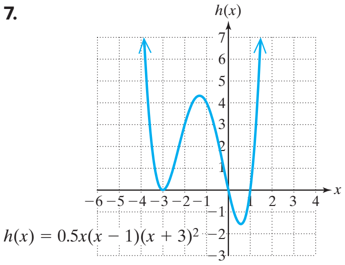
From the leading term, $0.005x^4$, we know that the end behavior should be up to the left and up to the right. Furthermore, the function has four real zeros (2, −3, 5, and −15), and should have four corresponding x-intercepts. Therefore, on the standard viewing window, the calculator does not show the key features of the graph.



By graphing f on the window [−20, 10, 2] by [−35, 10, 5], we see the end behavior displayed correctly, all four x-intercepts, and the turning points (there should be at most 3).

Answer

7.



SECTION 3.2 Practice Exercises

Prerequisite Review

For Exercises R.1–R.2, solve the equation.

R.1. $3x^3 + 21x^2 - 54x = 0$

R.2. $5x^3 + 6x^2 - 20x - 24 = 0$

For Exercises R.3–R.5, use transformations to graph the given function.

R.3. $m(x) = x^3 - 5$

R.4. $f(x) = (x + 2)^2 - 4$

R.5. $g(x) = (3x - 6)^2$

Concept Connections

1. A function defined by $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ where $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are real numbers and $a_n \neq 0$ is called a ________________ function.

2. The function given by $f(x) = -3x^5 + \sqrt{2}x + \frac{1}{2}x$ (is/is not) a polynomial function.

3. The function given by $f(x) = -3x^5 + 2\sqrt{x} + \frac{2}{x}$ (is/is not) a polynomial function.

4. A quadratic function is a polynomial function of degree ________________.

5. A linear function is a polynomial function of degree ________________.

6. The values of x in the domain of a polynomial function f for which $f(x) = 0$ are called the ________________ of the function.

7. What is the maximum number of turning points of the graph of $f(x) = -3x^6 - 4x^5 - 5x^4 + 2x^2 + 6$?

8. If the graph of a polynomial function has 3 turning points, what is the minimum degree of the function?

9. If c is a real zero of a polynomial function and the multiplicity is 3, does the graph of the function cross the x-axis or touch the x-axis (without crossing) at $(c, 0)$?

10. If c is a real zero of a polynomial function and the multiplicity is 6, does the graph of the function cross the x-axis or touch the x-axis (without crossing) at $(c, 0)$?

11. Suppose that f is a polynomial function and that $a < b$. If $f(a)$ and $f(b)$ have opposite signs, then what conclusion can be drawn from the intermediate value theorem?

12. What is the leading term of $f(x) = -\frac{1}{3}(x - 3)^4(3x + 5)^2$?

Objective 1: Determine the End Behavior of a Polynomial Function

For Exercises 13–20, determine the end behavior of the graph of the function. (See Example 1)

13. $f(x) = -3x^4 - 5x^2 + 2x - 6$

14. $g(x) = -\frac{1}{2}x^6 + 8x^4 - x^3 + 9$

15. $h(x) = 12x^5 + 8x^4 - 4x^3 - 8x + 1$

16. $k(x) = 11x^7 - 4x^2 + 9x + 3$

17. $m(x) = -4(x - 2)(2x + 1)^2(x + 6)^4$

18. $n(x) = -2(x + 4)(3x - 1)^3(x + 5)$

19. $p(x) = -2x^2(3 - x)(2x - 3)^3$

20. $q(x) = -5x^4(2 - x)^3(2x + 5)$

Objective 2: Identify Zeros and Multiplicities of Zeros

21. Given the function defined by $g(x) = -3(x - 1)^3(x + 5)^4$, the value 1 is a zero with multiplicity ________________, and the value -5 is a zero with multiplicity ________________.

22. Given the function defined by $h(x) = \frac{1}{2}x^5(x + 0.6)^3$, the value 0 is a zero with multiplicity ________________, and the value -0.6 is a zero with multiplicity ________________.

For Exercises 23–38, find the zeros of the function and state the multiplicities. (See Examples 2–4)

23. $f(x) = x^3 + 2x^2 - 25x - 50$
24. $g(x) = x^3 + 5x^2 - x - 5$
25. $h(x) = -6x^3 - 9x^2 + 60x$
26. $k(x) = -6x^3 + 26x^2 - 28x$
27. $m(x) = x^5 - 10x^4 + 25x^3$
28. $n(x) = x^6 + 4x^5 + 4x^4$
29. $p(x) = -3x(x + 2)^3(x + 4)$
30. $q(x) = -2x^4(x + 1)^3(x - 2)^2$
31. $t(x) = 5x(3x - 5)(2x + 9)(x - \sqrt{3})(x + \sqrt{3})$
32. $z(x) = 4x(5x - 1)(3x + 8)(x - \sqrt{5})(x + \sqrt{5})$
33. $c(x) = [x - (3 - \sqrt{5})][x - (3 + \sqrt{5})]$
34. $d(x) = [x - (2 - \sqrt{11})][x - (2 + \sqrt{11})]$
35. $f(x) = 4x^4 - 37x^2 + 9$
36. $k(x) = 4x^4 - 65x^2 + 16$
37. $n(x) = x^6 - 7x^4$
38. $m(x) = x^5 - 5x^3$

Objective 3: Apply the Intermediate Value Theorem

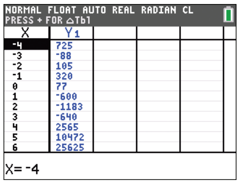
For Exercises 39–40, determine whether the intermediate value theorem guarantees that the function has a zero on the given interval. (See Example 5)

39. $f(x) = 2x^3 - 7x^2 - 14x + 30$
- **a.** [1, 2]
- **b.** [2, 3]
- **c.** [3, 4]
- **d.** [4, 5]

40. $g(x) = 2x^3 - 13x^2 + 18x + 5$
- **a.** [1, 2]
- **b.** [2, 3]
- **c.** [3, 4]
- **d.** [4, 5]

For Exercises 41–42, a table of values is given for $Y_1 = f(x)$. Determine whether the intermediate value theorem guarantees that the function has a zero on the given interval.

41. $Y_1 = 21x^4 + 46x^3 - 238x^2 - 506x + 77$
- **a.** [−4, −3]
- **b.** [−3, −2]
- **c.** [−2, −1]
- **d.** [−1, 0]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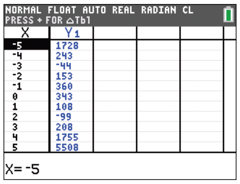
X	Y₁
-4	725
-3	-88
-2	105
-1	320
0	77
1	-600
2	-1183
3	-640
4	2565
5	10472
6	25625

X=-4

42. $Y_1 = 10x^4 + 21x^3 - 119x^2 - 147x + 343$
- **a.** [−4, −3]
- **b.** [−3, −2]
- **c.** [−2, −1]
- **d.** [−1, 0]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X	Y₁
-5	1728
-4	243
-3	-44
-2	153
-1	360
0	343
1	108
2	-99
3	208
4	1755
5	5508

X=-5

43. Given $f(x) = 4x^3 - 8x^2 - 25x + 50$,
- **a.** Determine if f has a zero on the interval [−3, −2].
- **b.** Find a zero of f on the interval [−3, −2].

44. Given $f(x) = 9x^3 - 18x^2 - 100x + 200$,
- **a.** Determine if f has a zero on the interval [−4, −3].
- **b.** Find a zero of f on the interval [−4, −3].

Objective 4: Sketch a Polynomial Function

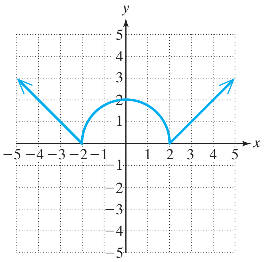
For Exercises 45–52, determine if the graph can represent a polynomial function. If so, assume that the end behavior and all turning points are represented in the graph.

a. Determine the minimum degree of the polynomial.

b. Determine whether the leading coefficient is positive or negative based on the end behavior and whether the degree of the polynomial is odd or even.

c. Approximate the real zeros of the function, and determine if their multiplicities are even or odd.

45.



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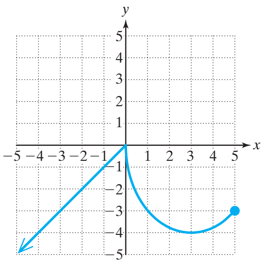

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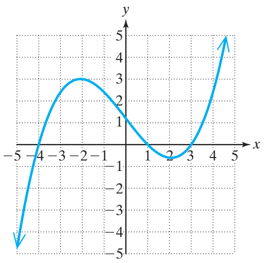

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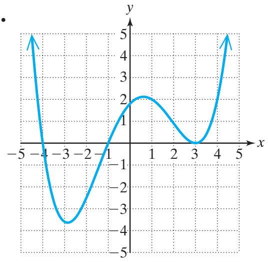

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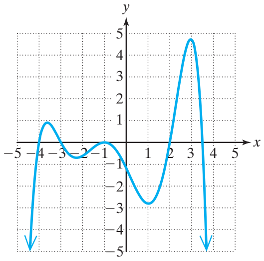

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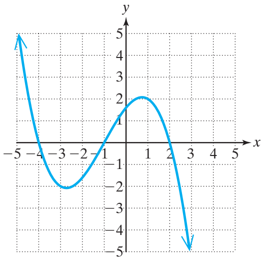

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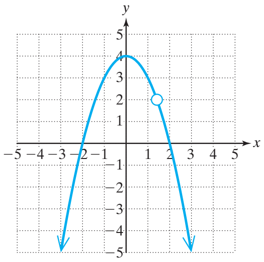

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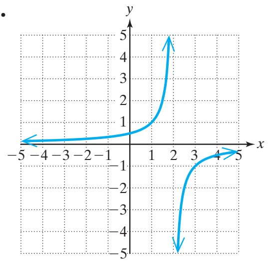


For Exercises 53–58,

a. Identify the power function of the form $y = x^n$ that is the parent function to the given graph.

b. In order, outline the transformations that would be required on the graph of $y = x^n$ to make the graph of the given function. See Section 2.6, page 236.

c. Match the function with the graph of i–vi.

53. $g(x) = -\frac{1}{3}x^6 - 2$

54. $f(x) = -\frac{1}{2}(x - 3)^4$

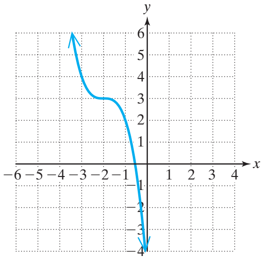
55. $k(x) = -(x + 2)^3 + 3$

56. $p(x) = 2(x + 4)^3 - 3$

57. $m(x) = (-x - 3)^5 + 1$

58. $n(x) = (-x + 3)^4 - 1$

i.



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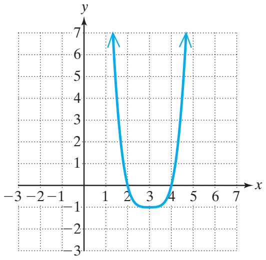

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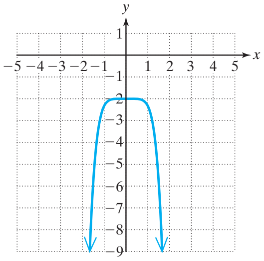

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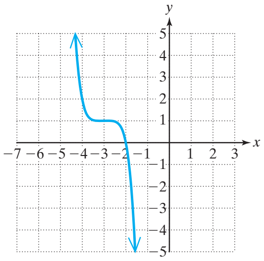

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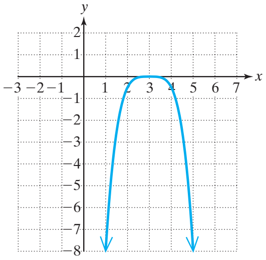

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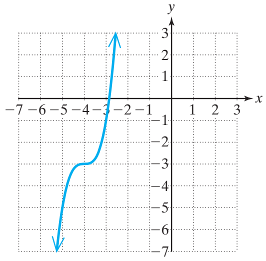


For Exercises 59–76, sketch the function. (See Examples 6–7)

59. $f(x) = x^3 - 5x^2$

60. $g(x) = x^5 - 2x^4$

61. $f(x) = \frac{1}{2}(x - 2)(x + 1)(x + 3)$

62. $h(x) = \frac{1}{4}(x - 1)(x - 4)(x + 2)$

63. $k(x) = x^4 + 2x^3 - 8x^2$

64. $h(x) = x^4 - x^3 - 6x^2$

65. $k(x) = 0.2(x + 2)^2(x - 4)^3$

66. $m(x) = 0.1(x - 3)^2(x + 1)^3$

67. $p(x) = 9x^5 + 9x^4 - 25x^3 - 25x^2$

68. $q(x) = 9x^5 + 18x^4 - 4x^3 - 8x^2$

69. $t(x) = -x^4 + 11x^2 - 28$

70. $v(x) = -x^4 + 15x^2 - 44$

71. $g(x) = -x^4 + 5x^2 - 4$

72. $h(x) = -x^4 + 10x^2 - 9$

73. $c(x) = 0.1x(x - 2)^4(x + 2)^3$

74. $d(x) = 0.05x(x - 2)^4(x + 3)^2$

75. $m(x) = -\frac{1}{10}(x + 3)(x - 3)(x + 1)^3$

76. $f(x) = -\frac{1}{10}(x - 1)(x + 3)(x - 4)^2$

Mixed Exercises

For Exercises 77–88, determine if the statement is true or false. If a statement is false, explain why.

77. The function defined by $f(x) = (x + 1)^5(x - 5)^2$ crosses the x-axis at 5.

78. The function defined by $g(x) = -3(x + 4)(2x - 3)^4$ touches but does not cross the x-axis at $\left(\frac{3}{2}, 0\right)$.

79. A third-degree polynomial has three turning points.

80. A third-degree polynomial has two turning points.

81. There is more than one polynomial function with zeros of 1, 2, and 6.

82. There is exactly one polynomial with integer coefficients with zeros of 2, 4, and 6.

83. The graph of a polynomial function with leading term of even degree is up to the far left and up to the far right.

84. If c is a real zero of an even polynomial function, then $-c$ is also a zero of the function.

85. The graph of $f(x) = x^3 - 27$ has three x-intercepts.

86. The graph of $f(x) = 3x^2(x - 4)^4$ has no points in Quadrants III or IV.

87. The graph of $p(x) = -5x^4(x + 1)^2$ has no points in Quadrants I or II.

88. A fourth-degree polynomial has exactly two relative minima and two relative maxima.

89. A rocket will carry a communications satellite into low Earth orbit. Suppose that the thrust during the first 200 sec of flight is provided by solid rocket boosters at different points during liftoff.

The graph shows the acceleration in G-forces (that is, acceleration in 9.8-m/sec^2 increments) versus time after launch.

a. Approximate the interval(s) over which the acceleration is increasing.

b. Approximate the interval(s) over which the acceleration is decreasing.

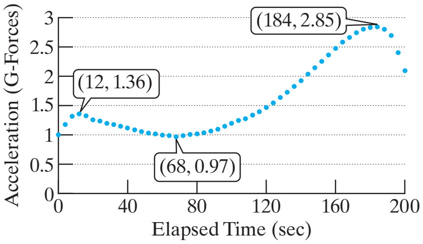
c. How many turning points does the graph show?

d. Based on the number of turning points, what is the minimum degree of a polynomial function that could be used to model acceleration versus time? Would the leading coefficient be positive or negative?

e. Approximate the time when the acceleration was the greatest.

f. Approximate the value of the maximum acceleration.

Acceleration in G-Forces vs. Time after Liftoff



90. Data from a 20-yr study show the number of new AIDS cases diagnosed among 20- to 24-yr-olds in the United States x years after the study began.

a. Approximate the interval(s) over which the number of new AIDS cases among 20- to 24-yr-olds increased.

b. Approximate the interval(s) over which the number of new AIDS cases among 20- to 24-yr-olds decreased.

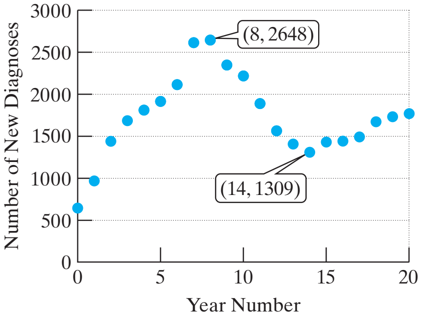
c. How many turning points does the graph show?

d. Based on the number of turning points, what is the minimum degree of a polynomial function that could be used to model the data? Would the leading coefficient be positive or negative?

e. How many years after the study began was the number of new AIDS cases among 20- to 24-yr-olds the greatest?

f. What was the maximum number of new cases diagnosed in a single year?

AIDS Diagnoses, 20- to 24-Yr-Olds United States, 1985–2005



Write About It

91. Given a polynomial function defined by $y = f(x)$, explain how to find the x-intercepts.

92. Given a polynomial function, explain how to determine whether an x-intercept is a touch point or a cross point.

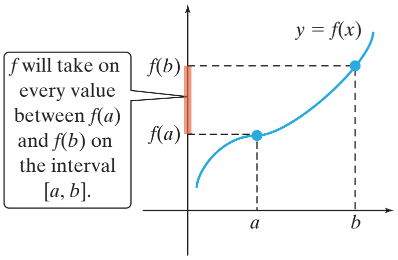
93. Write an informal explanation of what it means for a function to be continuous.

94. Write an informal explanation of the intermediate value theorem.

Expanding Your Skills

The intermediate value theorem given on page 306 is actually a special case of a broader statement of the theorem. Consider the following:

> Let f be a polynomial function. For $a < b$, if $f(a) \neq f(b)$, then f takes on every value between $f(a)$ and $f(b)$ on the interval $[a, b]$.



Use this broader statement of the intermediate value theorem for Exercises 95–96.

95. Given $f(x) = x^2 - 3x + 2$,

a. Evaluate $f(3)$ and $f(4)$.

b. Use the intermediate value theorem to show that there exists at least one value of x for which $f(x) = 4$ on the interval $[3, 4]$.

c. Find the value(s) of x for which $f(x) = 4$ on the interval $[3, 4]$.

96. Given $f(x) = -x^2 - 4x + 3$,

a. Evaluate $f(-4)$ and $f(-3)$.

b. Use the intermediate value theorem to show that there exists at least one value of x for which $f(x) = 5$ on the interval $[-4, -3]$.

c. Find the value(s) of x for which $f(x) = 5$ on the interval $[-4, -3]$.

Technology Connections

97. For a certain individual, the volume (in liters) of air in the lungs during a 4.5-sec respiratory cycle is shown in the table for 0.5-sec intervals. Graph the points and then find a third-degree polynomial function to model the volume $V(t)$ for t between 0 sec and 4.5 sec. (*Hint*: Use a CubicReg option or polynomial degree 3 option on a graphing utility.)

Time (sec)	Volume (L)
0.0	0.00
0.5	0.11
1.0	0.29
1.5	0.47
2.0	0.63
2.5	0.76
3.0	0.81
3.5	0.75
4.0	0.56
4.5	0.20

98. The torque (in ft-lb) produced by a certain automobile engine turning at x thousand revolutions per minute is shown in the table. Graph the points and then find a third-degree polynomial function to model the torque $T(x)$ for $1 \leq x \leq 5$.

Engine speed (1000 rpm)	Torque (ft-lb)
1.0	165
1.5	180
2.0	188
2.5	190
3.0	186
3.5	176
4.0	161
4.5	142
5.0	120

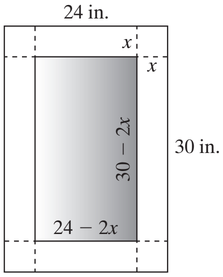
99. A solar oven is to be made from an open box with reflective sides. Each box is made from a 30-in. by 24-in. rectangular sheet of aluminum with squares of length x (in inches) removed from each corner. Then the flaps are folded up to form an open box.

a. Show that the volume of the box is given by

$$V(x) = 4x^3 - 108x^2 + 720x \quad \text{for } 0 < x < 12.$$

b. Graph the function from part (a) and use a "Maximum" feature on a graphing utility to approximate the length of the sides of the squares that should be removed to maximize the volume. Round to the nearest tenth of an inch.

c. Approximate the maximum volume. Round to the nearest cubic inch.



For Exercises 100–101, two viewing windows are given for the graph of $y = f(x)$. Choose the window that best shows the key features of the graph.

100. $f(x) = 2(x - 0.5)(x - 0.1)(x + 0.2)$

a. $[-10, 10, 1]$ by $[-10, 10, 1]$

b. $[-1, 1, 0.1]$ by $[-0.05, 0.05, 0.01]$

101. $g(x) = 0.08(x - 16)(x + 2)(x - 3)$

a. $[-10, 10, 1]$ by $[-10, 10, 1]$

b. $[-5, 20, 5]$ by $[-50, 30, 10]$

For Exercises 102–103, graph the function defined by $y = f(x)$ on an appropriate viewing window.

102. $k(x) = \frac{1}{100}(x - 20)(x + 1)(x + 8)(x - 6)$

103. $p(x) = (x - 0.4)(x + 0.5)(x + 0.1)(x - 0.8)$