تدريب ص 223

① في كل من الحالات الآتية، تحقق أن F تابع أصلي للتابع f على المجال I.

.1 $f(x)=\tan^2(x)$ $\quad F(x)=\tan x - x$ $\quad I=\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

$F'(x)=1+\tan^2 x-1=\tan^2 x=f(x)$

.2 $f(x)=\cos x - x\sin x$ $\quad F(x)=x\cos x$

$F'(x)=\cos x - x\sin x = f(x)$

.3 $f(x)=\frac{2(x^4-1)}{x^3}$ $\quad F(x)=\left(x+\frac{1}{x}\right)^2$

$F'(x)=2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)$

$=2\left(x-\frac{1}{x}+\frac{1}{x}-\frac{1}{x^3}\right)=\frac{2(x^4-x^2+x^2-1)}{x^3}$

$=\frac{2(x^4-1)}{x^3}=f(x)$

.4 $f(x)=\frac{2x-1}{x^2(x-1)^2}$ $\quad F(x)=-\frac{1}{x(x-1)}$

$F'(x)=-\frac{[(2x-1)(-1)]}{x^2(x-1)^2}=\frac{2x-1}{x^2(x-1)^2}=f(x)$

.5 $f(x)=\ln x$ $\quad F(x)=x\ln x - x$

$F'(x)=\ln x+\frac{1}{x}x-1=\ln x=f(x)$

.6 $f(x)=\frac{1}{x\ln x}$ $\quad F(x)=\ln(\ln x)$

$F'(x)=\frac{\frac{1}{x}}{\ln x}=\frac{1}{x\ln x}=f(x)$

② في كل من الحالات الآتية، تحقق أن F و G تابعان أصليان للتابع f نفسه على المجال I.

.1 $F(x)=\frac{x^2+3x-1}{x-1}$ $\quad G(x)=\frac{x^2+7x-5}{x-1}$

ط ١

$G'(x)=\frac{(2x+7)(x-1)-(x^2+7x-5)}{(x-1)^2}=\frac{x^2-2x-2}{(x-1)^2}$

$F'(x)=\frac{(2x+3)(x-1)-x^2-3x+1}{(x-1)^2}=\frac{x^2-2x-2}{(x-1)^2}$

F, G أصليان للتابع f

ط ٢ إذا كان فرقهما عدداً ثابتاً فهما لنفس التابع

$G(x)-F(x)=\frac{4x-4}{x-1}=\frac{4(x-1)}{x-1}=4$

.2 $F(x)=\tan^2 x$ $\quad G(x)=\frac{1}{\cos^2 x}$

$F(x)=\tan^2 x$ $\quad G(x)=1+\tan^2 x$

ومنه F, G أصليان لنفسه

.3 $F(x)=-\frac{1}{x^2+1}$ $\quad G(x)=\frac{5+3x^2}{2(1+x^2)}$

$F'(x)=-\frac{2x}{(x^2+1)^2}=-\frac{2x}{x^4+2x^2+1}$

$G'(x)=\frac{(6x)(2+2x^2)-(4x)(5+3x^2)}{(2+2x^2)^2}$

$=\frac{-8x}{4x^4+8x^2+4}=-\frac{2x}{(x^4+2x^2+1)}$

.4 $F(x)=\sin^2 x$ $\quad G(x)=2-\cos^2 x$

$G'(x)=-(2\cos x(-\sin x))=2\cos x\sin x$

$F'(x)=2\sin x\cos x$

②

5- $f(x)=\frac{2x+1}{(x^2+x)^2}$

$f(x)=(2x+1)(x^2-x)^{-2}$

$F(x)=-\frac{1}{(x^2+x)}$

6- $f(x)=\frac{4x-2}{\sqrt{x^2-x}}$

$f(x)=(4x-2)(x^2-x)^{\frac{1}{2}}=2(x-2)(x^2-x)^{-\frac{1}{2}}$

$F(x)=2.2(x^2-x)^{\frac{1}{2}}=4\sqrt{x^2-x}$

7- $f(x)=\frac{5}{4x-3}$ $\quad I=]-\infty,\frac{3}{4}[$

$f(x)=5.\frac{1}{4x-3}=5.\frac{1}{4}.\frac{4}{4x-3}$

$=\frac{5}{4}\ln|4x-3|=\frac{5}{4}\ln(3-4x)$

8- $f(x)=\frac{3x+1}{2x}$, $I=]0,+\infty[$

$=\frac{3}{2}+\frac{1}{2}.\frac{1}{x}$

$F(x)=\frac{3}{2}x+\frac{1}{2}\ln|x|$

9- $f(x)=\frac{x+1}{x-2}$ $\quad I=]-\infty,2[$

$x-2\,\overline{)\,x+1}$

$\mp x \pm 2$

3

$f(x)=1+\frac{3}{x-2}$

$F(x)=x+3\ln|x-2|$

$=x+3\ln(2-x)$

10- $f(x)=\frac{3x+2}{2x-1}$ $\quad I=]\frac{1}{2},+\infty[$

$f(x)=\frac{3x}{2x-1}+\frac{2}{2x-1}=\frac{3}{2}\frac{2}{2x-1}+\frac{2}{2x-1}$

$F(x)=\frac{3}{2}\ln|2x-1|+2\ln|2x-1|$

$=\frac{3}{2}\ln(2x-1)+2\ln(2x-1)$

③ أيكون التابعان F و G الآتيان تابعين أصليين للتابع f ذاته على $\mathbb{R}$؟

$G(x)=\sin x-3\sin^3 x$ و $f(x)=\sin(3x)-2\sin x$

$F'(x)=3\cos 3x-2\cos x$

$G'(x)=\cos x-3(3\sin^2 x\cos 3x)$

$=\cos x-9\sin^2 x\cos(3x)$

ومنه ليس لهما نفس التابع.

تدريب ص 227:

① جد تابعاً أصلياً للتابع f:

1- $f(x)=8x^3+6x^2-2x+3$

$F(x)=8\frac{x^4}{4}+6\frac{x^3}{3}-2\frac{x^2}{2}+3x$

$=2x^4+2x^3-x^2+3x$

2- $f(x)=\frac{1}{4x^4}$

$f(x)=x^{-4}$

$F(x)=\frac{1}{-3x^3}$

3- $f(x)=\sqrt[3]{x}+\frac{1}{\sqrt[3]{x}}-\frac{3}{x^2}$

$f(x)=x^{\frac{1}{3}}+x^{-\frac{1}{3}}-3x^{-2}$

$F(x)=\frac{3}{4}\sqrt[3]{x^4}+\frac{3}{2}x^{\frac{2}{3}}+\frac{2}{x}$

4- $f(x)=\frac{1}{1-2x+x^2}$

$f(x)=\frac{1}{(x-1)^2}=(x-1)^{-2}$

$F(x)=-\frac{1}{(x-1)}$

② جد تابعاً أصلياً على المجال :

1- $f(x) = \cos^2 3x$

$f(x) = \frac{1}{2} + \frac{1}{2}\cos 6x$

$F(x) = \frac{1}{2}x + \frac{1}{12}\sin 6x$

2- $f(x) = \cos^4 x$

$f(x) = (\cos^2 x)^2 = (\frac{1}{2} + \frac{1}{2}\cos 2x)^2$

$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$

$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}(\frac{1}{2} + \frac{1}{2}\cos 4x)$

$F(x) = \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\cos 4x$

$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\cos 4x$

3- $f(x) = \cos 3x \cdot \cos x$

$f(x) = \frac{1}{2}(\cos 4x + \cos 2x)$

$F(x) = \frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x$

4- $f(x) = \tan x$ $\quad I =]\frac{\pi}{2}, \frac{3\pi}{2}[$

$= \frac{\sin x}{\cos x} = -\frac{-\sin x}{\cos x}$

$F(x) = -\ln|\cos x| = -\ln(\cos x)$

5- $f(x) = \cot^2 x$, $I =]0, \pi[$

6- $f(x) = \cot x$ $\quad I =]0, \pi[$

$f(x) = \frac{\cos x}{\sin x}$

$F(x) = \ln(\sin x)$

7- $f(x) = \sqrt{(2x-1)^3}$

8- $f(x) = \frac{1}{\sqrt{3-2x}}$

$F(x) = (3-2x)^{-\frac{1}{2}} = -(-2) \cdot \frac{1}{2}(3-2x)^{-\frac{1}{2}}$

$= -\frac{\frac{1}{2}}{\frac{1}{2}}(3-2x)^{\frac{1}{2}} = -\sqrt{3-2x}$

9- $f(x) = x\sqrt[3]{(x^2+1)^2}$

$= x(x^2+1)^{\frac{2}{3}} = \frac{1}{2} \cdot 2x(x^2+1)^{\frac{2}{3}}$

$F(x) =$

10- $f(x) = \frac{x}{\sqrt{3-x^2}}$

$f(x) = x(3-x^2)^{-\frac{1}{2}} = -2x(-\frac{1}{2})(3-x^2)^{-\frac{1}{2}}$

$F(x) = -\frac{\frac{1}{2}}{\frac{1}{2}}\sqrt{3-x^2} = -\sqrt{3-x^2}$

مثال محلول ص 220 :

عين التابع الأصلي الذي ينعدم عند $x=1$ للتابع $f(x)=3x^2-x+1$ المعرف على $\mathbb{R}$.

$$F(x)=3\frac{x^3}{3}-\frac{x^2}{2}+x=x^3-\frac{x^2}{2}+x+k$$

$$F(1)=0 \quad 1-\frac{1}{2}+1+k=0$$

$$\frac{3}{2}+k=0 \Rightarrow k=-\frac{3}{2}$$

$$F(x)=x^3-\frac{x^2}{2}+x-\frac{3}{2}$$

مثال محلول ص 221 :

① أثبت أن التابع $F(x)=\frac{2}{3}x\sqrt{x}$ المعرف على المجال $[0,+\infty[$ تابع أصلي $f(x)=\sqrt{x}$ على المجال المفتوح $]0,+\infty[$.

② أيكون F تابعاً أصلياً للتابع $f(x)=\sqrt{x}$ على المجال $[0,+\infty[$ ؟

$$F'(x)=\frac{2}{3}\sqrt{x}+\frac{1}{2\sqrt{x}}\cdot\frac{2}{3}x=\frac{2}{3}\sqrt{x}+\frac{1}{3}\frac{x}{\sqrt{x}}$$

$$=\frac{2}{3}\sqrt{x}+\frac{1}{3}\sqrt{x}=\sqrt{x}=f(x)$$

$x \to \sqrt{x} \geq 0$ ، $x \to x>0$ ومنه لأن

اشتقاقيان على المجال $]0,+\infty[$

مثال محلول ص 224 :

جد F للتابع f على I :

① $f(x)=\frac{1}{x^3}$

$$f(x)=x^{-3} \Rightarrow F(x)=-\frac{1}{2}x^{-2}=-\frac{1}{2x^2}$$

② $f(x)=\sin^2 x$

$$=\frac{1}{2}-\frac{1}{2}\cos 2x$$

$$F(x)=\frac{1}{2}x-\frac{1}{4}\sin 2x$$

③ $f(x)=\cos 5x \cdot \sin x$

$$f(x)=\frac{1}{2}(\sin 6x \cdot \sin 4x)$$

$$F(x)=\frac{1}{2}\left(-\frac{1}{6}\cos 6x+\frac{1}{4}\cos 4x\right)$$

$$=\frac{1}{8}\cos 4x-\frac{1}{12}\cos 6x$$

④ $f(x)=\frac{3}{x}-5 \quad , I=]0,+\infty[$

$$=3\cdot\frac{1}{x}-5$$

$$F(x)=3\ln|x|-5x=3\ln x-5x$$

⑤ $f(x)=x^3-\frac{1}{x^2}$

$$F(x)=\frac{1}{4}x^4-\frac{x^{-1}}{-1}=\frac{1}{4}x^4+\frac{1}{x}$$

مثال حلول ص 226:

جد تابعاً أصلياً F للتابع f:

① $f(x)=(x-2)(x^2-4x+5)^3$, $I=\mathbb{R}$

$=\frac{1}{2}(2x-4)(x^2-4x+5)^3$

$F(x)=\frac{1}{2}(x^2-4x+5)^4\times\frac{1}{4}=\frac{1}{8}(x^2-4x+5)^4$

② $f(x)=\frac{2}{x+3}$ $I=]-\infty,-3[$

$=2\cdot\frac{1}{x+3}\Rightarrow F(x)=2\ln|x+3|=2\ln(-x-3)$

③ $f(x)=\frac{2x-1}{x^2-x+3}$ $I=\mathbb{R}$

$F(x)=\ln|x^2-x+3|=\ln(x^2-x+3)$

④ $f(x)=\frac{2x+1}{x-1}$, $I=]1,+\infty[$

$$\begin{array}{r|l} & 2 \\ \hline x-1 & 2x+1 \\ & \mp 2x \pm 2 \\ & 3 \end{array}$$

$f(x)=2+\frac{3}{x-1}$

$F(x)=2x+3\ln|x-1|$

$=2x+3\ln(x-1)$

⑤ $f(x)=xe^{x^2}$ $I=\mathbb{R}$

$=\frac{1}{2}\cdot 2xe^{x^2}\Rightarrow F(x)=\frac{1}{2}e^{x^2}$

⑥ $f(x)=\frac{1}{x\ln x}$, $I=]1,+\infty[$

$f(x)=\frac{\frac{1}{x}}{\ln(x)}$

$F(x)=\ln|\ln(x)|=\ln(\ln(x))$

مثال حلول ص 230:

احسب التكامل المحدد I:

1- $I=\int_{-1}^{1}\sqrt{(x+1)^3}\,dx$

$=\int_{-1}^{1}(x+1)^{\frac{3}{2}}dx=[\frac{2}{5}(x+1)^{\frac{5}{2}}]_{-1}^{1}$

$=\frac{2}{5}(2)^{\frac{5}{2}}-0=\frac{2}{5}\sqrt{2^5}=\frac{8}{5}\sqrt{2}$

2- $I=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\cos^2x\,dx$

$=[\frac{x}{2}+\frac{1}{4}\sin2x]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$

$=(\frac{\pi}{12}+\frac{\sqrt{3}}{8})-(\frac{\pi}{24}+\frac{1}{8})$

$=\frac{\pi}{12}-\frac{\pi}{24}+\frac{\sqrt{3}}{8}-\frac{1}{8}=\frac{\pi}{24}+\frac{\sqrt{3}-1}{8}$

3- $I=\int_{0}^{2}\frac{2}{x-3}\,dx$

$=[2\ln|x-3|]_0^2=[2\ln(3-x)]_0^2$

$=-2\ln3$

4- $I=\int_{0}^{2}|x^2-1|\,dx$

$x^2=1\Rightarrow x=\mp1$

x	$-\infty$	-1		0		1	$+\infty$
x^2-1	+	0	−			0	+

$I=\int_0^1(-x^2+1)dx+\int_1^2(x^2-1)dx$

$=[-\frac{x^3}{3}+x]_0^1+[\frac{x^3}{3}-x]_1^2$

$=\frac{2}{3}-0+\frac{2}{3}+\frac{2}{3}=\frac{6}{3}=2$

مثال محلول ص 231

احسب التكامل المحدد: $I=\int_0^1 xe^{-x}dx$

$u=x$	$v'=e^{-x}$
$u'=1$	$v=-e^{-x}$

$I=\int_0^1 u.v'dx=[u.v]_0^1-\int_0^1 u'.v\,dx$

$\int_0^1 xe^{-x}=[-xe^{-x}]_0^1-\int_0^1 -e^{-x}dx$

$=[-xe^{-x}]_0^1-\int_0^1 -e^{-x}dx$

$=[-xe^{-x}]_0^1-[-e^{-x}]_0^1=-\frac{1}{e}-[(\frac{1}{e})-1]$

$=-\frac{1}{e}-\frac{1}{e}+1=\frac{e-2}{e}$

مثال محلول ص 233

احسب التكاملات:

1- $I=\int_0^1 \frac{1}{x^2-x-2}dx$, $R\setminus\{-1,2\}$

$f(x)=\frac{1}{x^2-x-2}=\frac{1}{(x+1)(x-2)}$

$\frac{1}{(x+1)(x-2)}=\frac{a}{(x+1)}_{(x-2)}+\frac{b}{(x-2)}_{(x+1)}$

$1=a(x-2)+b(x+1)\Rightarrow 1=ax-2a+bx+b$

$1=x(a+b)-2a+b$

$a+b=0$ ---(1) $2a+b=1$ ---(2)

نجمع (1) و (2)

$3a=1\Rightarrow a=\frac{1}{3}$

$b=-\frac{1}{3}$ نعوض في (1)

ومنه: $f(x)=\frac{1}{3(x+1)}-\frac{1}{3(x-2)}$

$I=\frac{1}{3}\int_0^1\frac{1}{x+1}dx-\frac{1}{3}\int_0^1\frac{1}{x-2}dx$

$I=\frac{1}{3}[\ln|x-2|]_0^1-\frac{1}{3}[\ln|x+1|]_0^1$

لكن

$x+1>0$, $x-2<0$ على المجال $[0,1]$

$\Rightarrow I=\frac{1}{3}[\ln(2-x)]_0^1-\frac{1}{3}[\ln(x+1)]_0^1$

$=\frac{1}{3}(-\ln 2)-\frac{1}{3}\ln 2=-\frac{3}{2}\ln 2$

2- $I=\int_0^1 \frac{2x+1}{x^2+3x+2}dx$

$\frac{2x+1}{(x+1)(x+2)}=\frac{a}{(x+1)}+\frac{b}{(x+2)}$

$2x+1=a(x+2)+b(x+1)$

نعوض $x=-2$

$-4+1=-b\Rightarrow b=3$

نعوض $x=-1$

$-1=a\Rightarrow a=-1$

$\Rightarrow f(x)=\frac{3}{x+2}-\frac{1}{x+1}$

$I=3\int_0^1\frac{1}{x+2}dx-\int_0^1\frac{1}{x+1}dx$

$x+1>0$, $x+2>0$ على المجال $[0,1]$

$I=3[\ln(x+2)]_0^1-[\ln(x+1)]_0^1$

$=3\ln 3-3\ln 2-\ln 2$

$=3\ln 3-4\ln 2=\ln(\frac{27}{16})$

مثال حلول ص 234 :

$$I=\int_0^1 \frac{4x^3-3x}{2x^2-3x-2}\,dx$$

الحل

$$\begin{array}{r|l} & 2x+3 \\ \hline 2x^2-3x-2 & 4x^3-3x \\ & \mp 4x^3 \mp 6x^2 \pm 4x \\ & 6x^2+x \\ & \mp 6x^2 \mp 9x \mp 6 \\ & 10x+6 \end{array}$$

$$f(x)=2x+3+\frac{10x+6}{2x^2-3x-2}$$

$$M=\int_0^1 2x+3\,dx$$
$$=[x^2+3x]_0^1+J$$
$$=4+J$$

$$\frac{5x+3}{x^2-\frac{3}{2}x+1}=\frac{a}{(x-2)}+\frac{b}{(x+\frac{1}{2})}$$

$$5x+3=a(x+\tfrac{1}{2})+b(x-2)$$

نعوض $x=-\frac{1}{2}$ $\Longrightarrow$ $b=-\frac{1}{5}$

$x=2$ $\Longrightarrow$ $a=\frac{26}{5}$

$$\frac{5x+3}{x^2-\frac{3}{2}x+1}=\frac{26}{5}\cdot\frac{1}{(x-2)}-\frac{1}{5}\cdot\frac{1}{(x+\frac{1}{2})}$$

$$I=\frac{26}{5}\int_0^1\frac{1}{(x-2)}\,dx-\frac{1}{5}\int_0^1\frac{1}{(x+\frac{1}{2})}\,dx$$

على $[0,1]$: $x-2<0$, $x+\frac{1}{2}>0$

$$I=\frac{26}{5}[\ln(2-x)]_0^1-\frac{1}{5}[\ln(x+\tfrac{1}{2})]_0^1$$

$$=-\frac{26}{5}\ln2-[\frac{1}{5}\ln\frac{3}{2}-\frac{1}{5}\ln\frac{1}{2}]$$

$$=-\frac{26}{5}\ln2-[\frac{1}{5}\ln3-\cancel{\frac{1}{5}\ln2}+\cancel{\frac{1}{5}\ln2}]$$

$$=-\frac{26}{5}\ln2-\frac{1}{5}\ln3 = J$$

نعوض في I :

$$I=4-\frac{26}{5}\ln2-\frac{1}{5}\ln3$$

مثال حلول ص 235 :

$I=]0,+\infty[$ عين تابعاً أصلياً على $f(x)=\ln x$

$$f(t)=\ln(t)$$
$$F(x)=\int_1^x f(t)\,dt=\int_1^x \ln t\,dt$$

$$\begin{array}{l|l} u=\ln(t) & v'=1 \\ u'=\frac{1}{t} & v=t \end{array}$$

$$\int_1^x \ln(t)=[t\ln t]_1^x-\int_1^x t\cdot\frac{1}{t}\,dx$$
$$=x\ln x-x+1$$

تدرب ص 236+235 :

احسب التكاملات الآتية:

① $$I=\int_{\frac{3\pi}{2}}^{2\pi}\sqrt{2-2\cos2x}\,dx$$

$$I=\int_{\frac{3\pi}{2}}^{2\pi}\sqrt{2(1-\cos2x)}\,dx$$

$$I=\int_{\frac{3\pi}{2}}^{2\pi}\sqrt{2\cdot2\sin^2x}\,dx$$

$$=\int_{\frac{3\pi}{2}}^{2\pi}|2\sin x|\,dx$$

$$=\int_{\frac{3\pi}{2}}^{2\pi}-2\sin x\,dx=[2\cos x]_{\frac{3\pi}{2}}^{2\pi}$$

$$=2\cos2\pi-2\cos\frac{3\pi}{2}=2\times1-0=2$$

6- $N = \int_0^{\frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$

$= [\ln(\cos x + \sin x)]_0^{\frac{\pi}{4}}$

$= \ln\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \ln(1+0)$

$= \ln\sqrt{2} - 0 = \frac{1}{2}\ln 2$

② احسب التكاملات الآتية باستعمال تكامل بالتجزئة:

1- $I = \int_1^e x \ln x \, dx$

$u = \ln x$	$v' = x$
$u' = \frac{1}{x}$	$v = \frac{x^2}{2}$

$\int u v' dx = [u \cdot v]_1^e - \int u' \cdot v \, dx$

$= \left[\frac{1}{2} x^2 \ln x\right]_1^e - \int_1^e \frac{1}{x} \cdot \frac{x^2}{2}\, dx$

$= \left[\frac{1}{2} x^2 \ln x\right]_1^e - \left[\frac{1}{4} x^2\right]_1^e$

$= \left(\frac{1}{2}e^2\right) - \left[\left(\frac{1}{4}e^2\right) - \frac{1}{4}\right]$

$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}(e^2+1)$

2- $J = \int_0^{\pi} (x-1)\cos x \, dx$

$u = x-1$	$v' = \cos x$
$u' = 1$	$v = \sin x$

$\int_0^{\pi} u \cdot v' dx = [u \cdot v]_0^{\pi} - \int_0^{\pi} u' \cdot v \, dx$

$= [(x-1)\sin x]_0^{\pi} + [\cos x]_0^{\pi}$

$= (0-0) + (-1-1) = -2$

3- $K = \int_0^1 (x+2) e^x \, dx$

$u = x+2$	$v' = e^x$
$u' = 1$	$v = e^x$

② $\int_{-1}^{2} x|x-1| \, dx$

$x - 1 = 0 \Rightarrow x = 1$

x	$-\infty$	-1	1	2	$+\infty$
$x-1$	$-$		0	$+$	
إشارة					

$= \int_{-1}^{1} x(1-x)\, dx + \int_1^2 x(x-1)\, dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^2$

$= \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$

$= \frac{1}{6} - \frac{5}{6} + \frac{2}{3} + \frac{1}{6} = \frac{-3+4}{6} = \frac{1}{6}$

3- $K = \int_0^1 (e^{2x} - e^{-2x})\, dx$

$= \left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}\right] = \left(\frac{1}{2}e^2 + \frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^0 + \frac{1}{2}e^0\right)$

$= \frac{1}{2}e^2 + \frac{1}{2}e^{-2} - 1 = \frac{1}{2}(e^2 + e^{-2} - 2)$

4- $L = \int_{-2}^{-1} \frac{2x-1}{x-1}\, dx$

نجري القسمة الإقليدية

$\Rightarrow L = \int_{-2}^{-1} 2 + \frac{1}{x-1}\, dx$

على $[-1,-2]$ $x - 1 < 0$

$L = [2x + \ln(1-x)]_{-2}^{-1} = (-2 + \ln 2) - (-4 + \ln 3)$

$= -2 + \ln 2 + 4 - \ln 3 = 2 + \ln\frac{2}{3}$

5- $M = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x \, dx$

$= [-\ln|\cos x|]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = [-\ln(\cos x)]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

لأن $\cos x > 0$ في المجال $]\frac{\pi}{6}, \frac{\pi}{3}[$

$= -\ln\frac{1}{2} - \left(-\ln\frac{\sqrt{3}}{2}\right) = \ln 2 + \ln\frac{\sqrt{3}}{2} = \ln\sqrt{3}$

$N = \int_0^{\pi} e^x \sin x \, dx$

نعوض من N في ①:

$N = [-e^x \cos x]_0^{\pi} + \left(-\frac{1}{2}(e^{\pi}+1)\right)$

$= [(-e^{\pi}(-1)) - (-1)(1)] - \frac{1}{2}(e^{\pi}+1)$

$= e^{\pi} - \frac{1}{2}e^{\pi} + 1 - \frac{1}{2} = \frac{1}{2}(e^{\pi} - 1)$

③ جد تابعاً $f(x) \longrightarrow F(x)$ على المجال I.

1- $f(x) = x \cdot \cos x \qquad I = \mathbb{R}$

$u = x$	$v' = \cos x$
$u' = 1$	$v = \sin x$

$F(x) = \int u \cdot v' dx = [u \cdot v] - \int u' \cdot v \, dx$

$= x \sin x - \int \sin x$

$F(x) = x \sin x + \cos x$

2- $f(x) = x \sin 2x \qquad I = \mathbb{R}$

$u = x$	$v' = \sin 2x$
$u' = 1$	$v = -\frac{1}{2}\cos 2x$

$F(x) = [-\frac{1}{2}x \cos 2x] - \int -\frac{1}{2}\cos 2x \, dx$

$= -\frac{1}{2}x \cos 2x + \frac{1}{4}\sin 2x$

3- $f(x) = x^2 e^x \qquad I = \mathbb{R}$

$u = x^2$	$v' = e^x$
$u' = 2x$	$v = e^x$

$F(x) = [x^2 e^x] - \int 2x e^x dx$ ——①

$u = 2x$	$v' = e^x$
$u' = 2$	$v = e^x$

$\int 2x e^x dx = [2x e^x] - \int 2e^x dx$

$= 2x e^x - 2e^x$

نعوض في ①:

$F(x) = x^2 e^x - 2x e^x + 2e^x$

EVA≋ $= e^x(x^2 - 2x + 2)$

$\int_0^1 u \cdot v' dx = [u \cdot v]_0^1 - \int_0^1 u' \cdot v \, dx$

$= [(x+2)e^x]_0^1 - \int_0^1 e^x dx$

$= [(3e-2)] - [e^x]_0^1 = (3e-2) - (e-1)$

$= 3e - 2 - e + 1 = 2e - 1$

4- $L = \int_0^{\frac{\pi}{3}} x \sin 3x \, dx$

$u = x$	$v' = \sin 3x$
$u' = 1$	$v = -\frac{1}{3}\cos 3x$

$= [-\frac{1}{3}x \cos 3x]_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} \frac{1}{3}\cos 3x \, dx$

$= [-\frac{1}{3}\cos 3x]_0^{\frac{\pi}{3}} + [\frac{1}{9}\sin 3x]_0^{\frac{\pi}{3}}$

$= [-\frac{\pi}{9}(-1) - 0] - 0 = \frac{\pi}{9}$

5- $M = \int_0^{\pi} e^x \cos x \, dx$

$u = e^x$	$v' = \cos x$
$u' = e^x$	$v = \sin x$

$\int_0^{\pi} u \cdot v' = [u \cdot v]_0^{\pi} - \underbrace{\int_0^{\pi} e^x \sin x}_{N}$ ——①

$N = \int_0^{\pi} e^x \sin x$

$u = e^x$	$v' = \sin x$
$u' = e^x$	$v = -\cos x$

$N = [-e^x \cos x]_0^{\pi} + \int_0^{\pi} e^x \cos x \, dx$

$= [-e^x \cos x]_0^{\pi} + M$

نعوض في ①:

$M = [e^x \sin x]_0^{\pi} - [-e^x \cos x]_0^{\pi} - M$

$2M = [e^x \sin x]_0^{\pi} + [e^x \cos x]_0^{\pi}$

$2M = 0 - 0 + [e^{\pi}(-1) - 1]$

$2M = -e^{\pi} - 1 \Rightarrow M = -\frac{1}{2}(e^{\pi}+1)$

4- $f(x) = x^2 \ln x \qquad I =]0,+\infty[$

$u = \ln x$	$v' = x^2$
$u' = \frac{1}{x}$	$v = \frac{x^3}{3}$

$F(x) = [\frac{x^3}{3}\ln x] - \int \frac{1}{3x} x^3 \, dx$

$= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 = \frac{1}{9}x^3[3\ln x - 1]$

$= \frac{1}{9}x^3[\ln x^3 - 1] = \frac{1}{9}x^3[\ln x^3 - \ln e]$

$= \frac{1}{9}x^3 \ln(\frac{x^3}{e})$

5- $f(x) = x^2 \sin 2x \qquad I = \mathbb{R}$

$u = x^2$	$v' = \sin 2x$
$u' = 2x$	$v = -\frac{1}{2}\cos 2x$

$F(x) = [-\frac{1}{2}x^2\cos 2x] - \int -x\cos 2x \, dx$ —— (1)

$u = x$	$v' = \cos 2x$
$u' = 1$	$v = \frac{1}{2}\sin 2x$

$F(x) = [\frac{x}{2}\sin 2x] - \int \frac{1}{2}\sin 2x \, dx$

$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x$

نعوض في (1):

$F(x) = -\frac{1}{2}x^2\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x$

$= \cos 2x[-\frac{x^2}{2} + \frac{1}{4}] + \frac{x}{2}\sin 2x$

$= (\frac{-2x^2+1}{4})\cos 2x + \frac{x}{2}\cos 2x$

6- $f(x) = x^2 . \cos 3x$

$u = x^2$	$v' = \cos 3x$
$u' = 2x$	$v = \frac{1}{3}\sin 3x$

$F(x) = \frac{x^2}{3}\sin 3x - \int \frac{2}{3}x\sin 3x \, dx$ —— (1)

$u = \frac{2}{3}x$	$v' = \sin 3x$
$u' = \frac{2}{3}$	$v = -\frac{1}{3}\cos 3x$

$\int \frac{2}{3}x\sin 3x = -\frac{2}{9}x\cos 3x - \int -\frac{2}{9}\cos 3x$

$= -\frac{2}{9}x\cos 3x + \frac{2}{27}\sin 3x$

نعوض في (1)

$F(x) = \frac{x^3}{3}\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x$

(4) جد تابعاً أصلياً :

1- $f(x) = \frac{x+3}{x^2-1} \qquad I =]1,+\infty[$

$f(x) = \frac{x+3}{(x+1)(x-1)}$

$\frac{x+3}{(x+1)(x-1)} = \frac{a}{x+1} + \frac{b}{x-1}$

$x+3 = a(x-1) + b(x+1)$

نعوض $x = 1$

$4 = 2b \Rightarrow b = 2$

نعوض $x = -1$

$2 = -2a \Rightarrow a = -1$

$\Rightarrow f(x) = -\frac{1}{x+1} + \frac{2}{x-1}$

$F(x) = -\ln|x+1| + 2\ln|x-1|$

على I: $x-1>0$ و $x+1>0$

$F(x) = 2\ln(x-1) - \ln(x+1)$

2- $f(x) = \frac{x+1}{x^2-4} \qquad I =]-\infty,-2[$

$\frac{x+1}{(x-2)(x+2)} = \frac{a}{x-2} + \frac{b}{x+2}$

$x+1 = a(x+2) + b(x-2)$

$2x-1 = a(x+1) + xb$

نفرض $x=0$

$-1 = a \Rightarrow a = -1$

نفرض $x=-1$

$-3 = -b \Rightarrow b = 3$

$f(x) = -\frac{1}{x} + \frac{3}{x+1}$

$F(x) = -\ln|x| + 3\ln|x+1|$

على I: $x+1>0$, $x<0$

$F(x) = -\ln(-x) + 3\ln(x+1)$

5- $f(x) = \frac{x^3}{x^2-x-2}$ $\quad I =]2,+\infty[$

$\frac{x^3}{(x-2)(x+1)} = \frac{a}{(x-2)} + \frac{b}{(x+1)}$

$x^3 = a(x+1) + b(x-2)$

$b = \frac{1}{3} \Leftarrow x=-1$

$a = \frac{8}{3} \Leftarrow x=2$

$f(x) = \frac{1}{3}(8\ln|x-2| + \ln|x+1|)$

على I: $x-2>0$, $x+1>0$

$f(x) = \frac{1}{3}(8\ln(x-2) + \ln(x+1))$

6- $f(x) = \frac{2x-1}{(x+2)^2}$ $\quad I =]-\infty,-2[$

$f(x) = \frac{2x+4-5}{(x+2)^2} = \frac{2(x+2)}{(x+2)^2} - \frac{5}{(x+2)^2}$

$= 2\ln(-x-2) - \frac{5}{x+2}$

نفرض $x=2$

$3 = 4a \Rightarrow a = \frac{3}{4}$

نفرض $x=-2$

$-1 = -4b \Rightarrow b = \frac{1}{4}$

$f(x) = \frac{3}{4(x-2)} + \frac{1}{4(x+2)}$

$F(x) = \frac{1}{4}(3\ln|x-2| + \ln|x+2|)$

على I: $x-2<0$, $x+2<0$

$\Rightarrow F(x) = \frac{1}{4}(3\ln(2-x) + \ln(-2-x))$

3- $f(x) = \frac{x}{x^2-x-6}$ $\quad I =]-2,3[$

$\frac{x}{(x-3)(x+2)} = \frac{a}{x-3} + \frac{b}{x+2}$

$x = a(x+2) + b(x-3)$

نفرض $x=-2$

$-2 = -5b \Rightarrow b = \frac{2}{5}$

نفرض $x=3$

$3 = 5a \Rightarrow a = \frac{3}{5}$

$\Rightarrow f(x) = \frac{1}{5}(3\ln|x+2| + 2\ln|x-3|)$

على I: $x+2>0$, $x-3<0$

$f(x) = \frac{1}{5}(3\ln(x+2) + 2\ln(3-x))$

4- $f(x) = \frac{2x-1}{x^2+x}$ $\quad I =]-1,0[$

$\frac{2x-1}{x(x+1)} = \frac{a}{x} + \frac{b}{x+1}$

مثال محلول ص 237 :

[1] في حالة $b \geq 0$ نحقق المتراجحات :

$\sin b \leq b$

ثم أثبت أن $\cos(b) \geq 1 - \frac{b^2}{2}$

و $\sin(b) \geq b - \frac{b^3}{6}$

نعلم أن $\cos(x) \leq 1$

$$\int_0^b \cos(x)\,dx \leq \int_0^b 1\,dx$$

$$[\sin x]_0^b \leq [x]_0^b$$

$$\sin(b) - \sin(0) \leq b - 0$$

$$\sin(b) \leq b$$

لدينا $\sin(x) \leq x$

$$\int_0^b \sin x\,dx \leq \int_0^b x\,dx$$

$$[-\cos(x)]_0^b \leq \left[\frac{x^2}{2}\right]_0^b$$

$$-\cos(b) + \cos(0) \leq \frac{b^2}{2} - 0$$

$$-\cos(b) + 1 \leq \frac{b^2}{2}$$

$$-\cos(b) \leq -1 + \frac{b^2}{2} \qquad \times(-1)$$

$$\cos(b) \geq 1 - \frac{b^2}{2}$$

لدينا $\cos(x) \geq 1 - \frac{x^2}{2}$

$$\int_0^b \cos x\,dx \geq \int_0^b 1 - \frac{x^2}{2}\,dx$$

$$[\sin x]_0^b \geq \left[x - \frac{x^3}{6}\right]_0^b$$

$$\sin b - 0 \geq b - \frac{b^3}{6} - 0$$

$$\sin b \geq b - \frac{b^3}{6}$$

[2] ... تمرينات ومسائل ص 244

[1] جد تابعاً أصلياً:

1- $f(x) = 1 - \frac{1}{x^2} + \frac{3}{x}$, $I =]0, +\infty[$

$F(x) = x + \frac{1}{x} + 3\ln(x)$

2- $f(x) = \frac{2}{\sqrt{1-2x}}$ $I =]-\infty, \frac{1}{2}[$

$f(x) = (-) -2(1-2x)^{-\frac{1}{2}} = -\frac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}}$

$F(x) = -2\sqrt{1-2x}$

3- $f(x) = (2x-1)^3$ $I = \mathbb{R}$

$F(x) = \frac{1}{8}(2x-1)^4$

4- $f(x) = (2x-1)^3$

$F(x) = \frac{1}{8}(2x-1)^4$

5- $f(x) = \frac{1}{(1-3x)^2}$

$f(x) = (1-3x)^{-2}$

$F(x) = -\frac{1}{3}\frac{(1-3x)^{-1}}{-1} = \frac{1}{3(1-3x)}$

6- $f(x) = \frac{x-1}{(x^2-2x-3)^2}$ $I =]-1, 3[$

$f(x) = (x-1)(x^2-2x+3)^{-2}$

$f(x) = \frac{1}{2}(2x-2)(x^2-2x-3)^{-2}$

$F(x) = -\frac{1}{2(x^2-2x-3)}$

[2] جد تابعاً أصلياً:

1- $f(x) = \cos x(\sin^2 x - 3\sin x)$

$= \cos x \sin^2 x - 3\sin x \cos x$

$= \frac{1}{3}\sin^3 x - \frac{3}{2}\sin^2 x$

2- $f(x) = \frac{1}{x-4}$ $\quad I =]4, +\infty[$

$F(x) = \ln(x-4)$

3- $f(x) = \frac{2}{\cos^2 x} - 1$

$F(x) = 2\tan x - x$

5- $f(x) = 2e^{3x-1}$

$f(x) = 2 \cdot \frac{1}{3} 3e^{3x-1}$

$F(x) = \frac{2}{3} e^{3x-1}$

6- $f(x) = \frac{2x-1}{x+1}$ $\quad I =]-1, +\infty[$

$$\begin{array}{r|l} & 2 \\ x+1 & 2x-1 \\ & \mp 2x \mp 2 \\ & -3 \end{array}$$

$f(x) = 2 - \frac{3}{x+1}$

$F(x) = 2x - 3\ln(x+1)$

[3] جد تابعاً أصلياً يحقق الشرط:

1- $f(x) = \frac{2}{x^2} + x$, $F(1) = 0$

$f(x) = 2x^{-2} + x$

$F(x) = -\frac{2}{x} + \frac{x^2}{2} + K$

نعوض $x = 1$, $y = 0$

$0 = -\frac{2}{1} + \frac{1}{2} + K \Rightarrow K = \frac{3}{2}$

$F(x) = -\frac{2}{x} + \frac{x^2}{2} + \frac{3}{2}$

2- $f(x) = \frac{1}{(2x+1)^2}$ $\quad F(0) = 0$

$f(x) = \frac{1}{2} 2(2x+1)^{-2}$

$F(x) = \frac{-1}{4x+2} + K$

$F(0) = 0 \Rightarrow F(x) = -\frac{1}{4x+2} + \frac{1}{2}$

3- $f(x) = \sin(2x - \frac{\pi}{4})$, $F(\frac{\pi}{2}) = 0$

$f(x) = -\frac{1}{2} \cdot 2\sin(2x - \frac{\pi}{4})$

$F(x) = -\frac{1}{2}\cos(2x - \frac{\pi}{4}) + K$

نعوض $x = \frac{\pi}{2}$, $y = 0$

$0 = -\frac{1}{2}\cos(\pi - \frac{\pi}{4}) + K$

$0 = -\frac{1}{2}(-\cos\frac{\pi}{4}) + K$

$= -\frac{1}{2}(-\frac{\sqrt{2}}{2}) + K$

$\Rightarrow K = -\frac{\sqrt{2}}{4}$

4- $f(x) = \sin x \cdot \cos^2 x$, $F(\frac{\pi}{2}) = 0$

$f(x) = -(-\sin x) \cdot \cos^2 x$

$F(x) = -\frac{1}{3}\cos^3 x + K$

$F(\frac{\pi}{2}) = -\frac{1}{3}\cos^3\frac{\pi}{2} + K = 0$

$F(x) = -\frac{1}{3}\cos^3 x.$

5- $f(x) = \frac{x}{(x^2-1)^2}$ $\quad F(0) = 0$

$f(x) = x(x^2-1)^{-2} = \frac{1}{2} \cdot 2x(x^2-1)^{-2}$

$F(x) = -\frac{1}{2(x^2-1)} - \frac{1}{2}$

5- $I = \int_1^2 \frac{x^3}{x^4+2} \, dx$

$= \int_1^2 \frac{1}{4} \cdot \frac{4x^3}{x^4+2} \, dx = \left[\frac{1}{4} \ln(x^2+2)\right]_1^2$

$= \left[\frac{1}{4} \ln 8\right] - \left[\frac{1}{4} \ln 3\right] = \frac{1}{4} \ln 6$

6- $I = \int_0^{\pi} \sin\left(x+\frac{\pi}{4}\right) dx$

$= \left[-\cos\left(x+\frac{\pi}{4}\right)\right]_0^{\pi} = \sqrt{2}$

7- $I = \int_{-2}^{-1} \frac{x-3}{x} \, dx$

$= \int_{-2}^{-1} 1 - \frac{3}{x} \, dx = \left[x - 3\ln(-x)\right]_{-2}^{-1}$

$= -1 + 2 + 3\ln 2 - 1 + \ln 8$

8- $I = \int_0^1 t e^{t^2-1} \, dt$

$= \left[\frac{1}{2} t^2 - 1\right]_0^1 = \left[\frac{1}{2} e^0\right] - \left[\frac{1}{2} e^{-1}\right] = \frac{e-1}{2e}$

9- $I = \int_0^2 \sqrt{2x+1} \, dx$

$= \left[\frac{1}{3} \sqrt{(2x+1)^3}\right]_0^2 = \frac{1}{3}\sqrt{125} - \frac{1}{3} = \frac{1}{3}(\sqrt{125} - 1)$

10- $I = \int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

$= \left[\ln(e^x + e^{-x})\right]_0^1 = \left[\ln e + \frac{1}{e}\right] - \left[\ln 2\right]$

$= \ln\left(\frac{e+2}{e}\right) - \ln 2$

6- $f(x) = -\frac{1}{3-x}$ $F(1) = 1$

$F(x) = \ln|3-x| + k$

$F(x) = \ln|3-x| + 1 - \ln 2$

[5] احسب التكاملات:

1- $I = \int_2^{-1} (x^2 - 4x + 3) \, dx$

$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_2^{-1} = \left[-\frac{1}{3} - 2 - 3\right] - \left[\frac{8}{3} - 8 + 6\right]$

$= -\frac{16}{3} - \frac{2}{3} = -\frac{18}{3} = -6$

2- $I = \int_2^{-1} (x-2)(x^2 - 4x + 3) \, dx$

$= \int_2^{-1} \frac{1}{2}(2x-4)(x^2-4x+3) \, dx = \left[\frac{1}{4}(x^2-4x+3)^2\right]_2^{-1}$

$I = \left[\frac{1}{4}(1+4+3)^2\right] - \left[\frac{1}{4}(4-8+3)^2\right]$

$= \frac{1}{4}(64) - \frac{1}{4} = \frac{64}{4} - \frac{1}{4} = \frac{63}{4}$

3- $I = \int_1^2 \left(t^2 + t - \frac{1}{t}\right) dt$

$= \left[\frac{t^3}{3} + \frac{t^2}{2} - \ln t\right]_1^2$

$= \left[\frac{8}{3} + 2 - \ln 2\right] - \left[\frac{1}{3} + \frac{1}{2}\right]$

$= \frac{8}{3} + 2\ln 2 - \frac{1}{3} - \frac{1}{2} = \frac{23}{6} - \ln 2$

4- $I = \int_0^3 \frac{dt}{\sqrt{1+t}}$

$I = \int_0^3 (1+t)^{-\frac{1}{2}} dt = \left[2\sqrt{1+t}\right]_0^3$

$= 2\sqrt{4} - 2\sqrt{1} = 4 - 1 = 3$

[6] ليكن f التابع المعرف على $D = \mathbb{R} \setminus \{-3\}$ وفق

$$f(x) = \frac{4x^2-5x+1}{x+3}$$

1- جد الأعداد a و b و c التي تحقق $f(x) = ax+b+\frac{c}{x+3}$ أياً يكن x من D.

2- احسب $J = \int_2^0 f(x)\,dx$

-1

$$\begin{array}{r|l} & 4x-17 \\ \hline x+3 & 4x^2-5x+1 \\ & \mp 4x^2 \mp 12x \\ \hline & -17x+1 \\ & \pm 17x \pm 51 \\ \hline & 52 \end{array}$$

$$f(x) = 4x-17+\frac{52}{x+3}$$

حيث:

$a = 4 \qquad b = -17$

$c = 52$

-2

$$J = \int_2^0 4x-17+52\frac{1}{x+3}\,dx$$

$$= \left[2x^2-17x+52\ln(x+3)\right]_2^0$$

$$= 52\ln 3 - [8-34+52\ln 5]$$

$$= 52\ln 3 + 26 - 52\ln 5$$

$$= 52\ln\frac{3}{5} + 26$$

[7] ليكن f التابع المعرف على $D = \mathbb{R} \setminus \{1\}$ وفق $f(x) = \frac{x^2}{(x-1)^2}$

1- جد الأعداد a و b و c التي تحقق $f(x) = a+\frac{b}{x-1}+\frac{c}{(x-1)^2}$ أياً يكن x من D.

2- احسب $J = \int_{-3}^0 f(x)\,dx$

-1

$$\frac{x^2}{(x-1)^2} = \frac{a(x-1)^2+b(x-1)+c}{(x-1)^2}$$

$$x^2 = ax^2-2ax+a+bx-b+c$$

$a = 1$

$-2a+b = 0 \Rightarrow b = 2$

$a-b+c = 0 \Rightarrow c = 1$

$$f(x) = 1+\frac{2}{(x-1)}+\frac{1}{(x-1)^2}$$

-2

$$J = \int_{-3}^0 1+\frac{2}{x-1}+\frac{1}{(x-1)^2}\,dx$$

$$= \left[x+2\ln(1-x)-\frac{1}{(x-1)}\right]_{-3}^0$$

$$= 1-\left(-3+2\ln 4+\frac{1}{4}\right) = \frac{15}{4}-\ln 16$$

[8] أثبت أن $\frac{1}{1+e^x} = 1-\frac{e^x}{1+e^x}$ ثم استنتج قيمة $J = \int_0^1 \frac{1}{1+e^x}\,dx$

$$1-\frac{e^x}{1+e^x} = \frac{1+e^x-e^x}{1+e^x} = \frac{1}{1+e^x}$$

$$J = \int_0^1 1-\frac{e^x}{1+e^x}\,dx = \left[x-\ln(1+e^x)\right]_0^1$$

$$= [1-\ln(1+e)]-[0-\ln 2] = 1-\ln(2+2e)$$

[14] في كل من الحالات الآتية احسب التكامل المعطى

① $I = \int_{-2}^0 \frac{x}{x-1}\,dx$

$$= \int_{-2}^0 \frac{x-1+1}{x-1}\,dx$$

$$= \int_{-2}^0 \left(1+\frac{1}{x-1}\right) = \left[x+\ln(1-x)\right]_{-2}^0$$

$$= 0+\ln 1-(-2+\ln 3) = -\ln 3+2$$

5- $I=\int_0^1 \frac{2x^3-3x-4}{x-2}\,dx$

$$\begin{array}{r|l} & 2x^2+4x+5 \\ \hline x-2 & 2x^3-3x-4 \\ & \mp 2x^3 \mp 4x^2 \\ \hline & 4x^2-3x-4 \\ & \mp 4x^2 \mp 8x \\ \hline & 5x-4 \\ & \mp 5x \mp 10 \\ \hline & 6 \end{array}$$

$=\int_0^1 (2x^2+4x+5+\frac{6}{x-2})\,dx$

$=[\frac{2}{3}x^3+2x^2+5x+6\ln(2-x)]_0^1$

$=\frac{2}{3}+2+5+0-(6\ln 2)$

$=\frac{23}{3}-6\ln 2$

6- $I=\int_1^2 \frac{8x^2-4}{4x^2-1}\,dx$

$$\begin{array}{r|l} & 2 \\ \hline 4x^2-1 & 8x^2-4 \\ & \mp 8x^2 \mp 2 \\ \hline & -2 \end{array}$$

$I=\int_1^2 2-\frac{2}{4x^2-1}$

$=\int_1^2 2-\frac{2}{(2x+1)(2x-1)}$

$\frac{2}{(2x+1)(2x-1)}=\frac{a}{2x+1}+\frac{b}{2x-1}$

$2=a(2x-1)+b(2x+1)$

$2=-2a+0 \quad a=-1 \Leftarrow x=-\frac{1}{2}$

$2=0+2b \quad b=1 \Leftarrow x=\frac{1}{2}$

$=\int_1 2-\frac{1}{2x+1}+\frac{1}{2x-1}$

2- $I=\int_0^2 \frac{4x-5}{2x+1}\,dx$

$$\begin{array}{r|l} & 2 \\ \hline 2x+1 & 4x-5 \\ & \mp 4x \mp 2 \\ \hline & -7 \end{array}$$

$I=\int_0^2 (2-\frac{7}{2x+1})\,dx$

$=\int_0^2 (2-\frac{7}{2}\,\frac{2}{2x+1})\,dx$

$=[2x-\frac{7}{2}\ln(2x+1)]_0^2$

$4-\frac{7}{2}\ln 5-(0-\frac{7}{2}\ln 1)=4-\frac{7}{2}\ln 5$

3- $I=\int_{-1}^2 \frac{2x}{x^2-9}\,dx$

$I=[\ln(9-x^2)]_{-1}^2$

$=\ln 5-\ln 8=\ln\frac{5}{8}$

4- $I=\int_0^3 \frac{x+2}{(x+1)^4}\,dx$

$=\int_0^3 \frac{x+1}{(x+1)^4}+\frac{1}{(x+1)^4}\,dx$

$=\int_0^3 ((x+1)^{-3}+(x+1)^{-4})\,dx$

$=[\frac{(x+1)^{-2}}{-2}+\frac{(x+1)^{-3}}{-3}]_0^3$

$=[\frac{1}{-2(x+1)^2}+\frac{1}{-3(x+1)^3}]_0^3$

$=\frac{1}{-32}+\frac{1}{-192}-(\frac{1}{-2}+\frac{1}{-3})$

$=\frac{-1}{32}-\frac{1}{192}+\frac{1}{2}+\frac{1}{3}=\frac{51}{64}$

17) ليكن f التابع المعرف على $\mathbb{R}$ وفق $f(x)=x^3e^{2x}$
جد تابعاً أصلياً F للتابع f على R بالصيغة $F(x)=P(x)e^{2x}$ ، حيث P تابع كثير الحدود.
نفرض F تابعاً أصلياً

$F(x)=(ax^3+bx^2+cx+d)e^{2x}$

$F'(x)=f(x)$

$(3ax^2+2bx+c)e^{2x}+2e^{2x}(ax^3+bx^2+cx+d)=x^3e^{2x}$

$3ax^2+2bx+c+2ax^3+2bx^2+2cx+2d=x^3$

$2ax^3+(3a+2b)x^2+(2b+2c)x+c+2d=x^3$

$2a=1 \Rightarrow a=\frac{1}{2}$

$3a+2b=0 \Rightarrow \frac{3}{2}+2b=0 \Rightarrow b=-\frac{3}{4}$

$2b+2c=0 \Rightarrow c=-b \Rightarrow c=\frac{3}{4}$

$c+2d=0 \Rightarrow d=-\frac{c}{2} \Rightarrow d=-\frac{3}{8}$

$F(x)=\left(\frac{1}{2}x^3-\frac{3}{4}x^2+\frac{3}{4}x-\frac{3}{8}\right)e^{2x}$

18) نريد حساب $I=\int_0^1 \frac{x^3}{1+x^2}dx$. احسب $J=\int_0^1 \frac{x}{1+x^2}dx$ ، ثم $I+J$ ، واستنتج I .

$J=\int_0^1 \frac{x}{1+x^2}dx \Rightarrow J=\int_0^1 \frac{2x\cdot\frac{1}{2}}{1+x^2}dx$

$=\left[\frac{1}{2}\ln(1+x^2)\right]_0^1=\frac{1}{2}\ln 2$

$I+J=\int_0^1 \frac{x^3}{1+x^2}dx+\int_0^1 \frac{x}{1+x^2}dx$

$=\int_0^1 x\,dx=\left[\frac{1}{2}x^2\right]_0^1=\frac{1}{2}$

$\frac{1}{2}=I+\frac{1}{2}\ln 2 \Rightarrow I=\frac{1}{2}(1-\ln 2)$

19) نريد حساب $I=\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+2\sin x}dx$ احسب $J=\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+2\sin x}dx$ ، ثم $I+J$ ، واستنتج I

$J=\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+2\sin x}dx=\int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\cdot 2\cos x}{1+2\sin x}dx$

$=\left[\frac{1}{2}\ln(1+2\sin x)\right]_0^{\frac{\pi}{2}}=\frac{1}{2}\ln 3$

$I+J=\int_0^{\frac{\pi}{2}} \frac{\cos x+\sin 2x}{1+2\sin x}dx$

$=\int_0^{\frac{\pi}{2}} \frac{\cos x+2\sin x\cos x}{1+2\sin x}$

$=\int_0^{\frac{\pi}{2}} \cos x\,dx=[\sin x]_0^{\frac{\pi}{2}}=1$

$1=J+\frac{1}{2}\ln 3 \Rightarrow J=1-\frac{1}{2}\ln 3$

$I=\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+2\sin x}=\ln e-\ln\sqrt{3}=\ln\frac{e}{\sqrt{3}}$

س20 ليكن التابع f المعرف على $\mathbb{R}$ وفق

$$f(x) = e^{2x}\cos x$$

1- احسب $f'(x)$ و $f''(x)$

2- عين عددين a و b يحققان المساواة

$f(x) = a f'(x) + b f''(x)$ أياً يكن x.

3- استنتج تابعاً أصلياً F للتابع f على $\mathbb{R}$.

$f(x) = e^{2x}\cos x$

$f'(x) = 2e^{2x}\cos x - e^{2x}\sin x = e^{2x}(2\cos x - \sin x)$

$f''(x) = 2e^{x}(2\cos x - \sin x) + (-2\sin x - \cos x)e^{2x}$

$= e^{2x}(3\cos x - 4\sin x)$

2- $f(x) = a f'(x) + b f''(x)$

$e^{2x}\cos x = a e^{2x}(2\cos x - \sin x) + b e^{2x}(3\cos x - 4\sin x)$

$\cos x = (2a+3b)\cos x - (a+4b)\sin x$

$2a+3b = 1$ --- (1) ، $a+4b = 0$ --- (2)

من (2):

$a = -4b$ --- (3)

نعوض في (1):

$-8b + 3b = 1$

$-5b = 1 \Rightarrow b = -\frac{1}{5}$

نعوض في (3)

$a = \frac{4}{5}$

$\Rightarrow f(x) = \frac{4}{5} f'(x) - \frac{1}{5} f''(x)$

3-

$F(x) = \frac{4}{5} f(x) - \frac{1}{5} f'(x)$

$= \frac{4}{5} e^{2x}\cos x - \frac{1}{5} e^{2x}(2\cos x - \sin x)$

$\Rightarrow F(x) = \frac{2}{5} e^{2x}\cos x + \frac{1}{5} e^{2x}\sin x$

مثال محلول رقم 239 :

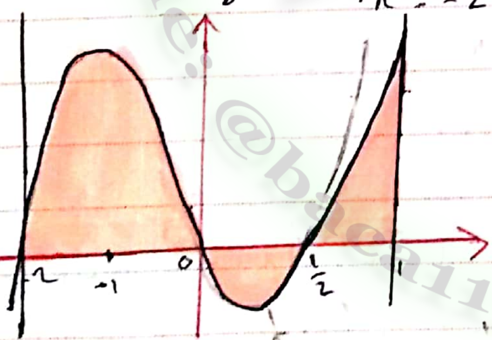
ليقطع C_f للتابع $f(x) = 4x(1-x)$ مع محور الفواصل عند $x=0$ ، $x=1$ عين مساحة السطح المحدود المحصور بين C_f ومحور الفواصل

$S = \int_0^1 4x - 4x^2\,dx = \left[2x^4 - \frac{4}{3}x^3\right]_0^1$

$= 2 - \frac{4}{3} = \frac{2}{3}$

مثال محلول رقم 240 :

ليكن C_f خط التابع $f(x) = 2x^3 + 3x^2 - 2x$ ولنحسب A مساحة السطح المحصور بين C ومحور الفواصل المستقيمين d_1 ، d_2 اللذين معادلتاهما $x=-2$ ، $x=1$



$A = \int_{-2}^{0} f(x)\,dx - \int_0^{\frac{1}{2}} f(x)\,dx + \int_{\frac{1}{2}}^{1} f(x)\,dx$

$= \left[\frac{1}{2}x^4 + x^3 - x^2\right]_{-2}^{0} - \left[\frac{1}{2}x^4 + x^3 - x^2\right]_0^{\frac{1}{2}} + \left[\frac{1}{2}x^4 + x^3 - x^2\right]_{\frac{1}{2}}^{1}$

$= 4 - \frac{3}{32} + \left[\frac{1}{2} + \frac{3}{32}\right] = 4 - \frac{3}{32} + \frac{19}{32} = \frac{75}{16}$

نشاط : p 242

ليكن f التابع المعرف على $\mathbb{R}$ بالعلاقة الآتية : $f(x) = x(1+e^{-x})$ وليكن C_f الخط البياني الممثل للتابع f. الهدف من هذا النشاط دراسة مساحة الحيز المحصور بين الخط البياني ومقاربه.

1-a- ادرس تغيرات التابع f.

b- تحقق أن المستقيم Δ الذي معادلته $y=x$ مستقيم مقارب للخط C_f بجوار $+\infty$، ادرس الوضع النسبي للخط C_f بالنسبة إلى المقارب.

c- ارسم Δ ، C_f.

2-a- ليكن λ عدداً حقيقياً موجباً تماماً. احسب $A(\lambda)$ مساحة الحيز المحصور بين C_f ، Δ والمستقيم الذي معادلته $x=\lambda$

b- ما نهاية $A(\lambda)$ عندما تسعى λ إلى $+\infty$ ؟

1-a- $\lim\limits_{x\to-\infty} f(x) = -\infty$ ، $\lim\limits_{x\to+\infty} f(x) = +\infty$

$f'(x) = 1+e^{-x} - e^{-x}.x = 1+e^{-x}(1-x)$

لدراسة إشارة f' ندرس $f''(x)$

$f''(x) = -e^{-x} + e^{-x}x - e^{-x} = e^{-x}(x-2)$

$f''(x) = 0 \Rightarrow x = 2$

$f'(2) = 1 - e^{-2}$

$\lim\limits_{x\to-\infty} f'(x) = +\infty$ ، $\lim\limits_{x\to+\infty} f'(x) = 1$

x	$-\infty$		2		$+\infty$
$f''(x)$		$-$	0	$+$	
$f'(x)$	$+\infty$	$\searrow$	$1-e^{-2}$	$\nearrow$	1

من جدول التغيرات نجد أن $f'(x) \geq 1-e^{-2} > 0$ أياً كانت $x\in\mathbb{R}$ فالتابع f متزايد تماماً.

x	$-\infty$		$+\infty$
f'		$+$	
f	$-\infty$	$\nearrow$	$+\infty$

b- $f(x) - y_\Delta = x(1+e^{-x}) - x = xe^{-x}$

$\lim\limits_{x\to+\infty} (f(x) - y_\Delta) = 0$

إذاً $y=x$ مقارب بجوار $+\infty$

ندرس إشارة $f(x) - y_\Delta$ من إشارة x

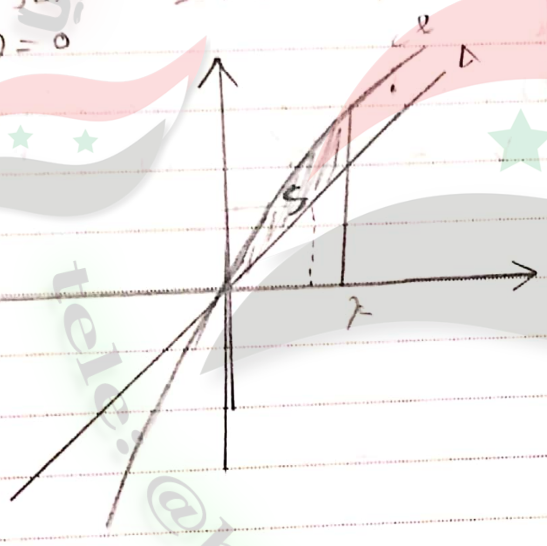
عندما $x>0$ فإن C_f فوق المقارب

عندما $x<0$ فإن C_f تحت المقارب

نحل :

$(f(x) - y_\Delta) = 0 \Rightarrow xe^{-x} = 0 \Rightarrow x = 0$

$f(0) = 0$



2-a- $S = \int_0^\lambda (f(x) - y_\Delta)\,dx$

$= \int_0^\lambda xe^{-x}\,dx$

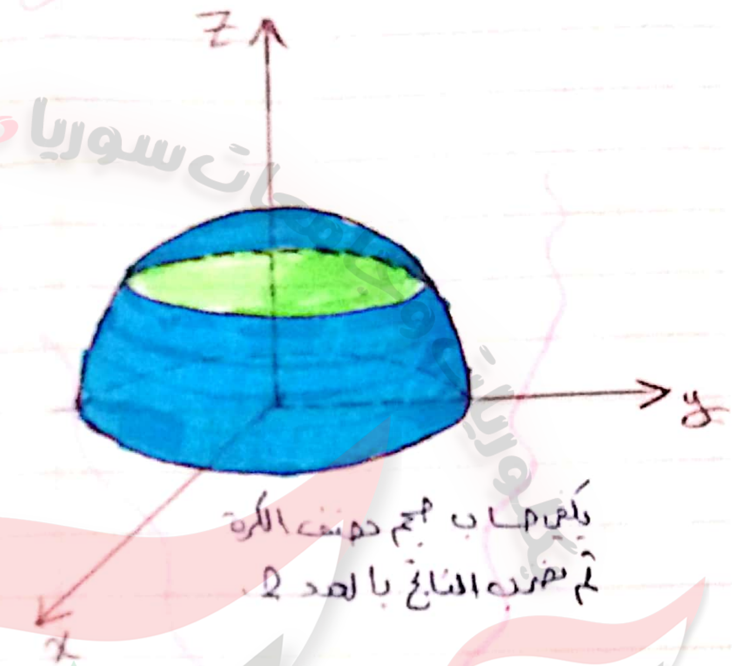
$u = x$	$v' = e^{-x}$
$u' = 1$	$v = -e^{-x}$

$S = [-xe^{-x}]_0^\lambda - \int_0^\lambda -e^{-x}\,dx$

$= -\lambda e^{-\lambda} - e^{-\lambda} + 1$

b- $\lim\limits_{\lambda\to+\infty} S = 1$

حجم كرة نصف قطرها R :



يكفي حساب حجم نصف الكرة ثم نضرب الناتج بالعدد 2

1- الشرائح: $A(Z) = \pi(R^2 - Z^2)$

2- استنتج عبارة $V = \frac{4}{3}\pi R^3$

$r^2 = R^2 - Z^2$ من فيثاغورث

مساحة المقطع

$A(Z) = \pi r^2 = \pi(R^2 - Z^2)$

حجم نصف الكرة

$V = \int_0^R \pi(R^2 - Z^2)\, dZ$

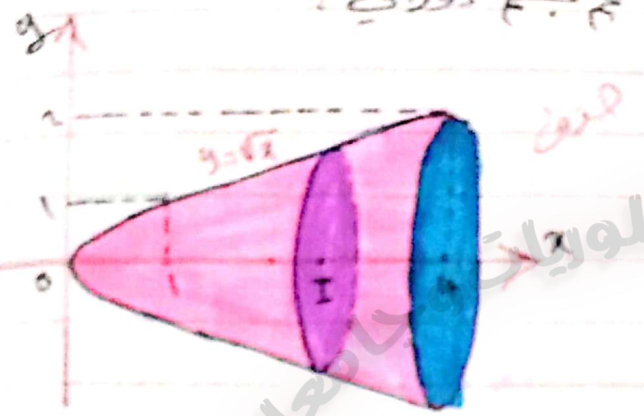
$= [\pi(R^2 Z - \frac{1}{3}Z^3)]_0^R = \pi[R^3 - \frac{1}{3}R^3] - 0$

$= \frac{2}{3}\pi R^2$

ومنه حجم الكرة

$V = 2 \times \frac{2}{3}\pi R^2 = \frac{4}{3}\pi R^2$

حجم جسم دوراني :



مساحة مقطع الجسم ... تساوي

$A(x) = \pi f^2(x)$

$A(x) = \pi(\sqrt{x})^2$

ومنه حجم الجسم

$V = \int_0^4 A(x)\,dx = \int_0^4 \pi x\, dx$

$= [\frac{1}{2}\pi x^2]_0^4 = 8\pi - 0 = 8\pi$

تكاملات وحالات

$\boxed{10}$ احسب التكاملات باستخدام التكامل بالتجزئة:

① $I = \int_1^e (x-1)\ln x\, dx$

$u = \ln x$	$v' = x - 1$
$u' = \frac{1}{x}$	$v = \frac{x^2}{2} - x$

$I = [\ln x(\frac{x^2}{2} - x)]_1^e - \int_1^e \frac{1}{x}(\frac{x^2}{2} - x)\,dx$

$= [\ln x(\frac{x^2}{2} - x)]_1^e - [\frac{1}{4}x^2 - x]_1^e$

$= \frac{e^2}{2} - e - [(\frac{e^2}{4} - e) - (\frac{1}{4} - 1)]$

$= \frac{e^2}{2} - e - \frac{e^2}{4} + e - \frac{3}{4} = \frac{e^2}{4} - \frac{3}{4} = \frac{1}{4}(e^2 - 3)$

② $I = \int_{\ln 1}^{\ln 3}$

$u = (x^2 - 1)$	$v' = e^x$
$u' = 2x$	$v = e^x$

$$I = [(x^2-1)e^x]_{\ln 2}^{\ln 3} - \int_{\ln 2}^{\ln 3} \underbrace{2xe^x}_{J}\, dx \quad \ldots ①$$

$u = 2x$	$v' = e^x$
$u' = 2$	$v = e^x$

$$J = [2x e^x]_{\ln 2}^{\ln 3} - \int_{\ln 2}^{\ln 3} 2e^x\, dx$$

$$= 6\ln 3 - 4\ln 2 - (6-4)$$

$$= 6\ln 3 - 4\ln 2 - 2$$

نعوض في ①:

$$I = [(x^2-1)e^x]_{\ln 2}^{\ln 3} - J$$

$$I = [3\ln^2 3 - 2\ln^2 2 - 1] - 6\ln 3 + 4\ln 2 + 2$$

$$= 3\ln^2 3 - 2\ln^2 2 - 6\ln 3 + 4\ln 2 + 1$$

$$= 3\ln 3(\ln 3 - 2) - 2\ln 2(\ln 2 - 2) + 1$$

③ $I = \int_0^1 (2x+1)e^{-x}\, dx$

$u = 2x+1$	$v' = e^{-x}$
$u' = 2$	$v = -e^{-x}$

$$I = [-(2x+1)e^{-x}]_0^1 - \int_0^1 -2e^{-x}\, dx$$

$$= [-(2x+1)e^{-x}]_0^1 - [2e^{-x}]_0^1 = (-\frac{3}{e} + 1) - (\frac{2}{e} - 2)$$

$$= 3 - \frac{5}{e}$$

④ $I = \int_1^2 (t-2)e^{2t}\, dt$

$u = t-2$	$v' = e^{2t}$
$u' = 1$	$v = \frac{1}{2}e^{2t}$

$$I = [\frac{1}{2}(t-2)e^{2t}]_1^2 - \int_1^2 \frac{1}{2}e^{2t}\, dt$$

$$= [\frac{1}{2}(t-2)e^{2t}]_1^2 - [\frac{1}{4}e^{2t}]_1^2 = -(\frac{1}{2}(-1)e^2) -$$

$$-[(\frac{1}{4}e^4) - (\frac{1}{4}e^2)] = \frac{1}{2}e^2 - \frac{1}{4}e^4 + \frac{1}{4}e^2 = \frac{1}{4}e^2(3 - e^2)$$