بسم الله الرحمن الرحيم

[واتقوا الله ويعلمكم الله والله بكل شئ عليم]

سورة البقرة الاية (282)

Welcome in Funny Mathematic World

اهلا وسهلا بـــكم فـــي عالـم الـرياضيات الـممتع

مع تحيات

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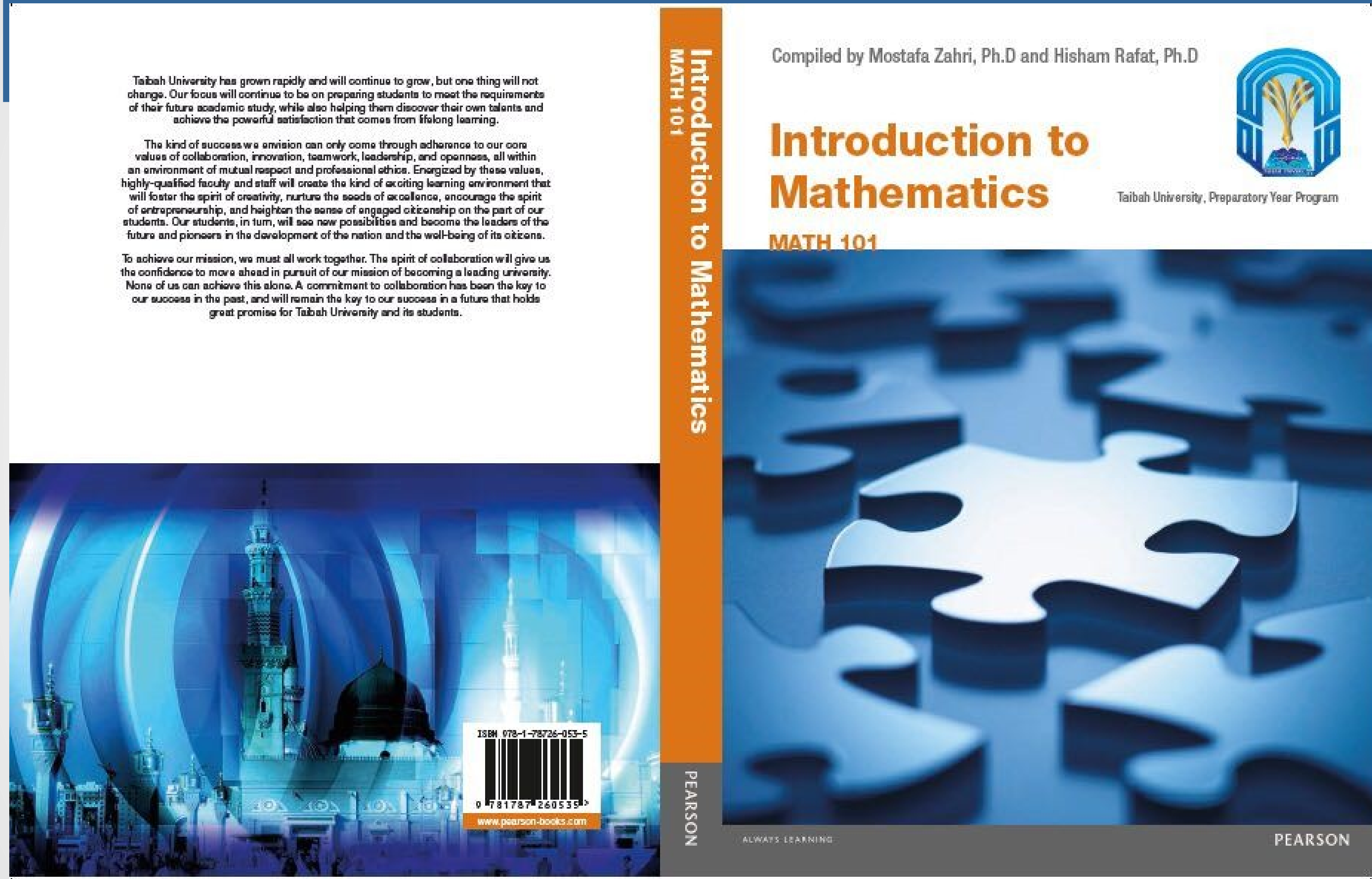
Taibah University has grown rapidly and will continue to grow, but one thing will not change. Our focus will continue to be on preparing students to meet the requirements of their future academic study, while also helping them discover their own talents and achieve the powerful satisfaction that comes from lifelong learning.
The kind of success we envision can only come through adherence to our core values of collaboration, innovation, teamwork, leadership, and openness, all within an environment of mutual respect and professional ethics. Energized by these values, highly-qualified faculty and staff will create the kind of exciting learning environment that will foster the spirit of creativity, nurture the seeds of excellence, encourage the spirit of entrepreneurship, and heighten the sense of engaged citizenship on the part of our students. Our students, in turn, will see new possibilities and become the leaders of the future and pioneers in the development of the nation and the well-being of its citizens.
To achieve our mission, we must all work together. The spirit of collaboration will give us the confidence to move ahead in pursuit of our mission of becoming a leading university. None of us can achieve this alone. A commitment to collaboration has been the key to our success in the past, and will remain the key to our success in a future that holds great promise for Taibah University and its students.
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MATH 101
PEARSON
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MATH 101
Taibah University, Preparatory Year Program
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- Final Exam (40 D).
- Total (100 D)

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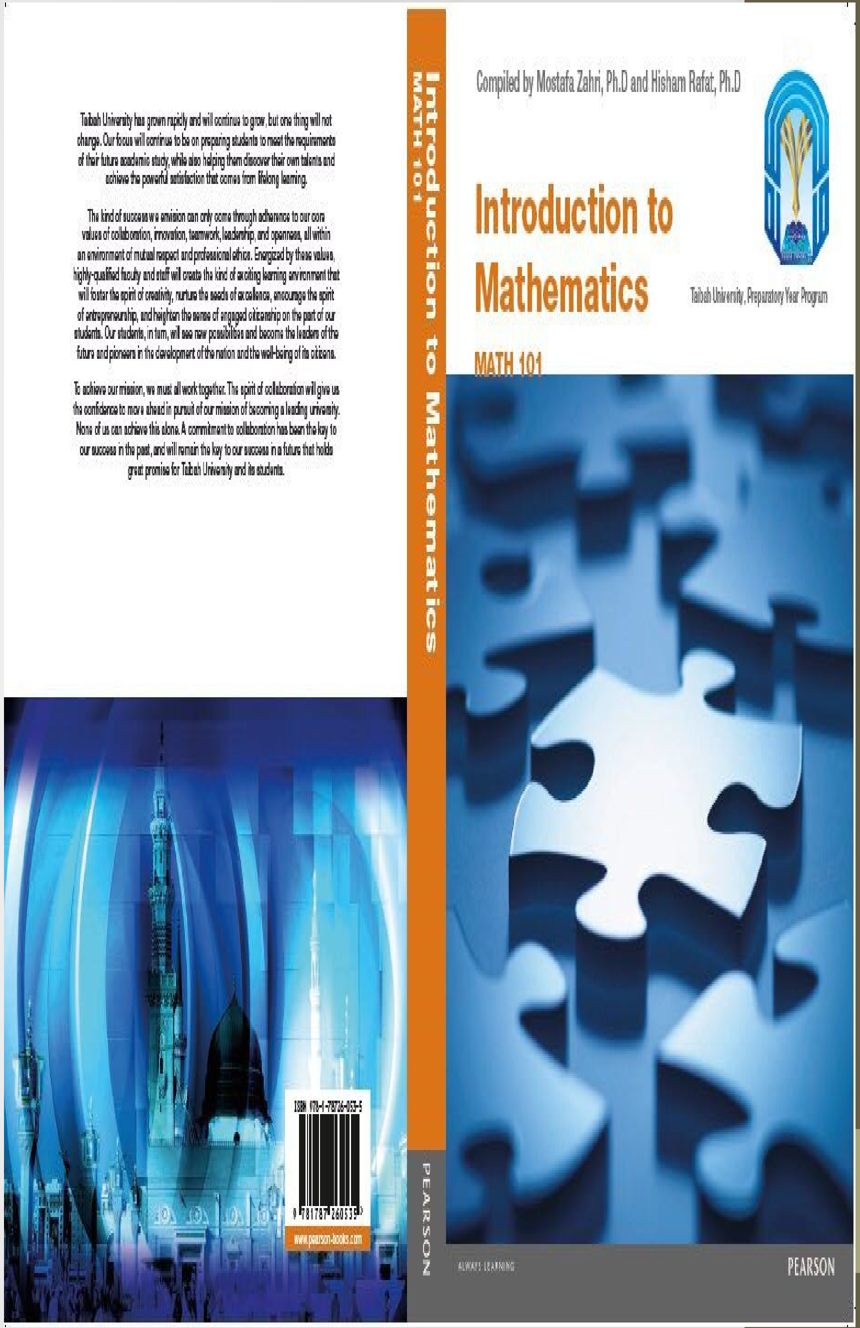
Review of Basic Concepts



1.1 Sets

- Basic Definitions
- Operations on Sets

Basic Definitions

Set: A set is a collection of objects.

The objects that belongs to a set are called the elements, or members, of the set.

*

Sets are commonly written using set braces{ }.

- **Any set has name: A,B,C,S,…**
- **Elements: a,b,c,…..**

- **Ex: S={1,2,3}, A={a,b,c},…**

- The order is not important.
- {1,2,3}={2,1,3}={3,1,2}

- Don’t repeat any element.
- {1,1,2,3} is False, {1,2,3} is True

Belongs to $\in$, Don't belongs to $\notin$

$$4 \in \{1, 2, 3, 4\}, a \in \{a, b, c\}$$

$$5 \notin \{1, 2, 3, 4\}, d \notin \{a, b, c\}$$

Set builder notation

S={1,2,3,4}= {the set containing the first four counting number }

={x|x is a natural number between 2 and 7}={3,4,5,6}

Finite and Infinite sets

A finite set is one that has a limited number of elements.

S={1,2,3,4}

A={1,2,3,….,20}

B

3,4,5,6}

Infinite set: is one that has no limited number of elements.

N={1,2,3,…} (Natural counting numbers)

Finite and Infinite sets

Infinite set:

O={1,3,5,…} (Odd numbers)

E={2,4,6,…} (Even numbers)

Between any two distinct natural numbers there are infinitely many fractions.

.

Sets

Example 1: **Using Set Notation and Terminology**

Identify each set as finite or infinite. Then determine whether 10 is an element of the set.

A={7,8,9,…,14}

B={1,1/4,1/16,1/64,…}

C={x|x is a fraction between 1 and 2}

D={x|xis a natural number between 9 and 11}

Sets

Solution:

A={7,8,9,…,14} is finite set, 10$\in A$

B={1,1/4,1/16,1/64,…} is infinite set, $10 \notin B$

C={x|x is a fraction between 1 and 2} is infinite set $10 \notin C$

D={x|xis a natural number between 9 and 11}

Is finite set and 10$\in D$.

Homework 1: **Listing the Elements of a Set**

Use set notation, and write the elements each set.

a) {x|x is a natural number less than 5}

b) {x|x is a natural number greater than 7 and less than 14}

Solution:

a) {1,2,3,4}

b) {8,9,10,11,12,13}

Especial Sets

1) The empty set : (the null set)

$$\emptyset = \{\ \ \}$$

2) The universal set U=contains all elements included in the discussion.

Subset and not subset

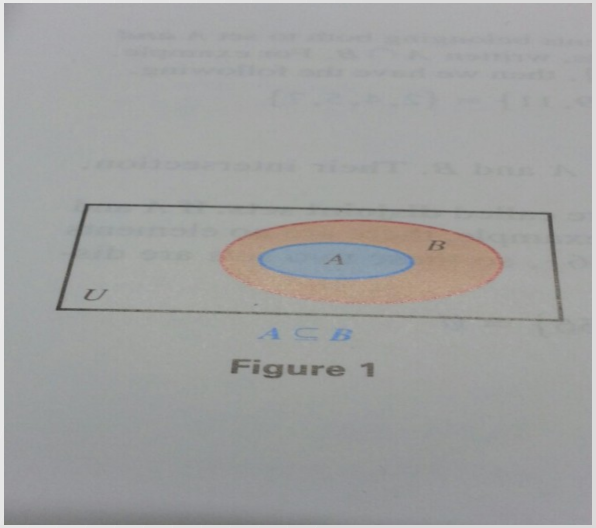
A$\subseteq B$ if all elements in A are elements in B.

1) A={2,5,9}, B={2,3,5,6,9,10}

A$\subseteq B$, B$\not\subseteq A$

2) S={1,2,3,4}, S$\subseteq N$.

3) $\emptyset \subseteq A$ $for\ any\ set\ A$.



Equal sets

A=B iff $A \subseteq B$ and $B \subseteq A$

$\{1,2,3\}=\{3,2,1\}$

But

$\{1,2,3\} \neq \{0,1,2,3\}$

Set Operations

Example 2:

Let U={1,3,5,7,9,11,13}, A={ 1,3,5,7,9,11}, B={1,3,7,9} , C={3,9,11}, and D={1,9}. Determine each statement True or False.

Solution

The complement of a set A المكملة

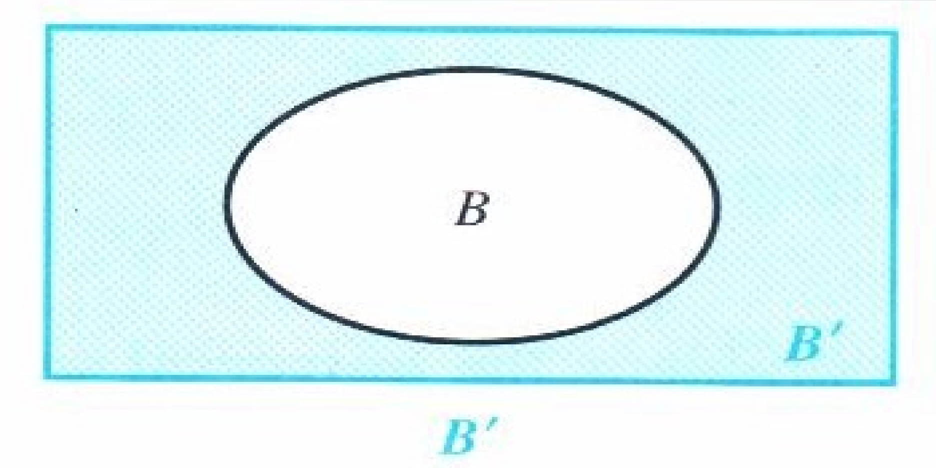
Homework 2.

Let U={1,2,3,4,5,6,7}, A={ 1,3,5,7}, B={3,4,6} ,

Find each set

Solution

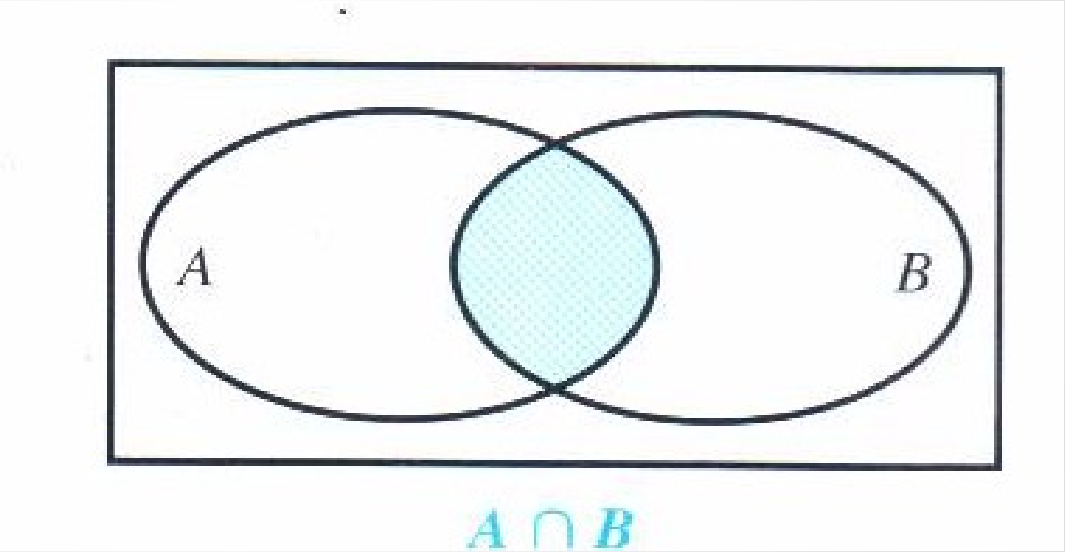
THE COMPLEMENT المكملة



THE INTERSECTION التقاطع

Ex:

THE INTERSECTION



Example 3: **Finding the Intersection of Two Sets**

Find each of the following.

Solution

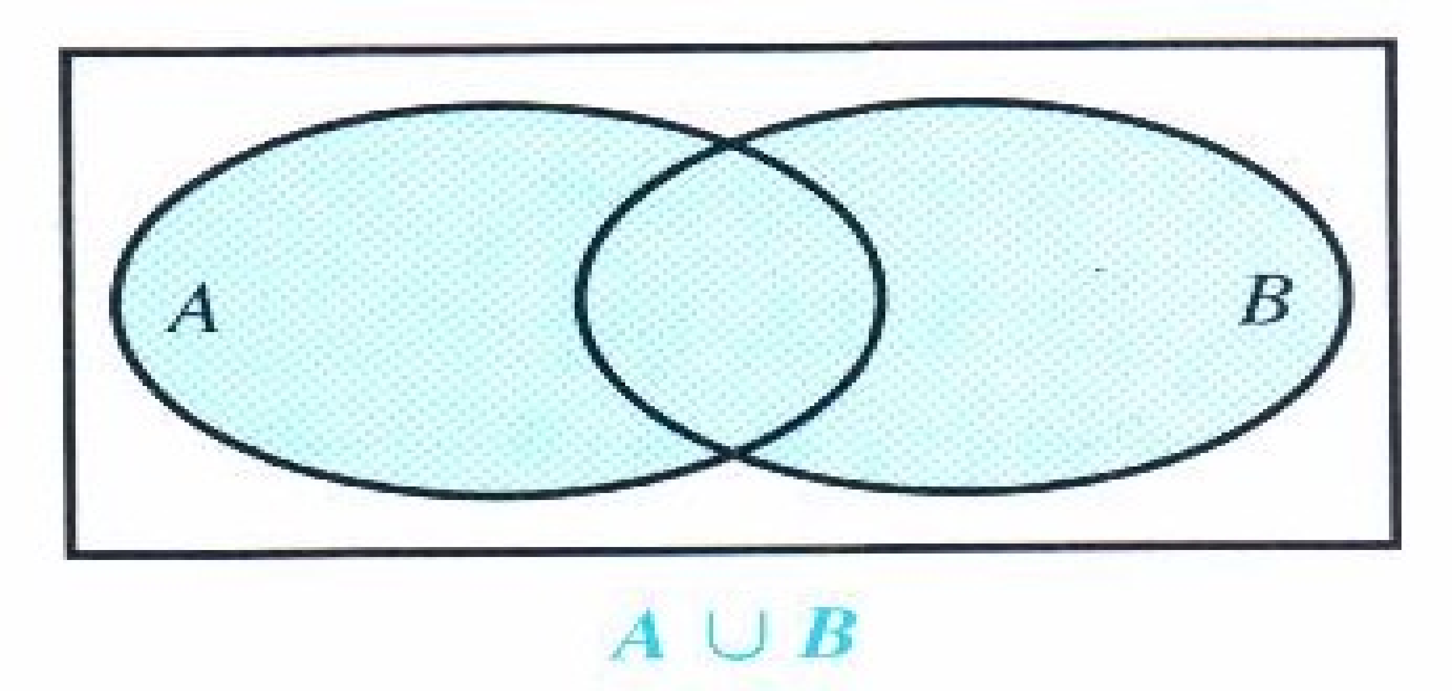
Notes: 1) If A

,

THE UNION الاتحاد

Ex:

THE UNION



Homework 3: **Finding the Union of Two Sets**

Find each of the following.

Solution

Notes: 1) If A

,

Set Operations

Let be sets, with universal set .

The complement of a set is the set of all elements in the universal set that do not belong to set

The intersection of a set A and B, Written is made up all the elements belongs to both set A and set B.

The union of sets A and B, written , is made up of all the elements belongs to set A or to set B.

1.2 Real Numbers and Their Properties

- Sets of numbers and the Number Line
- Exponents
- Order of Operations
- Properties of Real Numbers
- Order on the Number Line
- Absolute Value

1.2 Real Numbers and Their Properties

Sets of Numbers and the Number line.

1-Natural number N={1,2,3,…}

2-Whole numbers W={0,1,2,3,…}

3- Integers I={…,-3,-2,-1,0,1,2,3,…}

4- Rational numbers

Q=$\{\frac{p}{q} \mid p \text{ and } q \text{ are integers and } q \neq 0\}$

$$N \subseteq W \subseteq I \subseteq Q$$

1.2 Real Numbers and Their Properties

Rational numbers contains:

$2=\frac{2}{1}=\frac{4}{2}, \ldots, \quad \sqrt{4}=2, \sqrt{9}=3, \ldots \ldots$

$0=\frac{0}{1}=\frac{0}{2}, \ldots$

$\frac{1}{2}, \frac{2}{3}, \ldots.$

$0.75=\frac{3}{4},\ 0.758=\frac{758}{100}, \ldots$

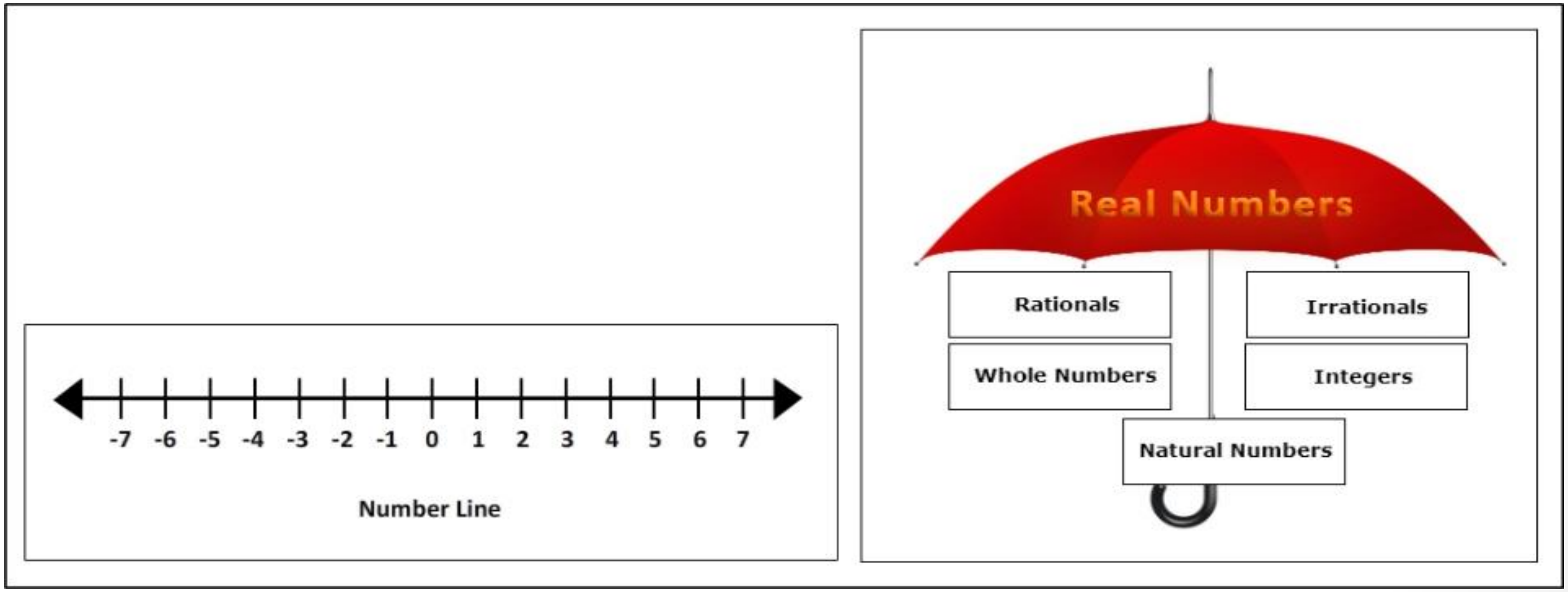
$0.\overline{6}=0.6666\ldots$

1.2 Real Numbers and Their Properties

Irrational numbers contains: Q

$\sqrt{2}, \sqrt{3}, \sqrt{5}, 0.4785 \ldots \ldots$

Real Numbers =R



1.2 Real Numbers and Their Properties

Set	Description
Natural numbers	{1, 2, 3, 4, …}
Whole numbers	{0, 1, 2, 3, 4, …}
Integers	{…,-3, -2, -1, 0, 1, 2, 3, …}
Rational numbers	$\{\frac{p}{q} \mid p$ and q are integers and $q \neq 0\}$
Irrational numbers	$\{x \mid x$ is real but not rational$\}$
Real numbers	$\{x \mid x$ corresponds to a point on a number line$\}$

Example 1: Identifying Sets of Numbers

Let $A = \left\{-8, -6, -\frac{12}{4}, -\frac{3}{4}, 0, \frac{3}{8}, \frac{1}{2}, 1, \sqrt{2}, \sqrt{5}, 6\right\}$.

List the elements from A that belong to each set.

a) Natural numbers b) Whole numbers c) Integers

d) Rational numbers e) Irrational numbers

f) Real numbers

Solution:

a) Natural numbers={1,6}

b) Whole numbers ={0,1,6}

c) Integers ={-8,-6, $-\frac{12}{4}$(or -3),0,1,6}

d) Rational numbers ={-8,-6, $-\frac{12}{4}$(or -3), $-\frac{3}{4}, 0, \frac{3}{8}, \frac{1}{2}, 1$,6}

e) Irrational numbers={$\sqrt{2}, \sqrt{5}$}

f) Real numbers= All elements of A.

The real numbers

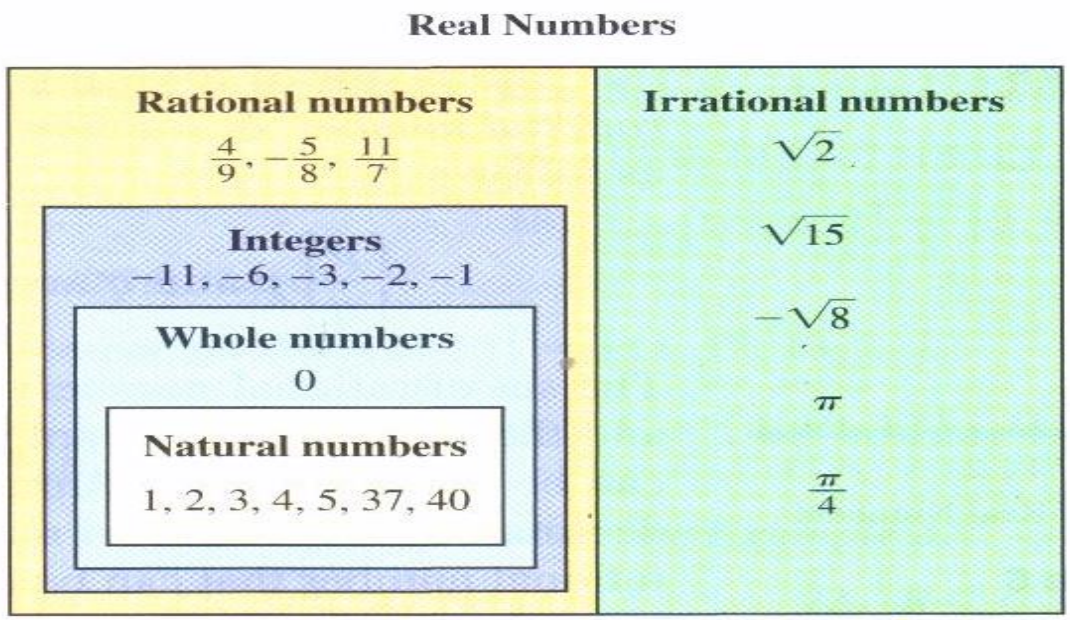


Figure 8

Exponents

- **Exponents**
- **What are ‘Exponents’?**
- Exponents:

$$\mathbf{a}^{n} = \mathbf{a} \cdot \mathbf{a} \cdot \mathbf{a} \cdot \mathbf{a} \cdot \ldots \cdot \mathbf{a}$$

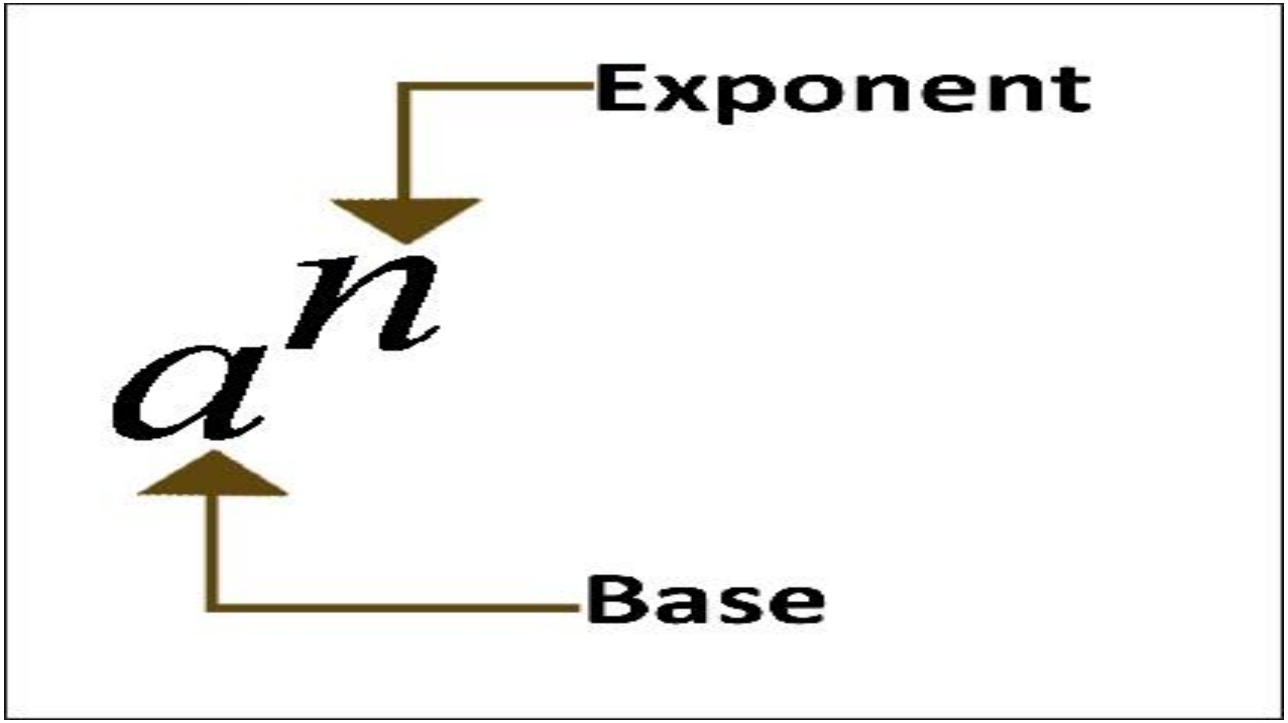
n factors of a

This is known as exponential notation. Put simply, $\mathbf{a}^n$ means **a** is multiplied by itself ***n*** times. In math, we say $\mathbf{a}^n$ is **a** to the ***n*** th power.

- In the expression a^n, **a** is known as the **base**, and ***n*** is known as the **exponent**

Exponents

- Exponents



Homework1

Evaluate each exponential expression , and identify the base and the exponent.

$a)\ 4^3$ b)$(-6)^2$ c)-6^2 d)4.3^2 e) $(4.3)^2$

Solution:

$a)\ 4^3$=4.4.4=64, the base is 4 and the exponent is 3

$b)\ (-6)^2 = (-6).(-6) = 36$ the base is (-6) the exponent is 2

$c)\ -6^2 = -6.6 = -36$, the base is 6 and the exponent is 2.

$d)\ 4.3^2 = 4.3.3 = 36$, the base is 3 and exponent is 2.

$e)\ (4.3)^2 = (4.3).(4.3) = 12.12 = 124$, the base is (4.3) and the exponent is 2.

Exponents

$$8^3 = 8 \times 8 \times 8 = 512$$

$$\left(-\frac{2}{7}\right)^4$$
$$= \left(-\frac{2}{7}\right)\left(-\frac{2}{7}\right)\left(-\frac{2}{7}\right)\left(-\frac{2}{7}\right)$$
$$= \frac{16}{2401}$$

Rule	Example
$a^0 = 1$	$9^0 = 1$ $4^0 = 1$
$a^n \times a^m = a^{n+m}$	$9^4 \times 9^3 = 9^7$ $4^{-2} \times 4^5 = 4^3$
$(a^n)^m = a^{nm}$	$(x^2)^3 = x^6$ $(5^2)^4 = 5^8 = 390{,}625$
$a^n \times b^n = (ab)^n$	$13^2 \times 3^2 = (13 \times 3)^2 = (39)^2 = 1521$ $y^8 \times z^8 = (yz)^8$
$\frac{1}{a^n} = a^{-n}$	$\frac{1}{7^5} = 7^{-5}$ $\frac{1}{p^3} = p^{-3}$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{7^{11}}{7^3} = 7^{11-8} = 7^3 = 343$ $\frac{g^{12}}{g^2} = g^{10}$
$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$	$\frac{5^3}{7^3} = \left(\frac{5}{7}\right)^3$ $\frac{k^6}{l^6} = \left(\frac{k}{l}\right)^6$

Order of operations

Rules for Order of Operations

Let's summarize the order in which we should perform operations when simplifying or evaluating expressions.

Step 1 Treat both parts of fractions separately

Work separately above and below each fraction bar

Step 2 Parentheses ()

Use the rules that follow within each set of parentheses or square brackets first. Start with the innermost set and work outward.

Step 3 Exponents x^y

Simplify all powers. Work from left to right.

Step 4 Roots √

Simplify all roots. Work from left to right.

Step 5 Multiplications and divisions × ÷

Do any multiplications or divisions in order. Work from left to right.

Step 6 Additions and subtractions + –

Do any additions or subtractions in order. Work from left to right.

Example 2:

Evaluate each expression.

$a)\ 6 \div 3 + 2^3 . 5$ $\quad b)\ (8+6) \div 7 \cdot 3 - 6$

$c)\ \frac{4+3^2}{6-5\cdot3}$ $\quad$ d) $\frac{-(-3)^3+(-5)}{2(-8)-5(3)}$

Solution:

$a)\ 6 \div 3 + 2^3 . 5 = 6 \div 3 + 8 \cdot 5 = 2 + 8 \cdot 5 = 2 + 40 = 42$

$b)\ (8+6) \div 7 \cdot 3 - 6 = 14 \div 7 \cdot 3 - 6 = 2 \cdot 3 - 6 = 6 - 6 = 0$

$c)\ \frac{4+3^2}{6-5\cdot3} = \frac{4+9}{6-15} = \frac{13}{-9}\ or = -\frac{13}{9}$

d) $\frac{-(-3)^3+(-5)}{2(-8)-5(3)} = \frac{-(-27)+(-5)}{2(-8)-5(3)} = \frac{27+(-5)}{-16-15} = \frac{22}{-31}\ or = -\frac{22}{31}$

Homework 2: Using order of Operations

Evaluate each expression for $x=-2, y=5, and\ z=-3$

$a)\ -4x^2-7y+4z$ $b)\ \frac{2(x-5)^2+4y}{z+4}$ $c)\ \frac{\frac{x}{2}-\frac{y}{5}}{\frac{3z}{9}+\frac{8y}{5}}$

Solution:

$$a)\ -4x^2-7y+4z=-4(-2)^2-7(5)+4(-3)$$

$$=-4(4)-35-12=-16-35-12=-63$$

$$b)\frac{2(x-5)^2+4y}{z+4}=\frac{2(-2-5)^2+4(5)}{-3+4}=\frac{2(-7)^2+4(5)}{1}$$

$$=2(49)+20=98+20=118$$

$$c)\ \frac{\frac{x}{2}-\frac{y}{5}}{\frac{3z}{9}+\frac{8y}{5}}=\frac{\frac{-2}{2}-\frac{5}{5}}{\frac{3\cdot(-3)}{9}+\frac{8(5)}{5}}=\frac{-1-1}{-1+8}=\frac{-2}{7}$$

Order of operations

Examples

1) $8 + (3 \times 7^2 + 5)$

$= 8 + (3 \times 49 + 5)$ Work within brackets, exponents

$= 8 + (117 + 5)$ Multiplication

$= 8 + 122$ Addition within bracket

=130 Addition

2) $(3^2 + (18 \div 9 + 3^2)) + 5^2$

$= (3^2 + (18 \div 9 + 9)) + 5^2$ Innermost bracket, exponent

$= (3^2 + (2 + 9)) + 5^2$ Division

$= (3^2 + 11) + 5^2$ Addition

$= (9 + 11) + 5^2$ Next bracket, exponent

x= 20 + 25 Addition within bracket, exponent

= **45** Addition

Properties of Real numbers

- **The Commutative Property of Addition**
- Look at this expression:

4 + 5 = 9 is the same as **5 + 4 = 9**

This is an example of **commutative property**. It means that we can move the numbers around in an addition sum and still get the same answer.

Properties of Real numbers

- **The Commutative Property of Multiplication and the Closure Property**
- As with addition, the commutative property works for multiplication too.

4 x 5 = 20 is the same as **5 x 4 = 20**

The commutative property of real numbers allows us to switch the order of numbers in additions and multiplications, making such operations simpler.

- The **commutative property** of real numbers allows us to switch the order of the terms in additions and multiplications without changing the answers.
- Another property of real numbers, the **closure property**, states that those answers will be real numbers.
- The sum of two real numbers is a real number – the additive closure property.
- The product of two real numbers is a real number – the multiplicative closure property.
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The Associative Property of Addition
Look at this expression:

3x + (4x + 6) is the same as **(3x + 4x) + 6**

This is an example of **associative property**.
The Associative Property of Multiplication
As with addition, the associative property works for multiplication too.

3 (2x) is the same as **(3 × 2)x**

The **associative property** of real numbers allows us to regroup numbers in additions and multiplications, making such operations simpler.

The commutative and associative properties we have described only apply to addition and multiplication. They do not work with subtraction or division.

This means, for example, that for any two numbers, a and b, where $a \neq$ b:

$$a - b \neq b - a \quad \text{and} \quad a \div b \neq b \div a$$

Additive Identity

Now let's consider the identity property. The identity property for addition is a number that when added to any number does not change the value of that number.

The additive identity for real numbers is 0. This means that adding 0 to any number doesn't change that number's value.

Example:

$3 + 0 = 3$

$970 + 0 = 970$

In general terms, there exists a unique real number 0 such that

This is known as the additive identity

Multiplicative Identity

Is there an identity property for multiplications too?

Yes, there is multiplicative identity. It means that when we multiply 1 by any number we get the same number, which means that it keeps its identity.

Example:

$4 \times 1 = 4$

$105 \times 1 = 105$

In general terms, there exists a unique real number 1 such that

This is known as the multiplicative identity

Summery

Property	Description	Equations
Commutative property	The sum or product of two real numbers is the same regardless of their order.	$a + b = b + a$ $ab = ba$
Closure property	The sum or product of two real numbers is a real number.	$a + b$ is a real number ab is a real number
Associative property	The sum or product of three real numbers is the same no matter which two are added or multiplied first.	$(a + b) + c = a + (b + c)$ $(ab)c = a(bc)$
Identity property	The sum of a real number and 0 is that real number, and the product of a real number and 1 is that real number.	$a + 0 = a$ $a \cdot 1 = a$
Inverse property	The sum of any real number and its negative is 0, and the product of any nonzero real number and its reciprocal is 1.	$a + (-a) = 0$ $a \cdot \frac{1}{a} = 1$ for $a \neq 0$
Distributive property	The product of a real number and the sum of two real numbers equals the sum of the products of the first number and each of the other numbers.	$a(b + c) = ab + bc$

Example 3 Simplifying Expression

Use the commutative and associative properties to simplify each expression

$a)\ 6 + (9 + x)$ $\quad b)\ \frac{5}{8}(16y)$ $\quad c)\ -10p(\frac{6}{5})$

Solution :

$a)\ 6 + (9 + x) = (6 + 9) + x = 15 + x$

$b)\ \frac{5}{8}(16y) = \left(\frac{5}{8} \cdot 16\right) y = 10y$

$c)\ -10p\left(\frac{6}{5}\right) = \frac{6}{5}(-10p) = \left[\frac{6}{5}(-10)\right] p = -12p$

Homework 3 Using the Distributive Property

Rewrite each expression using the distributive property and simplify, if possible.

$a)\ 3(x+y) \qquad b)\ -(m-4n)$

$c)\ \frac{1}{3}\left(\frac{4}{5}m-\frac{3}{2}n-27\right) \qquad d)\ 7p+21$

Solution :

$a)\ 3(x+y)=3x+3y$

$b)\ -(m-4n)=-m+4n$

Homework 3

Using the Distributive Property

$$c)\ \frac{1}{3}\left(\frac{4}{5}m-\frac{3}{2}n-27\right)$$

$$=\frac{1}{3}\cdot\frac{4}{5}m-\frac{1}{3}\cdot\frac{3}{2}n-\frac{1}{3}\cdot 27$$

$$=\frac{4}{15}m-\frac{1}{2}n-9$$

$$d)\ 7p+21=7(p+3)$$

Order on the Number Line

If the real number a is to the left of the real number b on a number line, then

a *is less than* b, *written* $a < b$

If a is to the right of b, then

a *is greater than* b, *written* $a > b$

Also we have

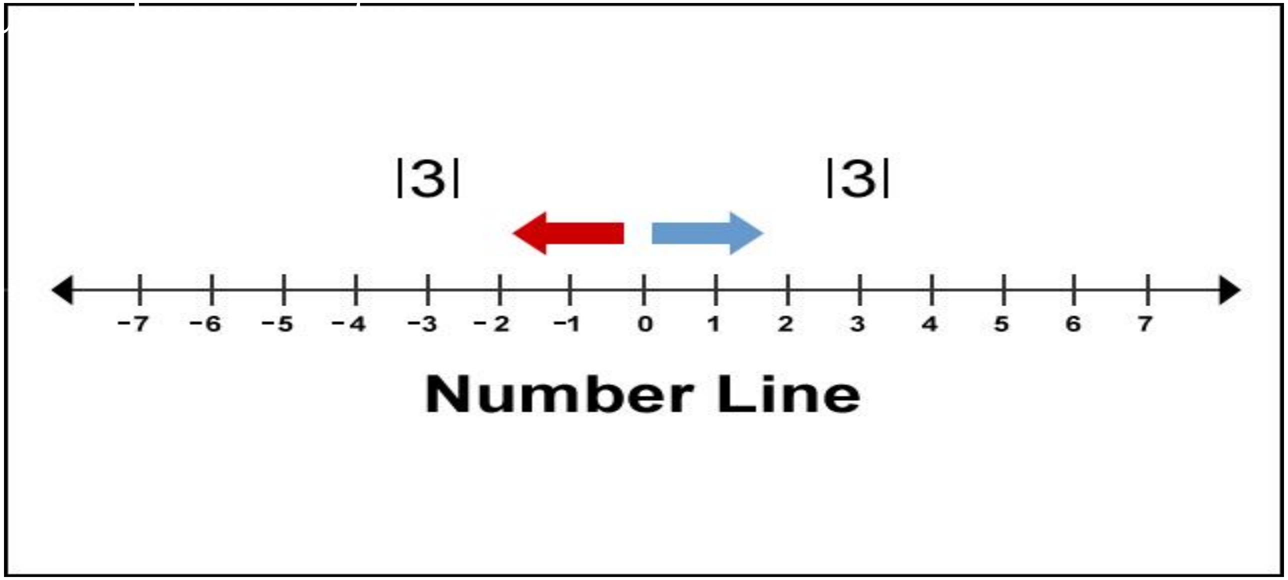
$a \leq b$ (a *is less than or equal* b)

$a \geq b$ (a *is greater than or equal* b)

$a < b < c$, b *is between* a *and* c.

Absolute Value

Absolute value is the distance of any number from 0 on a number line in any direction.
The direction doesn't change the distance; it is always positive.
So whether we are finding absolute value for negative or positive numbers, the absolute value is always positive.



Examples:

$|-3| = 3$, $|3| = 3$, $|-5| = 5$,

$$\left|-\frac{3}{8}\right| = \frac{3}{8}, -|-2| = -2, -|8| = -8$$

Homework4:

Let x=-6, and y=10. Evaluate each expression :

a) $|2x-3y|$ b) $\frac{2|x|-|3y|}{|xy|}$

Distance between points on a number line

If P and Q are two points on a number line with coordinates a and b, respectively, then the distance d(P,Q) between them is given by the following

d(P,Q)=|a-b|= or d(P,Q)=|a-b|

Example 5: Find the distance between -5 and 8.

Solution:

Use a=-5 and b=8

d(-5,8)=|-5-8|=|-13|=13

Or for a=8 and b=-5

d(8,-5)=|8-(-5)|=|8+5|=|13|=13

1.3 Polynomials

- Rules for Exponents
- Polynomials
- Addition and Subtraction
- Multiplication
- Division

Rules of Exponents

Rule	Math notation	Description
Product rule	$a^{m} \cdot a^{n} = a^{m+n}$	When multiplying powers of like bases, keep the base and add the exponents.
Power rule 1	$(a^{m})^{n} = a^{mn}$	To raise a power to a power, multiply the exponents.
Power rule 2	$(ab)^{m} = a^{m}b^{m}$	To raise a product to a power, raise each factor to that power.
Power rule 3	$\left(\frac{a}{b}\right)^{m} = \frac{a^{m}}{b^{m}} \quad b \neq 0$	To raise a quotient to a power, raise the numerator and the denominator.
Zero exponent	$a^{0} = 1 \quad a \neq 0$	A nonzero number to the power of zero equals 1.

Examples:

Find each product :

a) $y^4 . y^7$ $b)(6z^5)(9z^3)(2z^2)$

Solution:

a) $y^4 . y^7 = y^{4+7} = y^{11}$

$b)(6z^5)(9z^3)(2z^2)$=(6.9.2) $(z^5 . z^3 .)$=$108z^{10}$

Homework 1:

Simplify:

a) $(5^3)^2$ $b)$ $(3^4x^2)^3$ c) $(\frac{2^5}{b^4})^3$ d) $(\frac{-2m^6}{t^2z})^5$

Example 2 : Evaluate each power

a) 4^0 $b)$ $(-4)^0$ c)-4^0 $d) -(-4)^0$

$e)(7r)^0$

Solution : a) $4^0 = 1$ $b)$ $(-4)^0$=1

c)-$4^0 = -1$ $d) -(-4)^0 = -1$

$e)(7r)^0 = 1, r \neq 0$

Algebraic expression.

Any collection of numbers or variables joined by the basic operations of addition, subtractions multiplication or division and so on

$$-2x^2 + 3x, \frac{15y}{2y-3}, \sqrt{m^3 - 64},$$

$$(3a+b)^4$$

When a *false* statement such as −3 = 7 results, the equation is a contradiction, and the solution set is the **empty set** or **null set**, symbolized by $\emptyset$.

Term:

The product of a real number and one or more variables raised to powers

Example: 1) the term $-3m^4$

The coefficient is -3, the variable is m the power (degree) is 4

2) the term $-p^2$

The coefficient is -1, the variable is p the power (degree) is 2

Like Terms:

Are terms with the same variables each raised to the same powers

Example: 1) the terms $-3m^4, 6m^4, 4m^4$ are like terms

2) the terms $-3y^4, 6m^4, 4r^4$ are unlike terms.

Types of Polynomials:

1. One term is called Monomial. $-10r^6s^8$
2. Two terms is called Binomial. $29x^{11} + 8x^{15}$
3. Three terms is called Trinomial. $9p^7 - 4p^3 + 8p^2$
4. More than three terms is called None of These

$5a^3b^7 - 3a^5b^5 + 4a^2b^9 - a^{10}$

Addition and Subtraction

- We use $3m^5 - 7m^5 = (3-7)m^5 = -4m^5$
- **Example 3: Adding and subtracting polynomials:**

Add or subtract , as indicated.

- $a)\ (2y^4 - 3y^2 + y) + (4y^4 + 7y^2 + 6y)$
- b) $(-3m^3 - 8m^2 + 4) - (m^3 + 7m^2 - 3)$
- c) $(8m^4p^5 - 9m^3p^5) + (11m^4p^5 + 15m^3p^5)$
- d) $4(x^2 - 3x + 7) - 5(2x^2 - 8x - 4)$
- **Solution:**
- $a)\ (2y^4 - 3y^2 + y) + (4y^4 + 7y^2 + 6y) = (2+4)y^4 + (-3+7)y^2 + (1+6)y = 6y^4 + 4y^2 + 7y$
- b) $(-3m^3 - 8m^2 + 4) - (m^3 + 7m^2 - 3) = (-3-1)m^3 + (-8-7)m^2 + [4-(-3)] = -4m^3 - 15m^2 + 7$

Addition and subtraction

c) $(8m^4p^5 - 9m^3p^5) + (11m^4p^5 + 15m^3p^5)$
$= 19m^4p^5 + 6m^3p^5$

d) $4(x^2 - 3x + 7) - 5(2x^2 - 8x - 4) = (4 - 10)\,x^2$
$+ (-12 + 40)\text{x} + 28 + 20 = -6x^2 + 28x + 48.$

•Multiplication of Polynomials

- **Multiplication of Polynomials**
- There are several methods for multiplying polynomials. The choice of method depends on the type of polynomials being multiplied together.
- One of the easiest methods of multiplying polynomials is to use the concept of distribution property.
- $-3x(4x^2 - x + 10) = -12x^2 + 3x^2 - 30x$

•Multiplication of Polynomials

For example: :(Product horizontal)

$(3x-4)(2x^2-3x+5)=$

$=(3x-4)(2x^2)-(3x-4)(3x)+(3x-4)(5)$

$= 3x(2x^2)-4(2x^2)-3x(3x)-(-4)(3x)$
$\quad +3x(5)-4(5)$

$= 6x^3-8x^2-9x^2+12x+15x-20$

$= 6x^3-17x^2+27x-20$

•Multiplication of Polynomials

For example:(Product vertically)

$$(2x^2 - 3x + 5)$$
$$(3x - 4)$$

$$-8x^2 + 12x - 20 \leftarrow (-4)(2x^2 - 3x + 5)$$
$$6x^3 - 9x^2 + 15x \qquad \leftarrow (3x)\ (2x2 - 3x + 5)$$

$$6x^3 - 17x^2 + 27x - 20$$

•Multiplication of Polynomials

Homework 3: **Multiplication of Polynomials**

Multiply $(3p^2 - 4p + 1)(p^3 + 2p - 8)$

Solution:

$(3p^2 - 4p + 1)(p^3 + 2p - 8)$

$= (3p^2)(p^3) + (3p^2)(2\text{p}) + (3p^2)(-8)$

$+(-4p)(p^3) + (-4p)(2\text{p}) + (-4p)(-8)$

$+(1)(p^3) + (1)(2\text{p}) + (1)(-8)$

$= 3p^5 + 6p^3 - 24p^2 - 4p^3 - 8p^2 + 32p$

$+p^3 + 2\text{p} + -8 = 3p^5 + 3p^3 - 32p^2 + 34p - 8$

•Multiplication of Polynomials

FOIL method: (First, Outside, Inside, Last)

Example 4:

Find each product:

a) $(6m+1)(4m-3)$ b) $(2\text{x}+7)(2\text{x}-7)$

c) $r^2(3r+2)(3r-2)$

Solution:

a) $(6m+1)(4m-3) = 6m(4m) + 6m(-3) + 1(4m) + 1(-3) = 24m^2 - 14m - 3$

b) $(2\text{x}+7)(2\text{x}-7) = 4x^2 - 14x + 14x - 49 = 4x^2 - 49.$

•Multiplication of Polynomials

Solution:

c) $r^2(3r+2)(3r-2)$ =

=$r^2(9r^2 - 6\text{r} + 6\text{r} - 4)$

$= r^2(9r^2 - 4)$

$= 9r^4 - 4r^2$

•Special product

Product of the sum and difference of two terms:

$$(x + y)(x - y) = x^2 - y^2$$

Square of a binomial :

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$

•Special product

Homework 4: Using the special product

Find each product:

a) $(3p + 11)(3p - 11)$

b) $(5m^3 - 3)\,(5m^3 + 3)$

c) $(9k - 11r^3)(9k + 11r^3)$

d) $(2m + 5)^2$

e) $(3x - 7y^4)^2$

Solution:

a) $(3p + 11)(3p - 11) = (3p)^2 - (11)^2 = 9p^2 - 121$

•Special product

Homework 4: Using the special product

b) $(5m^3 - 3)\left(5m^3 + 3\right) =$

$= \left(5m^3\right)^2 - (3)^2 = 25m^6 - 9$

c) $\left(9k - 11r^3\right)\left(9k + 11r^3\right) = (9k)^2 - \left(11r^3\right)^2$

$= 81k^2 - 121r^6$

$d)\ (2m + 5)^2$

$= (2m)^2 + 2(2m)(5) + (5)^2$

$= 4m^2 + 20m + 25$

$e)\ (3x - 7y^4)^2$

$= (3x)^2 - 2(3x)\left(7y^4\right) + \left(7y^4\right)^2$

$= 9x^2 - 42xy^4 + 49y^8$

•Special product

Example 5: Multiplying more complicated Binomials

Find each product.

a) $[(3p-2)+5q][(3p-2)-5q]$ b) $(x+y)^3$ c) $(2a+b)^4$

Solution :

a) $[(3p-2)+5q][(3p-2)-5q] = (3p-2)^2-(5q)^2 = 9p^2-12p+4-25q^2$

b) $(x+y)^3 = (x+y)^2(x+y) = (x^2+2xy+y^2)(x+y) = x^3+2x^2\text{y}+\text{x}y^2$
$+x^2\text{y}+2\text{x}y^2+y^3 = x^3+3x^2\text{y}+3\text{x}y^2+y^3$

c) $(2a+b)^4 = (2a+b)^2(2a+b)^2 = (4a^2+4ab+b^2)(4a^2+4ab+b^2)$
$= 16a^4+32a^3b+24a^2b^2+8ab^3+b^4$

•Division

Example 6: **Dividing Polynomials with Missing Terms**

Divide $3x^3 - 2x^2 - 150$ *by* $x^2 - 4$.

Solution :-

$$\begin{array}{r} 3x - 2 \\ x^2 + 0x - 4 \,\big|\, 3x^3 - 2x^2 + 0x - 150. \\ 3x^3 + 0x^2 - 12x \leftarrow \textit{change the sign} \\ \hline -2x^2 + 12x - 150 \\ -2x^2 + 0x + 8 \leftarrow \textit{change the sign} \\ \hline 12x - 158 \leftarrow \textit{Remainder} \end{array}$$

$$\frac{3x^3 - 2x^2 - 150}{x^2 - 4} = 3x - 2 + \frac{12x - 158}{x^2 - 4}$$

•Division

Homework 5: **Dividing Polynomials**

Divide $4m^3 - 8m^2 + 5m + 6$ by $2m - 1$.

Solution :-

$$
\begin{array}{r l}
 & 2m^2 - 3m + 1 \\
2m - 1 \,| & 4m^3 - 8m^2 + 5m + 6. \\
 & 4m^3 - 2m^2 \quad \leftarrow \textit{change the sign} \\
\hline
 & -6m^2 + 5m + 6 \\
 & -6m^2 + 3m \quad \leftarrow \textit{change the sign} \\
\hline
 & 2m + 6 \\
 & 2m - 1 \leftarrow \textit{change the sign} \\
\hline
 & 7 \leftarrow \textit{Remainder}
\end{array}
$$

•Division

Homework 5: Dividing Polynomials

Divide $4m^3 - 8m^2 + 5m + 6$ *by* $2m - 1$.

Solution :

$$\frac{4m^3 - 8m^2 + 5m + 6}{2m - 1.} = 2m^2 - 3m + 1 + \frac{7}{2m - 1}$$

•Division

Homework 5: Dividing Polynomials

Divide $4m^3 - 8m^2 + 5m + 6$ *by* $2m - 1$.

Solution :

$$\frac{4m^3 - 8m^2 + 5m + 6}{2m - 1.} = 2m^2 - 3m + 1 + \frac{7}{2m - 1}$$

1.4 Factoring Polynomials

- Factoring Out the Greatest Common Factor
- Factoring by Grouping
- Factoring Trinomials
- Factoring Binomials
- Factoring by Substitution.

Factoring Out the Greatest Common Factor

Example 1 : Factoring Out the Greatest Common Factor

Factor out the greatest common factor from each polynomial GCF.

a) $9y^5 + y^2$ b) $6x^2t + 8xt + 12t$

c) $14(m+1)^3 - 28(m+1)^2 - 7(m+1)$

Solution :

a) $9y^5 + y^2 = y^2(9y^3) + y^2(1) = y^2(9y^3 + 1)$, **GCF=$y^2$**

b) $6x^2t + 8xt + 12t = 2t(x^2 + 4x + 6)$, **GCF=2t**

c) $14(m+1)^3 - 28(m+1)^2 - 7(m+1)$

$= 7(m+1)[2(m+1)^2 - 4(m+1) - 1]$

$= 7(m+1)[2(m^2 + 2m + 1) - 4m - 4 - 1]$

$= 7(m+1)[2m^2 + 4m + 2 - 4m - 4 - 1]$

$= 7(m+1)[2m^2 - 3]$

Factoring by Grouping

$$ax + ay + 6x + 6y = (ax + ay) + (6x + 6y)$$
$$= a(x + y) + 6(x + y) = (x + y)(a + 6)$$

Homework 1 : Factoring by Grouping

Factor each polynomial by grouping.

a) $mp^2 + 7m + 3p^2 + 21$ b) $2y^2 + az - 2z - ay^2$

c) $4x^3 + 2x^2 - 2x - 1$

Solution :

a) $mp^2 + 7m + 3p^2 + 21 = (mp^2 + 7m) + (3p^2 + 21)$
$$= m(p^2 + 7) + 3(p^2 + 7) = (p^2 + 7)(m + 3)$$

b) $2y^2 + az - 2z - ay^2 = 2y^2 - ay^2 + az - 2z = (2y^2 - ay^2) + (az - 2z)$
$$= y^2(2 - a) + z(a - 2) = -y^2(a - 2) + z(a - 2) = (a - 2)(z - y^2)$$

c) $4x^3 + 2x^2 - 2x - 1 = (4x^3 + 2x^2) + (-2x - 1) = 2x^2(2x + 1) - (2x + 1)$
$$= (2x + 1)(2x^2 - 1)$$

Factoring Trinomials

As shown here, factoring is the opposite of multiplication.

$$\xrightarrow[\textbf{Multiplication}]{}$$

$$\mathbf{(2x+1)(3x-4) = 6x^2 - 5x - 4}$$

$$\xleftarrow[]{\textbf{Factoring}}$$

<u>Example 2</u>: Factoring Trinomials

Factor each trinomials.

$\mathbf{a)\,4y^2 - 11y + 6}$ $\quad \mathbf{b)\,6p^2 - 7p - 5}$ $\quad \mathbf{c)\,4x^2 + 13x - 18}$

$\mathbf{d)\,16y^3 + 24y^2 - 16y}$

<u>Solution :</u>

$\mathbf{a)\,4y^2 - 11y + 6 =}$

$\mathbf{(2y-1)(2y-6) = 4y^2 - 14y + 6}$ **<u>Incorrect</u>**

$\mathbf{(2y-2)(2y-3) = 4y^2 - 10y + 6}$ **<u>Incorrect</u>**

$\mathbf{(y-2)(4y-3) = 4y^2 - 11y + 6}$ <u>correct</u>

Factoring Trinomials

Therefore:

$4y^2 - 11y + 6 = (y - 2)(4y - 3)$

Check:

$(y - 2)(4y - 3) = 4y^2 - 3y - 8y + 6 = 4y^2 - 11y + 6 \quad (True)$

b) $6p^2 - 7p - 5 =$

$(2p - 5)(3p + 1) = 6p^2 - 13p - 5$ **Incorrect**

$(3p - 5)(2p + 1) = 6p^2 - 7p - 5$ **Correct**

Check:

$(3p - 5)(2p + 1) = 6p^2 + 3p - 10p - 5 = 6p^2 - 7p - 5 (True)$

Factoring Trinomials

$c)\ 4x^2 + 13x - 18$

$(2x+9)(x-2) = 2x^2 + 5x - 18$ **Incorrect**

$(2x-3)(x+6) = 2x^2 + 9x - 18$ **Incorrect**

$(2x-1)(x+18) = 2x^2 + 35x - 18$ **Incorrect**

Additional trials are also unsuccessful. Thus, this trinomial cannot be factored with integer coefficients and is Prime.

$d)\ 16y^3 + 24y^2 - 16y = 8y(2y^2 + 3y - 2)$ Factor out the GCF, 8y

$= 8y(2y-1)(y+2)$ *Factor the trinomial.*

Factoring Perfect square Trinomials

$$x^2 + 2xy + y^2 = (x + y)^2$$

$$x^2 - 2xy + y^2 = (x - y)^2$$

Homework 2:

a) $16p^2 - 40pq + 25q^2$ b) $36x^2y^2 + 84xy + 49$

Solution:

a) $16p^2 - 40pq + 25q^2 =$

$= (4p)^2 - 2(4p)(5q) + (5q)^2 = (4p - 5q)^2$

$b)\ 36x^2y^2 + 84xy + 49 = (6xy)^2 + 2(6xy)(7) + 7^2 = (6xy + 7)^2$

Factoring Binomials

Difference of Squares $x^2 - y^2 = (x+y)(x-y)$

Difference of Cubes $x^3 - y^3 = (x-y)(x^2+xy+y^2)$

Sum of Cubes $x^3 + y^3 = (x+y)(x^2-xy+y^2)$

Example 3: Factoring Difference of Squares.

a) $4m^2 - 9$ *b)* $256k^4 - 625m^4$ *c)* $(a+2b)^2 - 4c^2$

d) $x^2 - 6x + 9 - y^4$ *e)* $y^2 - x^2 + 6x - 9$

d)

Solution:

a) $16p^2 - 40pq + 25q^2 =$

$= (4p)^2 - 2(4p)(5q) + (5q)^2 = (4p-5q)^2$

Factoring Binomials

Difference of Squares $x^2 - y^2 = (x + y)(x - y)$

Difference of Cubes $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Sum of Cubes $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Example 3: Factoring Difference of Squares.

$a)\ 4m^2 - 9$ $b)\ 256k^4 - 625m^4$ $c)\ (a + 2b)^2 - 4c^2$

$d)\ x^2 - 6x + 9 - y^4$ $e)\ y^2 - x^2 + 6x - 9$

d)

Solution:

c)$(a + 2b)^2 - 4c^2 = (a + 2b - 2c)(a + 2b + 2c)$

$d)\ x^2 - 6x + 9 - y^4 = \left(x^2 - 6x + 9\right) - y^4 = (x - 3)^2 - \left(y^2\right)^2$

$= [x - 3 - y][x - 3 + y].$

Factoring Binomials

Difference of Squares $x^2 - y^2 = (x + y)(x - y)$

Difference of Cubes $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Sum of Cubes $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Homework 3: Factoring Sums or Difference of Cubes

Factor each polynomial.

a) $x^3 + 27$ $\quad$ *b)* $m^3 - 64n^3$ $\quad$ *c)* $8q^6 + 125p^9$

Solution:

a) $x^3 + 27 = x^3 + (3)^3 = (x + 3)(9 - 3x + 9)$

b) $m^3 - 64n^3 = m^3 - (4n)^3 = (m - 4n)(m^2 + 4mn + 16n^2)$

c) $8q^6 + 125p^9 = (2q^2)^3 + (5p^3)^3$

$= (2q^2 + 5p^3)(4q^4 - 10q^2p^3 + 25p^6)$

Factoring by Substitution

EXAMPLE 4: Factoring by Substitution :

Factor each polynomial.

a) $10(2a-1)^2-19(2a-1)-15$ b) $(2a-1)^3+8$

c) $6z^4-13z^2-5$

Solution:

a) $10(2a-1)^2-19(2a-1)-15=10u^2-19u-15$

$=(5u+3)(2u-5)=[5(2a-1)+3][2(2a-1)-5]$

$=(10a-2)(4a-7)=2(5a-1)(4a-7)$.

b) $(2a-1)^3+8=u^3+(2)^3=(u+2)(u^2-2u+4)$

$=(2a-1+2)\left((2a-1)^2-2(2a-1)+4\right)$

$=(2a+1)(4a^2-4a+1-4a+2+4)$

$=(2a+1)(4a^2-8a+7)$

c) $6z^4-13z^2-5=6u^2-13u^2-5=(2u-5)(3u+1)$

1.5 Rational Expressions

- Rational Expression
- Lowest Terms of a Rational Expression
- Multiplication and Division
- Addition and Subtraction
- Complex Fractions.

Rational Expression

The quotient of two polynomials p and Q , with $Q \neq 0$,is a rational expression :

$$\frac{x+6}{x+2}, \quad \frac{(x+6)(x+4)}{(x+2)(x+4)}, \quad \frac{2p^2+7p-4}{5p^2+20p}$$

The domain of a rational expression is the set of real numbers for which the expression is defined.

Example 1 : Finding the domain.

Find the domain of the rational expression

a) $\frac{x+6}{x+2}$, **b)** $\frac{(x+6)(x+4)}{(x+2)(x+4)}$

Rational Expression

Solution :

a) $\frac{x+6}{x+2}$,

The solution of the equation : $x+2=0$ *is excluded from the domain*

$$x=-2$$

$$Domain=\mathbb{R}\backslash\{-2\}=\{x|x\neq-2\}=(-\infty,-2)\cup(-2,\infty)$$

b) $\frac{(x+6)(x+4)}{(x+2)(x+4)}$

The solution of the equation $(x+2)(x+4)=0$ *is excluded from the domain*

$$x=-2,-4$$

$$Domain=\mathbb{R}\backslash\{-2,-4\}=\{x|x\neq-2,-4\}=(-\infty,-4)\cup(-4,-2)\cup(-2,\infty)$$

Rational Expression

Lowest Terms of Rational Expression:

A rational expression $\frac{a}{b}$ is written in lowest terms when the greatest common factor of its numerator a and denominator b is 1.

Examples: $\frac{2}{3}, \frac{3}{5}, \frac{7}{8}, \ldots .$ *are in lowest terms*

$\frac{2}{4}, \frac{5}{10}, \frac{3}{15}, \ldots\ldots$ *are not in lowest terms*

Fundamental Principle of Fractions:

$$\frac{ac}{bc} = \frac{a}{b} \quad (b \neq 0, c \neq 0), \qquad \left(\frac{a+c}{b+c} \neq \frac{a}{b}\right)$$

Examples: $\frac{14}{21} = \frac{2\cdot 7}{3\cdot 7} = \frac{2}{3}, \quad \frac{25}{15} = \frac{5\cdot 5}{3\cdot 5} = \frac{5}{3}, \quad \frac{5}{8} = \frac{2+3}{5+3} \neq \frac{2}{5}$

Rational Expression

Homework 1:

Write each rational expression in lowest terms:

$$a)\ \frac{2x^2+7x-4}{5x^2+20x} \qquad b)\ \frac{6-3x}{x^2-4}$$

Solution:

$$a)\ \frac{2x^2+7x-4}{5x^2+20x}=\frac{(2x-1)(x+4)}{5x(x+4)}=\frac{2x-1}{5x}$$

$$\mathrm{b)}\ \frac{6-3\mathrm{x}}{\mathrm{x}^2-4}=\frac{3(2-x)}{(x+2)(x-2)}=\frac{-3(x-2)}{(x+2)(x-2)}=\frac{-3}{x+2}$$

Multiplication and Division

Multiplication and Division:

For fractions $\frac{a}{b}, \frac{c}{d}$ $(b \neq 0, d \neq 0)$, the following hold.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad \textit{and} \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \quad (c \neq 0)$$

Examples: $\frac{2}{7} \cdot \frac{3}{5} = \frac{2 \cdot 3}{7 \cdot 5} = \frac{6}{35}$, $\frac{4}{3} \div \frac{5}{7} = \frac{4}{3} \cdot \frac{7}{5} = \frac{4 \cdot 7}{3 \cdot 5} = \frac{28}{15}$

Example 2: Multiplying or Dividing Rational Expression s

Multiply or divided , as indicated

a) $\frac{2y^2}{9} \cdot \frac{27}{8y^5}$

b) $\frac{3m^2-2m-8}{3m^2+14m+8} \cdot \frac{3m+2}{3m+4}$

c) $\frac{3p^2+11p-4}{24p^3-8p^2} \div \frac{9p+36}{24p^4-36p^3}$

d) $\frac{x^3-y^3}{x^2-y^2} \cdot \frac{2x+2y+xz+yz}{2x^2+2y^2+zx^2+zy^2}$

Multiplication and Division

Solution:

a) $$\frac{2y^2}{9}\cdot\frac{27}{8y^5} = \frac{2y^2\cdot 27}{9\cdot 8y^5} = \frac{2\cdot 9\cdot 3\cdot y^2}{9\cdot 2\cdot 4\cdot y^2\cdot y^3} = \frac{3}{4y^3}$$

b) $$\frac{3m^2-2m-8}{3m^2+14m+8}\cdot\frac{3m+2}{3m+4} = \frac{(m-2)(3m+4)}{(m+4)(3m+2)}\cdot\frac{3m+2}{3m+4} = \frac{(m-2)(3m+4)(3m+2)}{(m+4)(3m+2)(3m+4)}$$

$$= \frac{m-2}{m+4}$$

c) $$\frac{3p^2+11p-4}{24\text{p}^3-8p^2} \div \frac{9p+36}{24p^4-36p^3} = \frac{(p+4)(3p-1)}{8p^2(3p-1)} \div \frac{9(p+4)}{12p^3(2p-3)}$$

$$= \frac{(p+4)(3p-1)}{8p^2(3p-1)}\cdot\frac{12p^3(2p-3)}{9(p+4)} = \frac{(p+4)(3p-1)12p^3(2p-3)}{8p^2(3p-1)9(p+4)}$$

$$= \frac{12p^3(2p-3)}{8p^2\cdot 9} = \frac{3\cdot 4\cdot p^2\cdot p(2p-3)}{2\cdot 4\cdot 3\cdot 3p^2} = \frac{p(2p-3)}{6}$$

Solution:

$$d)\ \frac{x^3 - y^3}{x^2 - y^2} \cdot \frac{2x + 2y + xz + yz}{2x^2 + 2y^2 + zx^2 + zy^2}$$

$$= \frac{(x-y)(x^2+xy+y^2)}{(x-y)(x+y)} \cdot \frac{2(x+y)+z(x+y)}{2(x^2+y^2)+z(x^2+y^2)}$$

$$= \frac{(x-y)(x^2+xy+y^2)}{(x-y)(x+y)} \cdot \frac{(x+y)(2+z)}{(x^2+y^2)(2+z)}$$

$$= \frac{(x^2+xy+y^2)}{(x^2+y^2)}$$

Addition and Subtraction

For fractions $\frac{a}{b}, \frac{c}{d}$ $(b \neq 0, d \neq 0)$, the following hold.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \textit{and} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

Examples:

$$\frac{2}{7} + \frac{3}{5} = \frac{2 \cdot 5 + 7 \cdot 3}{7 \cdot 5} = \frac{31}{35},$$

$$\frac{4}{3} - \frac{5}{7} = \frac{4 \cdot 7 - 3 \cdot 5}{3 \cdot 7} = \frac{28 - 15}{21} = \frac{13}{21}$$

Homework 2: Addition and Subtraction

Add or subtract, as indicated

$a)\ \frac{5}{9x^2}+\frac{1}{6x}$ $b)\ \frac{y}{y-2}+\frac{8}{2-y}$ $c)\ \frac{3}{(x-1)(x+2)}-\frac{1}{(x+3)(x-4)}$

Solution:

$$a)\ \frac{5}{9x^2}+\frac{1}{6x}=\frac{5\cdot 6x+1\cdot 9x^2}{9x^2\cdot 6x}=\frac{3x(10+3x)}{3x\cdot 18x}$$

$$=\frac{10+3x}{18x}$$

$$b)\ \frac{y}{y-2}+\frac{8}{2-y}=\frac{y}{y-2}-\frac{8}{y-2}=\frac{y-8}{y-2}$$

Homework 2: Addition and Subtraction

Add or subtract, as indicated

a) $\frac{5}{9x^2} + \frac{1}{6x}$ *b)* $\frac{y}{y-2} + \frac{8}{2-y}$ *c)* $\frac{3}{(x-1)(x+2)} - \frac{1}{(x+3)(x-4)}$

Solution:

$$c)\ \frac{3}{(x-1)(x+2)} - \frac{1}{(x+3)(x-4)} =$$

$$\frac{3(x+3)(x-4) - 1(x-1)(x+2)}{(x-1)(x+2)(x+3)(x-4)} = \frac{3x^2 - 3x - 36 - x^2 - x + 2}{(x-1)(x+2)(x+3)(x-4)}$$

$$= \frac{2x^2 - 4x - 34}{(x-1)(x+2)(x+3)(x-4)} = \frac{2(x^2 - 2x - 17)}{(x-1)(x+2)(x+3)(x-4)}$$

Complex Fractions

Example 3: Simplifying Complex Fractions

Simplify each complex fraction.

$$a)\ \frac{6-\frac{5}{k}}{1+\frac{5}{k}} \qquad b)\ \frac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}}$$

Solution:

$$a)\ \frac{6-\frac{5}{k}}{1+\frac{5}{k}} = \frac{k(6-\frac{5}{k})}{k(1+\frac{5}{k})} = \frac{6k-5}{k+5}$$

$$b)\ \frac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}}$$

Complex Fractions

Solution:

$$b)\ \frac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}} = \frac{a(a+1)(\frac{a}{a+1}+\frac{1}{a})}{a(a+1)(\frac{1}{a}+\frac{1}{a+1})}$$

$$= \frac{a(a+1)(\frac{a}{a+1})+a(a+1)\frac{1}{a}}{a(a+1)(\frac{1}{a})+a(a+1)(\frac{1}{a+1})} = \frac{a^2+(a+1)}{(a+1)+a}$$

$$= \frac{a^2+a+1}{2a+1}$$

1.6 Rational Exponents

- Negative Exponents and the Quotient Rule
- Rational Exponents
- Complex Fractions Revisited
- Radical Notation
- Simplified Radicals
- Operations with Radicals
- Rationalizing Denominators

Negative Exponents

Suppose that a is a nonzero real number and n any integer.

$$a^{-n} = \frac{1}{a^n}, \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Example 1 : Using the Definition of a negative exponent.

Evaluate each expression. In part (d) and (e), write the expression without negative exponents. Assume all variables represent nonzero real numbers.

$$\mathbf{a})\ 4^{-2} \qquad b)\ -4^{-2} \qquad c)\ \left(\frac{2}{5}\right)^{-3} \qquad d)\ (xy)^{-3}$$

$$e)\ xy^{-3}$$

Negative Exponents and the Quotient Rule

Solution:

a) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

$b) -4^{-2} = -\frac{1}{4^2} = -\frac{1}{16}$

$c)\ (\frac{2}{5})^{-3} = \frac{1}{(\frac{2}{5})^3} = \frac{1}{\frac{8}{125}} = 1 \div \frac{8}{125} = 1 \cdot \frac{125}{8} = \frac{125}{8}$

$or \left(\frac{2}{5}\right)^{-3} = \left(\frac{5}{2}\right)^{3} = \frac{125}{8}$

Negative Exponents and the Quotient Rule

Solution:

$$d)\ (xy)^{-3} = \frac{1}{(xy)^3} = \frac{1}{x^3y^3}$$

$$e)\ xy^{-3} = x \cdot \frac{1}{y^3} = \frac{x}{y^3}.$$

Negative Exponents and the Quotient Rule

Quotient Rule

Suppose that m and n are integer and a is a nonzero real number.

$$\frac{a^m}{a^n} = a^{m-n}$$

Homework 1 : Using the Quotient Rule.

Simplify each expression. Assume all variables represent nonzero real numbers.

a) $\frac{12^5}{12^3}$ b) $\frac{a^5}{a^{-8}}$ c) $\frac{16m^{-9}}{12m^{11}}$ d) $\frac{25r^7z^5}{10r^9z}$

Negative Exponents and the Quotient Rule

Solution:

a) $$\frac{12^5}{12^3} = 12^{5-3} = 12^2 = 144$$

b) $$\frac{a^5}{a^{-8}} = a^{5-(-8)} = a^{5+8} = a^{13}$$

c) $$\frac{16m^{-9}}{12m^{11}} = \frac{4}{3}m^{-9-11} = \frac{4}{3}m^{-20} = \frac{4}{3m^{20}}$$

d) $$\frac{25r^7z^5}{10r^9z^1} = \frac{25}{10}r^{7-9}z^{5-1} = \frac{5}{2}r^{-2}z^4 = \frac{5z^4}{2r^2}$$

Negative Exponents and the Quotient Rule

Example 2 : Using the Rules for Exponents.

Simplify each expression. Write answers without negative exponents. Assume all variables represent nonzero real numbers.

a) $3x^{-2}(4^{-1}x^{-5})^2$ b) $\dfrac{12p^3q^{-1}}{8p^{-2}q}$ c) $\dfrac{(3x^2)^{-1}(3x^5)^{-2}}{(3^{-1}x^{-2})^2}$

Solution:

a) $3x^{-2}(4^{-1}x^{-5})^2 = 3x^{-2}4^{-2}x^{-10} = 3 \cdot 4^{-2} \cdot x^{-2-10} = \dfrac{3}{4^2 \cdot x^{2+10}}$

$= \dfrac{3}{16x^{12}}$

b) $\dfrac{12p^3q^{-1}}{8p^{-2}q} = \dfrac{12}{8}\dfrac{p^3}{p^{-2}}\dfrac{q^{-1}}{q} = \dfrac{3}{2} \cdot p^{3-(-2)}\, q^{-1-1} = \dfrac{3}{2} \cdot p^5\, q^{-2} = \dfrac{3p^5}{2q^2}$

Negative Exponents and the Quotient Rule

$c)\ \dfrac{(3x^2)^{-1}(3x^5)^{-2}}{(3^{-1}x^{-2})^2}$

$= \dfrac{3^{-1}x^{-2}3^{-2}x^{-10}}{3^{-2}x^{-4}}$

$= \dfrac{3^{-1-2}x^{-2-10}}{3^{-2}x^{-4}}$

$= \dfrac{3^{-3}x^{-12}}{3^{-2}x^{-4}}$

$= 3^{-3-(-2)}x^{-12-(-4)}$

$= 3^{-1}x^{-8}$

$= \dfrac{1}{3x^8}$

Rational Exponents

The expression $a^{\frac{1}{n}}$

$a^{\frac{1}{n}}, n\ Even$

If n is an even positive integer, and if $a > 0$, then $a^{\frac{1}{n}}$ is the positive real number whose nth power is a, That is $(a^{\frac{1}{n}})^n = a$. (In this case $a^{\frac{1}{n}}$ is the principle nth root of a)

$a^{\frac{1}{n}}, n\ Odd$

If n is an odd positive integer, and a is any nonzero real number, then $a^{\frac{1}{n}}$ is the positive or negative real number whose nth power is a, That is $(a^{\frac{1}{n}})^n = a$.

For all positive integers n, $0^{\frac{1}{n}} = 0$

Rational Exponents

Homework 2 : Using the definition of $a^{\frac{1}{n}}$

Evaluate each expression.

a) $36^{\frac{1}{2}}$ b) $-100^{\frac{1}{2}}$ c) $-(225)^{\frac{1}{2}}$ d) $625^{1/4}$

e) $(-1296)^{\frac{1}{4}}$ f) $-1296^{\frac{1}{4}}$ g) $(-27)^{\frac{1}{3}}$ h) $-32^{\frac{1}{5}}$

Solution:

a) $36^{\frac{1}{2}} = 6$ b) $-100^{\frac{1}{2}} = -10$ c) $-(225)^{\frac{1}{2}} = -15$

d) $625^{1/4} = 5$ e) $(-1296)^{\frac{1}{4}} =$ *is not real number*

f) $-1296^{\frac{1}{4}} = -6$ g) $(-27)^{\frac{1}{3}} = -3$ h) $-32^{\frac{1}{5}} = -2$

Rational Exponents

The expression $a^{\frac{m}{n}}$

Let m be any integer, n be any positive integer , and a be any real number for which $a^{\frac{1}{n}}$ is a real number.

$$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$$

Example 3 : Using the definition of $a^{\frac{m}{n}}$

Evaluate each expression.

a) $125^{\frac{2}{3}}$ $b)32^{\frac{7}{5}}$ $c)\ -81^{\frac{3}{2}}$ $d)\ (-27)^{2/3}$

$e)16^{\frac{-3}{4}}$ $f)(-4)^{\frac{5}{2}}$

Rational Exponents

$$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}}$$

Solution

a) $125^{\frac{2}{3}} = (125^{\frac{1}{3}})^2 = 5^2 = 25$

$b) 32^{\frac{7}{5}} = (32^{\frac{1}{5}})^7 = 2^7 = 128$

$c) -81^{\frac{3}{2}} = -(81^{\frac{1}{2}})^3 = -9^3 = -729$

$d) (-27)^{2/3} = [(-27)^{1/3}]^2 = (-3)^2 = 9$

$e) 16^{\frac{-3}{4}} = \frac{1}{16^{\frac{3}{4}}} = \frac{1}{(16^{\frac{1}{4}})^3} = \frac{1}{2^3} = \frac{1}{8}$

$f) (-4)^{\frac{5}{2}} = \textit{is not a real number}.$

$\textit{This is because } (-4)^{\frac{1}{2}} \textit{ is not real number}$

Definitions and Rules for Exponents

Suppose that r and s represent rational numbers. The results here are valid for all positive numbers a and b.

Product rule 1 $a^r \cdot a^s = a^{r+s}$

Product rule 2 $(a^r)^s = a^{rs}$

Quotient rule $\frac{a^r}{a^s} = a^{r-s}$

$(ab)^r = a^r b^r, \qquad (\frac{a}{b})^r = \frac{a^r}{b^r}$

Negative exponent $a^{-r} = \frac{1}{a^r}$

Rational Exponents

Homework 3 : Using the Rules for exponents

Simplify each expression. Assume all variables represent positive real numbers.

a) $\dfrac{27^{\frac{1}{3}} \cdot 27^{\frac{5}{3}}}{27^3}$ $b)81^{\frac{5}{4}} \cdot 4^{-\frac{3}{2}}$ $c)6y^{\frac{2}{3}} \cdot 2y^{\frac{1}{2}}$

$d)\left(\dfrac{3m^{5/6}}{y^{\frac{3}{4}}}\right)^2\left(\dfrac{8y^3}{m^6}\right)^{\frac{2}{3}}$

$e)m^{\frac{2}{3}}\left(m^{\frac{7}{3}} + 2m^{\frac{1}{3}}\right)$

Rational Exponents

Solution:

a) $\dfrac{27^{\frac{1}{3}} \cdot 27^{\frac{5}{3}}}{27^3} = \dfrac{27^{\frac{1}{3}+\frac{5}{3}}}{27^3} = \dfrac{27^{\frac{6}{3}}}{27^3} = \dfrac{27^2}{27^3} = 27^{2-3} = 27^{-1}$

$= \dfrac{1}{27}$

b) $81^{\frac{5}{4}} \cdot 4^{-\frac{3}{2}} = \left(81^{\frac{1}{4}}\right)^5 \cdot \dfrac{1}{4^{\frac{3}{2}}} = (3)^5 \cdot \dfrac{1}{\left(4^{\frac{1}{2}}\right)^3} = 243 \cdot \dfrac{1}{2^3}$

$= \dfrac{243}{8}$

c) $6y^{\frac{2}{3}} \cdot 2y^{\frac{1}{2}} = 12y^{\frac{2}{3}+\frac{1}{2}} = 12y^{\frac{4+3}{6}} = 12y^{\frac{7}{6}}$

Rational Exponents

Solution:

$$d)\left(\frac{3m^{5/6}}{y^{\frac{3}{4}}}\right)^2\left(\frac{8y^3}{m^6}\right)^{\frac{2}{3}}=\frac{3^2m^{5\cdot 2/6}}{y^{\frac{3\cdot 2}{4}}}\frac{8^{\frac{2}{3}}y^{3\cdot\frac{2}{3}}}{m^{6\cdot\frac{2}{3}}}=\frac{9m^{5/3}}{y^{\frac{3}{2}}}\frac{4y^2}{m^4}$$

$$=36m^{\frac{5}{3}-4}y^{2-\frac{3}{2}}=36m^{\frac{5-12}{4}}y^{\frac{4-3}{2}}=36m^{\frac{-7}{4}}y^{\frac{1}{2}}$$

$$=\frac{36y^{\frac{1}{2}}}{m^{\frac{7}{4}}}$$

$$e)m^{\frac{2}{3}}(m^{\frac{7}{3}}+2m^{\frac{1}{3}})=m^{\frac{2}{3}+\frac{7}{3}}+2m^{\frac{2}{3}+\frac{1}{3}}=m^{\frac{9}{3}}+2m^{\frac{3}{3}}=m^3+2m^1$$

$$=m^3+2m$$

Example 4 : Factoring Expressions with Negative or Rational Exponents

Factor out the least power of the variable or variable expression. Assume all variables represent positive real numbers.

a) $12x^{-2} - 8x^{-3}$ b) $4m^{\frac{1}{2}} + 3m^{\frac{3}{2}}$ c) $(y-2)^{\frac{-1}{3}} + (y-2)^{\frac{2}{3}}$

SOLUTION

a) $12x^{-2} - 8x^{-3} = 4x^{-3}\left(3x^{-2-(-3)} - 2x^{-3-(-3)}\right) = 4x^{-3}(3x-2)$

b) $4m^{\frac{1}{2}} + 3m^{\frac{3}{2}} = m^{\frac{1}{2}}(4+3m)$

c) $(y-2)^{\frac{-1}{3}} + (y-2)^{\frac{2}{3}} = (y-2)^{\frac{-1}{3}}[1 + (y-2)]$

$= (y-2)^{\frac{-1}{3}}(y-1)$

Complex Fractions Revisited

Negative exponents are sometimes used to write complex fractions.

Homework 4 : Simplifying a Fraction with Negative Exponents

Simplify $\frac{(x+y)^{-1}}{x^{-1}+y^{-1}}$. Write the results with only positive exponents.

SOLUTION

$$\frac{(x+y)^{-1}}{x^{-1}+y^{-1}} = \frac{1}{(x+y)(x^{-1}+y^{-1})} = \frac{1}{xx^{-1}+xy^{-1}+yx^{-1}+yy^{-1}}$$

$$= \frac{1}{1+\frac{x}{y}+\frac{y}{x}+1} == \frac{1}{\frac{x}{y}+\frac{y}{x}+2}$$

Radical Notation

In this section we used rational exponents to express roots. An alternative notation for roots is radical notation.

Radical Notation for $a^{1/n}$:

Suppose that a is areal number, n is a positive integer, and $a^{1/n}$ is a real number.

$$a^{1/n} = \sqrt[n]{a}$$

Radical Notation for $a^{m/n}$:

Suppose that a is areal number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number.

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

Radical Notation

In the radical $\sqrt[n]{a}$, the symbol $\sqrt[n]{}$ is **a radical symbol**

The number a **is the radicand**, and n **is the index**. We use the familiar notation $\sqrt{a}$ instead of $\sqrt[2]{a}$ for the square root.

For even of n(square roots, fourth roots, and so on), when a is positive, there are two nth roots, one positive and one negative. In such cases, the notation $\sqrt[n]{a}$ represents the positive root, the principal nth root. We write the negative root as $-\sqrt[n]{a}$.

Radical Notation

Example 5 : Evaluating Roots

Write each root using exponents and evaluate.

a) $\sqrt[4]{16}$ b) $-\sqrt[4]{16}$ c) $\sqrt[5]{-32}$ d) $\sqrt[3]{1000}$

e) $\sqrt[6]{\frac{64}{729}}$ f) $\sqrt[4]{-16}$

SOLUTION

a) $\sqrt[4]{16} = 16^{1/4} = 2$

b) $-\sqrt[4]{16} = -16^{1/4} = -2$

c) $\sqrt[5]{-32} = (-32)^{1/5} = -2$

d) $\sqrt[3]{1000} = (1000)^{\frac{1}{3}} = 10$

e) $\sqrt[6]{\frac{64}{729}} = (\frac{64}{729})^{\frac{1}{6}} = \frac{2}{3}$

f) $\sqrt[4]{-16}$ *is not a real number.*

Radical Notation

Homework 5 : Converting from Rational Exponents to Radicals

Write in radical form and simplify. Assume all variable expressions represent positive real numbers.

a) $8^{\frac{2}{3}}$ b) $(-32)^{\frac{4}{5}}$ c) $-16^{\frac{3}{4}}$ d) $x^{\frac{5}{6}}$

e) $3x^{\frac{2}{3}}$ f) $2p^{\frac{1}{2}}$ g) $(3a+b)^{\frac{1}{4}}$

SOLUTION

a) $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ *or* $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$

b) $(-32)^{\frac{4}{5}} = \sqrt[5]{(-32)^4} = \sqrt[5]{1048576} = 16$ *or* $(-32)^{\frac{4}{5}} = (\sqrt[5]{-32})^4$
$= (-2)^4 = 16$

Radical Notation

SOLUTION

$c)\ -16^{\frac{3}{4}} = -\left(\sqrt[4]{16}\right)^3 = -2^3 = -8$

$d)\ x^{\frac{5}{6}} = \sqrt[6]{x^5}$

$e)\ 3x^{\frac{2}{3}} = 3\sqrt[3]{x^2}$

$f)\ 2p^{\frac{1}{2}} = 2\sqrt{p}$

$g)\ (3a+b)^{\frac{1}{4}} = \sqrt[4]{(3a+b)}$

Radical Notation

CAUTION

It is not possible to distribute exponents over a sum, so in Homework 5(g), $(3a+b)^{\frac{1}{4}} \neq (3a)^{\frac{1}{4}} + b^{\frac{1}{4}}$

$$\sqrt[n]{x^n + y^n} \neq x + y$$

Radical Notation

Example 6 : Converting from Radicals to Rational Exponents

Write in exponential form. Assume all variable expressions represent positive real numbers.

a) $\sqrt[4]{x^5}$ b) $\sqrt{3y}$ c) $10(\sqrt[5]{z})^2$ d) $5\sqrt[3]{(2x^4)^7}$

e) $\sqrt{p^2+q}$

SOLUTION

a) $\sqrt[4]{x^5} = x^{\frac{4}{5}}$

b) $\sqrt{3y} = (3y)^{\frac{1}{2}}$

c) $10(\sqrt[5]{z})^2 = 10z^{\frac{2}{5}}$

d) $5\sqrt[3]{(2x^4)^7} = 5(2x^4)^{\frac{7}{3}} = 5 \cdot 2^{\frac{7}{3}} x^{\frac{28}{3}}$

e) $\sqrt{p^2+q} = (p^2+q)^{\frac{1}{2}}$

Evaluating $\sqrt[n]{a^n}$

suppose that a is areal number, If n is an even positive integer, then $\sqrt[n]{a^n} = |a|$

Example: $\sqrt{(-9)^2} = |-9| = 9$, $\sqrt{13^2} = |13| = 13$

suppose that a is areal number, If n is an odd positive integer, then $\sqrt[n]{a^n} = a$

Example: $\sqrt[5]{2^5} = 2$, $\sqrt[3]{(-8)^3} = -8$

Evaluating $\sqrt[n]{a^n}$

Homework 6. Using Absolute Value to simplify Roots

Simplify each expression.

a) $\sqrt{p^4}$ b) $\sqrt[4]{p^4}$ c) $\sqrt{16m^8r^6}$ d) $\sqrt[6]{(-2)^6}$

e) $\sqrt[5]{m^5}$ f) $\sqrt{(2k+3)^2}$ g) $\sqrt{x^2-4x+4}$

SOLUTION

a) $\sqrt{p^4} = \sqrt{(p^2)^2} = |p^2|$

b) $\sqrt[4]{p^4} = |p|$

c) $\sqrt{16m^8r^6} = \sqrt{(4m^4r^3)^2} = |4m^4r^3|$

d) $\sqrt[6]{(-2)^6} = |-2| = 2$

e) $\sqrt[5]{m^5} = m$

f) $\sqrt{(2k+3)^2} = |2k+3|$

g) $\sqrt{x^2-4x+4} = \sqrt{(x-2)^2} = |x-2|$

Rules for Radicals

Suppose that a and b represent real numbers, and m and n represent positive integers for which the indicated roots are real numbers.

Rule ***Description***

Product rule *The product of two roots is the root of the product.*

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

Quotient rule *The root of a quotient is the quotient of the roots*

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, (b \neq 0)$$

Power rule *The index of the root of a root is the product of their indexes*

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Example 7. Simplifying Radical Expressions

Simplify. Assume all variable expression represent positive real numbers.

a) $\sqrt{6}\cdot\sqrt{54}$ b) $\sqrt[3]{m}\cdot\sqrt[3]{m^2}$ c) $\sqrt{\frac{7}{64}}$ d) $\sqrt[4]{\frac{a}{b^4}}$

e) $\sqrt[7]{\sqrt[3]{2}}$ f) $\sqrt[4]{\sqrt{3}}$

SOLUTION

a) $\sqrt{6}\cdot\sqrt{54}=\sqrt{6\cdot 54}=\sqrt{324}=18$

b) $\sqrt[3]{m}\cdot\sqrt[3]{m^2}=\sqrt[3]{m^3}=m$ c) $\sqrt{\frac{7}{64}}=\frac{\sqrt{7}}{\sqrt{64}}=\frac{\sqrt{7}}{8}$

Rules for Radicals

SOLUTION

$$d)\ \sqrt[4]{\frac{a}{b^4}} = \frac{\sqrt[4]{a}}{\sqrt[4]{b^4}} = \frac{\sqrt[4]{a}}{b}$$

$$e)\ \sqrt[7]{\sqrt[3]{2}} = \sqrt[21]{2}$$

$$f)\ \sqrt[4]{\sqrt{3}} = \sqrt[4\cdot 2]{3} = \sqrt[8]{3}$$

Simplified Radicals

Simplified Radicals:

An expression with radicals is simplified when all of the following conditions are saisfied.

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions
3. No denominator contain a radical
4. Exponents in the radicand and the index of the radical have greatest common factor 1
5. All indicated operations have been performed (if possible)

Homework 7. Simplifying Radical

Simplify each radical.

a) $\sqrt{175}$ b) $-3\sqrt[5]{32}$ c) $\sqrt[3]{81x^5y^7z^6}$

SOLUTION

a) $\sqrt{175} = \sqrt{5.5.7} = 5\sqrt{7}$

b) $-3\sqrt[5]{32} = -3 \cdot 2 = -6$

c) $\sqrt[3]{81x^5y^7z^6} = \sqrt[3]{3 \cdot 3^3x^2 \cdot x^3 \cdot y^3 \cdot y^3 \cdot y \cdot z^3 \cdot z^3}$

$= \sqrt[3]{3 \cdot x^2 \cdot y}\sqrt[3]{(3xy^2z^2)^3} = 3xy^2z^2\sqrt[3]{3x^2y}$

Operations on Radicals

Radicals with the same radicand and the same index, such as $3\sqrt[4]{11pq}$ *and* $-7\sqrt[4]{11pq}$, are like radicals, On the other hand, examples of unlike radicals are as follows:

$2\sqrt{5},\ and\ 2\sqrt{3}$ ***radicands are different***

$2\sqrt{3}\ and\ 2\sqrt[3]{3}$ ***indexes are different.***

We add or subtract like radicals by using distributed property. Only like radicals can be combined. Sometimes we need to simplify radicals before adding or subtracting.

Rules for Radicals

Example 8. Adding and Subtracting Radical

Add or subtract, as indicated. Assume all variable expression represent positive real numbers.

a) $3\sqrt[4]{11pq} + \left(-7\sqrt[4]{11pq}\ \right)$

b) $\sqrt{98x^3y} + 3x\sqrt{32xy}$

c) $\sqrt[3]{64m^4n^5} - \sqrt[3]{-27m^{10}n^{14}}$

SOLUTION

a) $3\sqrt[4]{11pq} + \left(-7\sqrt[4]{11pq}\ \right) = -4\sqrt[4]{11pq}$

b) $\sqrt{98x^3y} + 3x\sqrt{32xy} = \sqrt{49 \cdot 2 \cdot x^2 \cdot x \cdot y} + 3x\sqrt{16 \cdot 2 \cdot x \cdot y}$

$= 7x\sqrt{2xy} + 3x(4)\sqrt{2xy} = (7x + 12x)\sqrt{2xy} = 19x\sqrt{2xy}$

Rules for Radicals

$$c)\ \sqrt[3]{64m^4n^5} - \sqrt[3]{-27m^{10}n^{14}}$$

$$= \sqrt[3]{(4mn)^3mn^2} - \sqrt[3]{(-3m^3n^4)^3mn^2}$$

$$= 4mn\sqrt[3]{mn^2} - (-3)m^3n^4\sqrt[3]{mn^2}$$

$$= (4mn + 3m^3n^4)\sqrt[3]{mn^2}$$

Rules for Radicals

Homework 8. *Simplifying Radicals*

Simplify each radicals. Assume all variables represent positive real numbers.

$a)\ \sqrt[6]{3^2}$ $b)\ \sqrt[6]{x^{12}y^3}$ $c)\ \sqrt[9]{\sqrt{6^3}}$

Solution

$a)\ \sqrt[6]{3^2} = 3^{\frac{2}{6}} = 3^{\frac{1}{3}} = \sqrt[3]{3}$

$b)\ \sqrt[6]{x^{12}y^3} = (x^{12}y^3)^{\frac{1}{6}} = x^{\frac{12}{6}}y^{\frac{3}{6}} = x^2y^{\frac{1}{2}} = x^2\sqrt{y}$

$c)\ \sqrt[9]{\sqrt{6^3}} = \sqrt[2\cdot 9]{6^3} = \sqrt[18]{6^3} = 6^{\frac{3}{18}} = 6^{\frac{1}{6}} = \sqrt[6]{6}$

Rules for Radicals

Example 9. ***Multiplying Radical Expressions***

Find each product

$a)\ (\sqrt{7}-\sqrt{10})(\sqrt{7}+\sqrt{10})$ $\quad$ $b)(\sqrt{2}+3)(\sqrt{8}-5)$

Solution

$$a)\ (\sqrt{7}-\sqrt{10})(\sqrt{7}+\sqrt{10}) = (\sqrt{7})^2-(\sqrt{10})^2 = 7-10 = -3$$

$$b)(\sqrt{2}+3)(\sqrt{8}-5) = \sqrt{2}(\sqrt{8})-\sqrt{2}(5)+3(\sqrt{8})-3(5)$$
$$= \sqrt{16}-5\sqrt{2}+3(2\sqrt{2})-15 = 4-5\sqrt{2}+6\sqrt{2}-15$$
$$= -11+\sqrt{2}$$

Rules for Radicals

Example 9. ***Multiplying Radical Expressions***

Find each product

$a)\ (\sqrt{7}-\sqrt{10})(\sqrt{7}+\sqrt{10})$ $\qquad b)(\sqrt{2}+3)(\sqrt{8}-5)$

Solution

$$a)\ (\sqrt{7}-\sqrt{10})(\sqrt{7}+\sqrt{10}) = (\sqrt{7})^2-(\sqrt{10})^2 = 7-10 = -3$$

$$b)(\sqrt{2}+3)(\sqrt{8}-5) = \sqrt{2}(\sqrt{8})-\sqrt{2}(5)+3(\sqrt{8})-3(5)$$
$$= \sqrt{16}-5\sqrt{2}+3(2\sqrt{2})-15 = 4-5\sqrt{2}+6\sqrt{2}-15$$
$$= -11+\sqrt{2}$$

Rationalizing Denominators

The third condition for a simplified radical requires that no denominator contain a radical. We achieve this by rationalizing the denominator- that is, multiplying by a form of 1.

<u>Homework 9</u>: Rationalizing Denominators

Rationalize each denominator

$$a)\ \frac{4}{\sqrt{3}} \qquad b)\ \sqrt[4]{\frac{3}{4}}$$

Solution

Rationalizing Denominators

Homework 9: **Rationalizing Denominators**

Rationalize each denominator

$$a)\ \frac{4}{\sqrt{3}} \qquad b)\ \sqrt[4]{\frac{3}{5}}$$

Solution

$$a)\ \frac{4}{\sqrt{3}} = \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

$$b)\ \sqrt[4]{\frac{3}{5}} = \frac{\sqrt[4]{3}}{\sqrt[4]{5}} = \frac{\sqrt[4]{3}}{\sqrt[4]{5}} \cdot \frac{\sqrt[4]{5}\cdot\sqrt[4]{5}\cdot\sqrt[4]{5}}{\sqrt[4]{5}\cdot\sqrt[4]{5}\cdot\sqrt[4]{5}} = \frac{\sqrt[4]{375}}{5}$$

Rationalizing Denominators

Example 10: Simplifying Radicals Expressions with Fractions

Simplify each Expression. Assume all variables represent positive real numbers

$$a)\ \frac{\sqrt[4]{xy^3}}{\sqrt[4]{x^3y^2}} \qquad b)\ \sqrt[3]{\frac{5}{x^6}} - \sqrt[3]{\frac{4}{x^9}}$$

Solution

$$a)\ \frac{\sqrt[4]{xy^3}}{\sqrt[4]{x^3y^2}} = \sqrt[4]{\frac{\mathbf{xy^3}}{\mathbf{x^3y^2}}} = \sqrt[4]{\frac{\mathbf{y}}{\mathbf{x^2}}} = \frac{\sqrt[4]{\mathbf{y}}}{\sqrt[4]{\mathbf{x^2}}} = \frac{\sqrt[4]{\mathbf{y}}}{\sqrt[4]{\mathbf{x^2}}} \cdot \frac{\sqrt[4]{\mathbf{x^2}}}{\sqrt[4]{\mathbf{x^2}}} = \frac{\sqrt[4]{\mathbf{x^2y}}}{\mathbf{x}}$$

$$b)\ \sqrt[3]{\frac{5}{x^6}} - \sqrt[3]{\frac{4}{x^9}} = \frac{\sqrt[3]{5}}{\sqrt[3]{x^6}} - \frac{\sqrt[3]{\mathbf{4}}}{\sqrt[3]{\boldsymbol{x^9}}} = \frac{\sqrt[3]{5}}{x^2} - \frac{\sqrt[3]{\mathbf{4}}}{\boldsymbol{x^3}} = \frac{x\sqrt[3]{5}}{x^3} - \frac{\sqrt[3]{\mathbf{4}}}{\boldsymbol{x^3}}$$

$$= \frac{x\sqrt[3]{5} - \sqrt[3]{\mathbf{4}}}{x^3}$$

Rationalizing a Binomial Denominators

Homework 10: **Rationalizing a Binomial Denominator**

Rationalize the denominator of

$$\frac{1}{1-\sqrt{2}}$$

Solution

$$\frac{1}{1-\sqrt{2}} = \frac{1}{1-\sqrt{2}} \cdot \frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{\left(1+\sqrt{2}\right)}{\left(1-\sqrt{2}\right)\left(1+\sqrt{2}\right)}$$

$$= \frac{\left(1+\sqrt{2}\right)}{1+\sqrt{2}-\sqrt{2}-\sqrt{2}\sqrt{2}} = \frac{\left(1+\sqrt{2}\right)}{1-2}$$

$$= -1-\sqrt{2}.$$