

MATH203 Calculus

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Power series representations of functions

Consider the power series

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots + x^{n-1} + \cdots$$

It is a geometric series with $a = 1$ and $r = x$, if $|x| < 1$, then the sum of series $s = \frac{1}{1-x}$, i.e. $1 + x + x^2 + \cdots + x^{n-1} + \cdots = \frac{1}{1-x}$ if $-1 < x < 1$

Remarks:

- 1 We can say that the function $f(x) = \frac{1}{1-x}$ is defined by the power series $\sum_{n=1}^{\infty} x^{n-1}$.
- 2 We can say that $\sum_{n=1}^{\infty} x^{n-1}$ is a power series representation for $f(x) = \frac{1}{1-x}$ if $|x| < 1$.
- 3 We can say $f(x) = \frac{1}{1-x}$ is representation of function as power series $\sum_{n=1}^{\infty} x^{n-1}$.

Theorem

Suppose that a power series $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$ then the function defined by

$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \dots$ is differentiable

continuous on $(-R, R)$ and

- 1 $f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \dots$. In other words, the series can be differentiated term by term.
- 2 $\int_0^x f(x) = C + a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \cdots + a_n \frac{x^{n+1}}{n+1} + \dots$. In other words, the series can be integrated term by term.
- 3 Note that whether we differentiate or integrate, the radius of convergence is preserved. However, convergence at the endpoints must be investigated every time.

Examples

Find a function representation f of the following power series:

$$(1): \sum_{n=0}^{\infty} (-1)^n x^n$$

$$(2): \sum_{n=0}^{\infty} x^n$$

$$(3): \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Solution:

Examples

Find a power series representation for the following functions:

$$(1): f(x) = \frac{1}{1-x^2}$$

$$(2): f(x) = \frac{1}{1-4x^2}$$

$$(3): f(x) = \frac{2}{(1+2x)^2}$$

Solution:

Note: All the power series considered in previous lecture are called power series centered at 0.

Examples

Find a power series representation for the following functions:

(1): $f(x) = \ln(1 + x)$

(2): $f(x) = e^x$

(3): $f(x) = \frac{x^3}{4-x^3}$

(4): $f(x) = \frac{5}{3-x}$ centered at (a) $c = 1$, (b) $c = 0$

(5): $f(x) = \frac{(x-3)^2}{4-2x}$ centered at $c = 3$

Solution: