Differentiation (1)

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(Product Rule)

Q#1

(c)

$y = \sin x \cos 2x$

$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos 2x) + \cos 2x \frac{d}{dx}(\sin x)$

$= \sin x\left(-\sin 2x \frac{d}{dx}(2x)\right) + \cos 2x \cos x$

$\Rightarrow \frac{dy}{dx} = -2\sin x \sin 2x + \cos x \cos 2x$

$\Rightarrow \frac{dy}{dx} = \cos x \cos 2x - 2\sin x \sin 2x$

(d) $y = x^3 e^{2x}$

$\frac{dy}{dx} = x^3 \frac{d}{dx}(e^{2x}) + e^{2x}\frac{d}{dx}(x^3)$

$= x^3 \times e^{2x}\frac{d}{dx}(2x) + e^{2x}(3x^{3-1})$

$\frac{dy}{dx} = 2x^3e^{2x} + 3x^2e^{2x}$

$\Rightarrow \frac{dy}{dx} = x^2e^{2x}(2x+3)$

Q#2

(a) $y = (t^2+1)\sin 4t$

$\frac{dy}{dt} = (t^2+1)\frac{d}{dt}(\sin 4t) + \sin 4t \frac{d}{dt}(t^2+1)$

$= (t^2+1)\cos 4t \frac{d}{dt}(4t) + \sin 4t\,(2t)$

$\Rightarrow \frac{dy}{dt} = \dot{y} = 4(t^2+1)(\cos 4t) + 2t\sin 4t$

(c) $y = (e^x + e^{-2x})(3x^2 - 2x)$

$\frac{dy}{dx} = (e^x + e^{-2x})\frac{d}{dx}(3x^2-2x) + (3x^2-2x)\frac{d}{dx}(e^x + e^{-2x})$

$= (e^x + e^{-2x})(6x-2) + (3x^2-2x)\left[e^x + e^{-2x}\frac{d}{dx}(-2x)\right]$

$\Rightarrow y' = (e^x + e^{-2x})(6x-2) + (3x^2-2x)(e^x - 2e^{-2x})$

$= e^x(6x-2) + e^{-2x}(6x-2) + e^x(3x^2-2x) - 2e^{-2x}(3x^2-2x)$ *

~~$= e^x[6x-2+3x^2-2x] + e^{-2x}[6x-2-2(3x^2-2x)]$~~

~~$= e^x[3x^2+4x-2] + e^{-2x}[6x-2-6x^2+4x]$~~

~~$\Rightarrow y' = e^x(3x^2+4x-2) - e^{-2x}(6x^2-10x+2)$~~

* $\Rightarrow y' = e^x(6x-2) + e^{-2x}(6x-2) + e^x(3x^2-2x) - 2e^{-2x}(3x^2-2x)$

$\Rightarrow y' = e^x[6x-2+3x^2-2x] + e^{-2x}[6x-2-6x^2+4x]$

$\Rightarrow y' = e^x(3x^2+4x-2) - e^{-2x}(6x^2-10x+2)$

Q#2 (e) $y = \frac{t^2+1}{e^t}$

$\Rightarrow y = \frac{t^2+1}{e^t}$

$\Rightarrow y = e^{-t}(t^2+1)$

$\Rightarrow \frac{dy}{dt} = e^{-t}\frac{d}{dt}(t^2+1) + (t^2+1)\frac{d}{dt}(e^{-t})$

$\Rightarrow \frac{dy}{dt} = e^{-t}(2t) + (t^2+1)e^{-t}\frac{d}{dt}(-t)$

$\Rightarrow \dot{y} = 2te^{-t} - e^{-t}(t^2+1)$

$\Rightarrow \dot{y} = -e^{-t}(t^2+1-2t)$

$\Rightarrow \dot{y} = -e^{-t}(t^2-2t+1)$

Q#3

(c) $y = \sin x \cos 2x$

$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos 2x) + \cos 2x \frac{d}{dx}(\sin x)$

$= \sin x(-\sin 2x)\frac{d}{dx}(2x) + \cos 2x \cos x$

$\frac{dy}{dx} = -2\sin x \sin 2x + \cos x \cos 2x$

$\frac{d^2y}{dx^2} = -2\left[\sin x \frac{d}{dx}(\sin 2x) + \sin 2x \frac{d}{dx}(\sin x)\right] + \cos x \frac{d}{dx}(\cos 2x) + \cos 2x \frac{d}{dx}(\cos x)$

$= -2\left[\sin x \cos 2x \frac{d}{dx}(2x) + \sin 2x(\cos x)\right] + \cos x(-\sin 2x)\frac{d}{dx}(2x) + \cos 2x(-\sin x)$

$\Rightarrow \frac{d^2y}{dx^2} = -2[\sin x \cos 2x(2) + \cos x \sin 2x] - \cos x \sin 2x(2) - \sin x \cos 2x$

$\Rightarrow \frac{d^2y}{dx^2} = -4\sin x \cos 2x - 2\cos x \sin 2x - 2\cos x \sin 2x - \sin x \cos 2x$

$\Rightarrow \frac{d^2y}{dx^2} = -5\sin x \cos 2x - 4\cos x \sin 2x$

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Q#3

(d) $y = x^3 e^{2x}$

$\frac{dy}{dx} = x^3 \frac{d}{dx}(e^{2x}) + e^{2x}\frac{d}{dx}(x^3)$

$= x^3 e^{2x}\frac{d}{dx}(2x) + e^{2x}(3x^2)$

$\frac{dy}{dx} = 2x^3e^{2x} + 3x^2e^{2x}$

$\frac{d^2y}{dx^2} = 2\left[\frac{d}{dx}(x^3e^{2x})\right] + 3\left[x^2\frac{d}{dx}(e^{2x}) + e^{2x}\frac{d}{dx}(x^2)\right]$

$\frac{d^2y}{dx^2} = 2\left[2x^3e^{2x} + 3x^2e^{2x}\right] + 3\left[x^2e^{2x}\frac{d}{dx}(2x) + e^{2x}(2x)\right]$

$\Rightarrow \frac{d^2y}{dx^2} = 4x^3e^{2x} + 6x^2e^{2x} + 3\left[2x^2e^{2x} + 2xe^{2x}\right]$

$\Rightarrow \frac{d^2y}{dx^2} = 4x^3e^{2x} + 6x^2e^{2x} + 6x^2e^{2x} + 6xe^{2x}$

$= 4x^3e^{2x} + 12x^2e^{2x} + 6xe^{2x}$

$\boxed{\frac{d^2y}{dx^2} = 2xe^{2x}[2x^2 + 6x + 3]}$

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Q# 1 (b)

$y = \frac{x}{e^x+1}$

$u = x \quad , v = e^x + 1$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = e^x$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\Rightarrow \frac{dy}{dx} = \frac{(e^x+1)(1) - x(e^x)}{(e^x+1)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{e^x + 1 - xe^x}{(e^x+1)^2}$

$\Rightarrow \boxed{\frac{dy}{dx} = y' = \frac{e^x + 1 - xe^x}{(e^x+1)^2}}$

(e) $y = \frac{\ln x}{x^2}$

$u = \ln x \quad , v = x^2$

$\frac{du}{dx} = \frac{1}{x} \quad \frac{dv}{dx} = 2x$

(2) $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2(\frac{1}{x}) - \ln x\,(2x)}{(x^2)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{x - 2x\ln x}{x^4}$

$\Rightarrow y' = \frac{x(1 - 2\ln x)}{x^{4}}$

$\Rightarrow \boxed{y' = \frac{1 - 2\ln x}{x^3}}$

Q# 2 (a)

$y = \frac{t^2 - 1}{t^2 + 1}$

$u = t^2 - 1 \quad , v = t^2 + 1$

$\frac{du}{dt} = 2t \quad , \frac{dv}{dt} = 2t$

$\frac{dy}{dt} = \frac{v\frac{du}{dt} - u\,dv/dt}{v^2}$

$\Rightarrow y' = \frac{(t^2+1)(2t) - (t^2-1)(2t)}{(t^2+1)^2}$

$\Rightarrow y' = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}$

$\Rightarrow \boxed{y' = \frac{4t}{(t^2+1)^2}}$

(d) $y = \frac{z + \sin z}{z + \cos z}$

$u = z + \sin z \quad ; \quad v = z + \cos z$

$\frac{du}{dz} = 1 + \cos z \quad ; \quad \frac{dv}{dz} = 1 - \sin z$

$\frac{dy}{dz} = \frac{v\,du/dz - u\,dv/dz}{v^2}$

$\Rightarrow \frac{dy}{dz} = \frac{(z + \cos z)(1 + \cos z) - (z + \sin z)(1 - \sin z)}{(z + \cos z)^2}$

$\Rightarrow \frac{dy}{dz} = \frac{(z + z\cos z + \cos z + \cos^2 z) - (z - z\sin z + \sin z - \sin^2 z)}{(z + \cos z)^2}$

$\Rightarrow \frac{dy}{dz} = \frac{z + z\cos z + \cos z + \cos^2 z - z + z\sin z - \sin z + \sin^2 z}{(z + \cos z)^2}$

$= \frac{z\cos z + z\sin z + \cos z - \sin z + \sin^2 z + \cos^2 z}{(z + \cos z)^2}$

$= \frac{z\cos z + \cos z + z\sin z - \sin z + 1}{(z + \cos z)^2}$

$\Rightarrow \boxed{\frac{dy}{dz} = \frac{1 + (z+1)\cos z + (z-1)\sin z}{(z + \cos z)^2}}$

Q# 2 (e)

$y = \frac{1 + x + x^2}{1 + x^3}$

$u = 1 + x + x^2$

$\frac{du}{dx} = 1 + 2x$

& $v = 1 + x^3$

$\frac{dv}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\Rightarrow \frac{dy}{dx} = \frac{(1+x^3)(1+2x) - (1+x+x^2)(3x^2)}{(1+x^3)^2}$

$= \frac{(1 + 2x + x^3 + 2x^4) - (3x^2 + 3x^3 + 3x^4)}{(1+x^3)^2}$

$= \frac{2x^4 + x^3 + 2x + 1 - 3x^4 - 3x^3 - 3x^2}{(1+x^3)^2}$

$= \frac{-x^4 - 2x^3 - 3x^2 + 2x + 1}{(1+x^3)^2}$

$= -\frac{(x^4 + 2x^3 + 3x^2 - 2x - 1)}{(1+x^3)^2}$

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Implicit Differentiation

$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$

Q#1 (d) Find $\frac{dy}{dx}$, given

$x^3 - y^3 = x + y$

Differentiate w.r.to x

$\frac{d}{dx}(x^3 - y^3) = \frac{d}{dx}(x+y)$

$\Rightarrow 3x^2 - 3y^2\frac{dy}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow 3x^2 - 1 = 3y^2\frac{dy}{dx} + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(3y^2+1) = 3x^2 - 1$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{3x^2-1}{3y^2+1}}$

Q#2 (b) Find $\frac{dx}{dt}$

$3\cos 2x - t^2 = 20$

$\frac{d}{dt}(3\cos 2x - t^2) = \frac{d}{dt}(20)$

$\Rightarrow -3\sin 2x \frac{d}{dt}(2x) - 2t = 0$

$\Rightarrow -6\sin 2x \frac{dx}{dt} = 2t$

$\Rightarrow \frac{dx}{dt} = \frac{-2t}{6\sin 2x}$

$\Rightarrow \boxed{\frac{dx}{dt} = \frac{-t}{3\sin 2x}}$

Q#3 (e) Find dy/dx

$2e^{y-x} = 3e^x + y^2$

$\frac{d}{dx}(2e^{y-x}) = \frac{d}{dx}(3e^x + y^2)$

$\Rightarrow 2e^{y-x}\frac{d}{dx}(y-x) = 3e^x + 2y\frac{dy}{dx}$

$\Rightarrow 2e^{y-x}\left(\frac{dy}{dx} - 1\right) = 3e^x + 2y\frac{dy}{dx}$

$\Rightarrow 2e^{y-x}\frac{dy}{dx} - 2e^{y-x} = 3e^x + 2y\frac{dy}{dx}$

$\Rightarrow 2e^{y-x}\frac{dy}{dx} - 2y\frac{dy}{dx} = 3e^x + 2e^{y-x}$

$\Rightarrow \frac{dy}{dx}(2e^{y-x} - 2y) = 3e^x + 2e^{y-x}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{3e^x + 2e^{y-x}}{2e^{y-x} - 2y}}$

Q#5 Find y'' given

$\sin x + \cos y = 1$

$\frac{d}{dx}(\sin x + \cos y) = \frac{d}{dx}(1)$

$\Rightarrow \cos x - \sin y\frac{dy}{dx} = 0 \Rightarrow \cos x = \sin y\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{\sin y}$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\cos x}{\sin y}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sin y \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sin y)}{(\sin y)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\sin y(-\sin x) - \cos x \cos y \frac{dy}{dx}}{\sin^2 y}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\sin x \sin y - \cos x \cos y\left(\frac{\cos x}{\sin y}\right)}{\sin^2 y}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{-\sin x \sin^2 y - \cos^2 x \cos y}{\sin y}}{\sin^2 y}$

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{-\sin x \sin^2 y - \cos^2 x \cos y}{\sin^3 y}}$

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12 Techniques and applications of differentiation

This chapter continues the study of differentiation started in Chapter 11. The table of derivatives in Chapter 11, although useful, is limited. In this chapter several techniques for expanding the range of functions we can differentiate are introduced.

Methods for differentiating products and quotients of functions are explained in Block 1. Some functions can be differentiated when the function is written in terms of a new variable. This gives rise to the chain rule in Block 2.

Although we have focused on differentiating a function of x, say $y(x)$, it is not always possible to express y explicitly in terms of x. For example, it is impossible to rearrange the equation $e^x + e^y = x^2 + y^3$ to obtain y by itself on the left-hand side. However, by using implicit differentiation as explained in Block 3 it is still possible to find an expression for the derivative, $\frac{dy}{dx}$.

If a function is expressed parametrically the derivative is found by differentiating parametrically. This is covered in Block 4. The final technique, logarithmic differentiation, is useful for finding the derivative of products of functions involving many factors.

Blocks 6 and 7 turn to some applications of differentiation. The calculation of equations of tangents and normals is treated in Block 6. Finally, the important topic of maximum and minimum values of a function closes the chapter.

BLOCK

1 The product rule and the quotient rule

1.1 Introduction

Chapter 11 introduced the concept of differentiation and the use of a table of derivatives. Clearly every possible function cannot be listed in a table. We need a set of rules, used in conjunction with the table of derivatives, to extend the range of functions which we can differentiate. The product rule and the quotient rule are two such rules.

1.2 The product rule

As its name tells us, the product rule helps us to differentiate a product of functions. Consider the function $y(x)$, where $y(x)$ is the product of two functions, $u(x)$ and $v(x)$, that is

$$y(x) = u(x)v(x)$$

For example, if $y(x) = x^2 \sin x$ then $u(x) = x^2$ and $v(x) = \sin x$. The product rule states:

KEY POINT

If

$$y(x) = u(x)v(x)$$

then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = u'v + uv'$$

Example 1.1

Find $\frac{dy}{dx}$ where $y = x^2 \sin x$.

Solution

We have

$$y = x^2 \sin x = uv$$

and so $u = x^2$ and $v = \sin x$. Hence

$$\frac{du}{dx} = 2x, \quad \frac{dv}{dx} = \cos x$$

Applying the product rule we have

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$
$$= 2x(\sin x) + x^2(\cos x)$$
$$= x(2\sin x + x\cos x)$$

Example 1.2

Find y' where $y = e^x \cos x$.

Solution

We have

$$y = e^x \cos x = uv$$

So

$$u = \boxed{}, \quad v = \boxed{}$$

and hence $e^x, \cos x$

$$\frac{du}{dx} = \boxed{}, \quad \frac{dv}{dx} = \boxed{}$$

Applying the product rule yields: $e^x, -\sin x$

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$
$$= e^x \cos x + \boxed{}$$
$$= e^x(\cos x - \sin x)$$

$e^x(-\sin x)$

Example 1.3

Find $\frac{d^2y}{dx^2}$ where $y = x^2 \ln x$.

Solution

We have

$$y = x^2 \ln x = uv$$

so

$$u = x^2, \quad v = \ln x$$

Then

$$\frac{du}{dx} = 2x, \quad \frac{dv}{dx} = \boxed{}$$

$\frac{1}{x}$

Applying the product rule, we have

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$= \boxed{}$$

$2x \ln x + x$

To obtain $\frac{d^2y}{dx^2}$ we differentiate $\frac{dy}{dx}$. Hence we need to find $\frac{d}{dx}(2x \ln x + x)$. The derivative of x is simply 1 so let us examine $\frac{d}{dx}(2x \ln x)$.

To find $\frac{d}{dx}(2x \ln x)$ we use the product rule, with $u = 2x$, $v = \ln x$. Then

$$\frac{du}{dx} = 2, \quad \frac{dv}{dx} = \frac{1}{x}$$

and so

$$\frac{d}{dx}(2x \ln x) = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$= 2 \ln x + 2x\left(\frac{1}{x}\right)$$

$$= 2 \ln x + 2$$

Finally

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x \ln x + x)$$

$$= \boxed{}$$

$2 \ln x + 3$

Exercises

1 Find $\frac{dy}{dx}$ where y is given by
(a) $x \cos x$ (b) xe^x (c) $\sin x \cos 2x$ (d) x^3e^{2x}
(e) $x^4 \sin 2x$

2 Calculate y' where y is given by
(a) $(t^2 + 1) \sin 4t$ (b) $(3t + 7)e^{-2t}$
(c) $(e^x + e^{-2x})(3x^2 - 2x)$ (d) $\sqrt{x}e^x$
(e) $\frac{t^2 + 1}{e^t}$

3 Find the second derivative of the functions in question 1.

Solutions to exercises

1 (a) $\cos x - x \sin x$ (b) $e^x(1 + x)$
(c) $\cos x \cos 2x - 2 \sin x \sin 2x$
(d) $x^2e^{2x}(3 + 2x)$
(e) $4x^3 \sin 2x + 2x^4 \cos 2x$

2 (a) $2t \sin 4t + 4(t^2 + 1)(\cos 4t)$
(b) $-e^{-2t}(6t + 11)$
(c) $(e^x - 2e^{-2x})(3x^2 - 2x)$ $+(e^x + e^{-2x})(6x - 2)$, which may be written as $e^x(3x^2 + 4x - 2)$ $-e^{-2x}(6x^2 - 10x + 2)$
(d) $e^x\left(\frac{x^{-1/2}}{2} + x^{1/2}\right)$
(e) $-e^{-t}(t^2 - 2t + 1)$

3 (a) $-x \cos x - 2 \sin x$
(b) $e^x(x + 2)$
(c) $-5 \sin x \cos 2x - 4 \cos x \sin 2x$
(d) $2xe^{2x}(2x^2 + 6x + 3)$
(e) $16x^3 \cos 2x - 4x^2(x^2 - 3) \sin 2x$

1.3 The quotient rule

The quotient rule shows us how to differentiate a quotient of functions, for example

$$\frac{\sin x}{x}, \quad \frac{t^2-1}{t^2+1}, \quad \frac{e^z+z}{\cos z}$$

The quotient rule may be stated thus:

KEY POINT

If

$$y(x) = \frac{u(x)}{v(x)}$$

then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{vu' - uv'}{v^2}$$

Example 1.4

Find y' given $y = \frac{\sin x}{x}$.

Solution

We have

$$y = \frac{\sin x}{x} = \frac{u}{v}$$

so

$$u = \sin x, \quad v = x$$

and so

$$\frac{du}{dx} = \cos x, \quad \frac{dv}{dx} = 1$$

Applying the quotient rule gives

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{x\cos x - \sin x(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

Example 1.5

Find y' given $y = \dfrac{t^3}{t+1}$.

Solution

We have

$$y = \frac{t^3}{t+1} = \frac{u}{v}$$

and so

$$u = t^3, \quad v = t+1, \quad u' = 3t^2, \quad v' = 1$$

Applying the quotient rule gives

$$\frac{dy}{dt} = \frac{vu' - uv'}{v^2}$$

$$= \boxed{}$$

$$\frac{(t+1)3t^2 - t^3(1)}{(t+1)^2}$$

which can be simplified to

$$\frac{dy}{dt} = \boxed{}$$

$$\frac{t^2(2t+3)}{(t+1)^2}$$

Exercises

1 Find $\dfrac{dy}{dx}$ where y is given by

(a) $\dfrac{e^x}{x}$ (b) $\dfrac{x}{e^x+1}$ (c) $\dfrac{\cos x}{\sin x}$ (d) $\dfrac{1-x}{1+x}$

(e) $\dfrac{\ln x}{x^2}$

2 Find y' when y is given by

(a) $\dfrac{t^2-1}{t^2+1}$ (b) $\dfrac{e^{2t}+t}{e^t-1}$ (c) $\dfrac{\sin 3t}{\cos t + t}$

(d) $\dfrac{z+\sin z}{z+\cos z}$ (e) $\dfrac{1+x+x^2}{1+x^3}$

Solutions to exercises

1 (a) $\dfrac{e^x(x-1)}{x^2}$ (b) $\dfrac{e^x+1-xe^x}{(e^x+1)^2}$

(c) $-\operatorname{cosec}^2 x$ (d) $\dfrac{-2}{(1+x)^2}$ (e) $\dfrac{1-2\ln x}{x^3}$

2 (a) $\dfrac{4t}{(t^2+1)^2}$

(b) $\dfrac{(e^t-1)(2e^{2t}+1)-(e^{2t}+t)e^t}{(e^t-1)^2}$

(c) $\dfrac{3(\cos t + t)\cos 3t - \sin 3t(-\sin t + 1)}{(\cos t + t)^2}$

(d) $\dfrac{(z+1)\cos z + (z-1)\sin z + 1}{(\cos z + z)^2}$

(e) $-\left(\dfrac{x^4+2x^3+3x^2-2x-1}{(x^3+1)^2}\right)$

End of block exercises

1 Find the derivative of each of the following:

(a) $(x-1)\sin 2x$ (b) $\dfrac{\sin 2x}{x-1}$ (c) $\dfrac{x-1}{\sin 2x}$

(d) $e^{2x}\sin 3x$ (e) $e^{-2x}\sin 3x$

2 Differentiate the following:

(a) $\dfrac{t^3-t^2}{t^2+1}$ (b) $3\sin 2x\cos x$ (c) $\dfrac{3\cos x}{\sin 2x}$

(d) $\dfrac{e^{3r}}{e^{2r}}$ (e) $(r+1)(r+\sin r)$

3 Find $\dfrac{dH}{dt}$ given

$$H = e^{2t}t^2\sin t$$

4 Find $\dfrac{dR}{dt}$ given

$$R = \frac{e^{2t}\sin t}{t^2}$$

Solutions to exercises

1 (a) $2(x-1)\cos 2x + \sin 2x$

(b) $\dfrac{2(x-1)\cos 2x - \sin 2x}{(x-1)^2}$

(c) $\dfrac{\sin 2x - 2(x-1)\cos 2x}{\sin^2 2x}$

(d) $e^{2x}(3\cos 3x + 2\sin 3x)$

(e) $e^{-2x}(3\cos 3x - 2\sin 3x)$

2 (a) $\dfrac{t(t^3+3t-2)}{(t^2+1)^2}$

(b) $6\cos x\cos 2x - 3\sin x\sin 2x$

(c) $-3\left(\dfrac{\sin 2x\sin x + 2\cos x\cos 2x}{\sin^2 2x}\right)$

(d) e^r (e) $(r+1)\cos r + \sin r + 2r + 1$

3 $e^{2t}[t^2\cos t + 2t(t+1)\sin t]$

4 $\dfrac{e^{2t}[2t\sin t + t\cos t - 2\sin t]}{t^3}$

BLOCK

2 The chain rule

2.1 Introduction

In Block 1 we saw how to differentiate products and quotients of functions. This block introduces the chain rule which allows us to differentiate an additional class of functions.

2.2 The chain rule

Suppose y is a function of z, that is $y = y(z)$, and that z is a function of x, that is $z = z(x)$. So

$$y = y(z) = y(z(x))$$

Hence y may be considered to be a function of x. For example, if $y(z) = 2z^2 + 3z$ and $z = \cos 2x$ then

$$y = 2(\cos 2x)^2 + 3(\cos 2x)$$

Since y can be considered as a function of x, then $\frac{dy}{dx}$ may be found. The chain rule helps us to find $\frac{dy}{dx}$. The chain rule states

KEY POINT

If $y = y(z)$ and $z = z(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

Example 2.1

Given $y = z^4$ and $z = 3x + 6$, find $\frac{dy}{dx}$.

Solution

We have

$$y = z^4 = (3x + 6)^4$$

and we seek $\frac{dy}{dx}$. Now

$$y = z^4, \quad z = 3x + 6$$

and so

$$\frac{dy}{dz} = 4z^3, \quad \frac{dz}{dx} = 3$$

Using the chain rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz}\frac{dz}{dx}\\ &= 4z^3(3)\\ &= 12z^3\\ &= 12(3x+6)^3\end{aligned}$$

Sometimes care must be taken to recognise the independent and dependent variables. Example 2.2 uses the chain rule but it may look unfamiliar.

Example 2.2

Given $z(y) = y^3$ and $y(x) = 2x^2 - x$ find $\frac{dz}{dx}$.

Solution

We have

$$z(y) = y^3 \quad \text{so} \quad \frac{dz}{dy} = 3y^2$$

and

$$y(x) = 2x^2 - x \quad \text{so} \quad \frac{dy}{dx} = 4x - 1$$

The chain rule has the form

$$\begin{aligned}\frac{dz}{dx} &= \frac{dz}{dy}\frac{dy}{dx}\\ &= 3y^2(4x-1)\\ &= 3(2x^2-x)^2(4x-1)\end{aligned}$$

Example 2.3

Given $y = (x^3 + x)^7$, find $\frac{dy}{dx}$.

Solution

We let $z = x^3 + x$ and so

$$y = (x^3 + x)^7 = z^7$$

Then $\frac{dy}{dz} = 7z^6$ and $\frac{dz}{dx} = 3x^2 + 1$. Applying the chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz}\frac{dz}{dx}\\ &= (7z^6)(3x^2+1)\\ &= 7(3x^2+1)(x^3+x)^6\end{aligned}$$

Example 2.4

Given $y = \sqrt{x^2+1}$ find $\frac{dy}{dx}$.

Solution

Let $z = x^2 + 1$ and then

$y = \sqrt{x^2+1} = \square$ $\sqrt{z}$

Then

$\frac{dy}{dz} = \square$, $\frac{dz}{dx} = \square$ $\frac{1}{2}z^{-1/2}$, $2x$

and applying the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

$= \square$ $\frac{1}{2}z^{-1/2}(2x)$

$$= \frac{x}{\sqrt{z}}$$

$= \square$ $\frac{x}{\sqrt{x^2+1}}$

Example 2.5

Find $\frac{dy}{dx}$ given $y = \ln(x^2 + x + 1)$.

Solution

Let $z = x^2 + x + 1$ so that

$y = \ln(x^2 + x + 1) = \square$ $\ln z$

Then

$$\frac{dy}{dz} = \boxed{}, \quad \frac{dz}{dx} = \boxed{}$$

$\frac{1}{z}$, $2x + 1$

So

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

$$= \boxed{}$$

$\frac{2x+1}{z}$

$$= \frac{2x+1}{x^2 + x + 1}$$

We note that in the final solution, the numerator is the derivative of the denominator. In general:

KEY POINT

If $y = \ln f(x)$ then

$$\frac{dy}{dx} = \frac{f'}{f}$$

Example 2.6

Given $y = \ln(1 - x)$ find $\frac{dy}{dx}$.

Solution

Here $f(x) = 1 - x$ and $f'(x) = -1$. So

$$\frac{dy}{dx} = \frac{-1}{1 - x}$$

which may be simplified to

$$\frac{dy}{dx} = \frac{1}{x - 1}$$

Example 2.7

Given $y = 5\ln(2t - 1)$ find $\frac{dy}{dt}$.

Solution

Here $f(t) = 2t - 1$ and so $f'(t) = 2$. Hence

$$\frac{dy}{dt} = 5\left(\frac{2}{2t - 1}\right)$$

$$= \frac{10}{2t - 1}$$

Example # 2.8 Given $y = \ln(e^x + \sin x)$

mple 2.8

en $y = \ln(e^x + \sin x)$ find $\frac{dy}{dx}$.

ution Solution

have we have

$$y = \ln(e^x + \sin x)$$
$$= \ln f(x) \quad \text{where } f(x) = e^x + \sin x$$

$$\frac{f'}{f} = \frac{e^x + \cos x}{e^x + \sin x}$$

$$\frac{dy}{dx} = \boxed{}$$

xercises Exercises

1 Differentiate each of the following functions:
(a) $(x^3+2)^6$ (b) $\sqrt{\sin x}$ (c) $(e^x+1)^7$
(d) $(\cos 2x)^5$ (e) $\ln(x+1)$

2 Find $\frac{dy}{dt}$ where y is given by
(a) $e^{(3t^2)}$ (b) $3e^{t^2}$ (c) $e^{\sin 2t}$ (d) $e^{2\sin t}$ (e) $2e^{\sin t}$

3 Find the rate of change of y at the specified point:
(a) $y = \ln(3t^2+5)$, $t = 1$
(b) $y = \sin(t^2)$, $t = 2$
(c) $y = \cos(t^3+1)$, $t = 1$
(d) $y = (t^3-1)^{2/3}$, $t = 2$
(e) $y = 4e^{\cos t}$, $t = \frac{\pi}{2}$

Solutions to exercises

1 (a) $18x^2(x^3+2)^5$ (b) $\frac{\cos x}{2\sqrt{\sin x}}$
(c) $7e^x(e^x+1)^6$
(d) $-10\sin 2x\cos^4 2x$ (e) $\frac{1}{x+1}$

2 (a) $6te^{3t^2}$ (b) $6te^{t^2}$ (c) $2\cos 2te^{\sin 2t}$
(d) $2\cos te^{2\sin t}$ (e) $2\cos te^{\sin t}$

3 (a) 0.75 (b) −2.6146 (c) −2.7279
(d) 4.1821 (e) −4

End of block exercises

1 Use the chain rule to differentiate each of the following functions:
(a) $y = (6x^3 - x)^4$ (b) $h = (t^4-1)^{1.3}$
(c) $v = \sqrt{9-2t}$ (d) $i = \sin(y^3)$
(e) $R = \cos(\sqrt{r})$

2 Find the derivative of each function
(a) $Y(t) = 5e^{\sin 2t}$ (b) $m(p) = 3\ln(p^4+2)$
(c) $H(r) = 5\sin(\pi r^2 + 1)$
(d) $x(t) = -3\cos\left(\frac{1}{t}\right)$ (e) $Q(s) = \frac{1}{\ln s}$

3 Evaluate $\frac{dy}{dx}$ at the specified value of x.
(a) $y = \sqrt{x + \sin x}$, $x = 1$
(b) $y = \sin(x^2+1)$, $x = 0.5$
(c) $y = e^{\sqrt{x}}$, $x = 1$
(d) $y = 2\cos\left(\frac{1}{x}\right)$, $x = 1$
(e) $y = \frac{1}{(3x^2+1)^4}$, $x = 0.5$

4 Differentiate the following functions where a, b and n are constants.
(a) $y = (at+b)^n$ (b) $y = e^{at+b}$
(c) $y = \sin(at+b)$
(d) $y = \cos(at+b)$ (e) $y = \ln(at+b)$

Solutions to exercises

1 (a) $4(18x^2 - 1)(6x^3 - x)^3$

(b) $\dfrac{4t^3(t^4 - 1)^{-2/3}}{3}$

(c) $-(9 - 2t)^{-1/2}$ (d) $3y^2 \cos(y^3)$

(e) $-\dfrac{1}{2}r^{-1/2} \sin \sqrt{r}$

2 (a) $10 \cos 2t\, e^{\sin 2t}$ (b) $\dfrac{12p^3}{p^4 + 2}$

(c) $10\pi r \cos(\pi r^2 + 1)$

(d) $-\dfrac{3}{t^2} \sin\left(\dfrac{1}{t}\right)$ (e) $\dfrac{-1}{s(\ln s)^2}$

3 (a) 0.5675 (b) 0.3153
(c) 1.3591 (d) 1.6829
(e) −0.7311

4 (a) $an(at + b)^{n-1}$ (b) ae^{at+b}
(c) $a \cos(at + b)$ (d) $-a \sin(at + b)$
(e) $\dfrac{a}{at + b}$

BLOCK 3 Implicit differentiation

3.1 Introduction

So far we have met many functions of the form $y = f(x)$, for example $y = x^2 + 3$, $y = \sin 2x$ and $y = e^{3x} - 2x$. Whenever y is equated to an expression involving only x terms we say that y is expressed **explicitly** in terms of x.

Sometimes we have an equation connecting x and y but it is impossible to write it in the form $y = f(x)$. Examples of this include $x^2 - y^3 + \sin x - \cos y = 1$, $\sin(x + y) + e^x + e^{-y} = x^3 + y^3$. In these cases we say that y is expressed **implicitly** in terms of x.

Whether y is expressed explicitly or implicitly in terms of x we can still differentiate to find the derivative $\frac{dy}{dx}$. If y is expressed explicitly in terms of x then $\frac{dy}{dx}$ will also be expressed explicitly in terms of x. If y is expressed implicitly in terms of x then $\frac{dy}{dx}$ will be expressed in terms of x and y.

3.2 Differentiating $f(y)$ with respect to x

If y is expressed implicitly in terms of x and we wish to find $\frac{dy}{dx}$, then we frequently need to differentiate a function of y with respect to x, that is, find $\frac{d}{dx}(f(y))$. To do this we use the chain rule. The examples illustrate the technique.

Example 3.1

Find $\frac{d}{dx}(y^3)$.

Solution

We use the chain rule. Let $z = y^3$ so that we wish to find $\frac{dz}{dx}$. Using the chain rule (see Example 2.2 in Block 2)

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

Now

$$z = y^3 \quad \text{and so} \quad \frac{dz}{dy} = 3y^2$$

Hence

$$\frac{dz}{dx} = 3y^2\frac{dy}{dx}$$

Hence

$$\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

The result of Example 3.1 can easily be extended to

KEY POINT

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$

Example 3.2

Find $\frac{d}{dx}(\sin y)$.

Solution

Let $z = \sin y$ so that we wish to find $\frac{dz}{dx}$. We know from the chain rule that

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

We have $z = \sin y$ so

$$\frac{dz}{dy} = \square$$ $\cos y$

and

$$\frac{d}{dx}(\sin y) = \square$$ $\cos y\frac{dy}{dx}$

Generalising the result of Example 3.2 we have

KEY POINT

$$\frac{d}{dx}(f(y)) = \frac{df}{dy}\frac{dy}{dx}$$

Example 3.3

Find $\frac{d}{dt}(\ln y)$.

Solution

We use the previous result with $f(y) = \ln y$. Then

$$\frac{d}{dt}(\ln y) = \frac{df}{dt} = \frac{df}{dy}\frac{dy}{dt} = \frac{1}{y}\frac{dy}{dt}$$

Exercises

1 Differentiate the following functions of y with respect to x:
(a) y^2 (b) $3y^4$ (c) $2y^2 - 3y + 1$ (d) $\frac{1}{y}$ (e) $\sqrt{y}$

2 Differentiate the following functions of y with respect to x.
(a) $\sin 2y$ (b) $3\cos y - y$ (c) e^{2y} (d) $2\cos 3y$
(e) $\sin^2 y$

3 Differentiate the following functions of y with respect to x:
(a) $(y+3)^4$ (b) $(y^2+3)^4$

Solutions to exercises

1 (a) $2y\frac{dy}{dx}$ (b) $12y^3\frac{dy}{dx}$ (c) $4y\frac{dy}{dx} - 3\frac{dy}{dx}$
(d) $-\frac{1}{y^2}\frac{dy}{dx}$ (e) $\frac{1}{2\sqrt{y}}\frac{dy}{dx}$

2 (a) $2(\cos 2y)\frac{dy}{dx}$ (b) $-3(\sin y)\frac{dy}{dx} - \frac{dy}{dx}$
(c) $2e^{2y}\frac{dy}{dx}$ (d) $-6(\sin 3y)\frac{dy}{dx}$
(e) $2\sin y\cos y\frac{dy}{dx}$

3 (a) $4(y+3)^3\frac{dy}{dx}$ (b) $8y(y^2+3)^3\frac{dy}{dx}$

3.3 Finding $\frac{dy}{dx}$ implicitly

We illustrate the technique.

Example 3.4

Find $\frac{dy}{dx}$ given $y^2 + x = x^3 + 3y$.

Solution

We differentiate each term w.r.t. x.

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}, \quad \frac{d}{dx}(x) = 1, \quad \frac{d}{dx}(x^3) = 3x^2, \quad \frac{d}{dx}(3y) = 3\frac{dy}{dx}$$

So

$$2y\frac{dy}{dx} + 1 = 3x^2 + 3\frac{dy}{dx}$$

Rearrangement yields

$$\frac{dy}{dx} = \frac{3x^2 - 1}{2y - 3}$$

Note that $\frac{dy}{dx}$ is given in terms of x and y.

Example 3.5

Find y' given

$$x^2y - 3xy^3 = y^2 + 7$$

Solution

We differentiate each term w.r.t. x.

To differentiate x^2y we use the product rule.

$$\frac{d}{dx}(x^2y) = 2xy + x^2\frac{dy}{dx}$$

To differentiate $3xy^3$ we again use the product rule.

$$\frac{d}{dx}(3xy^3) = \boxed{}$$

$$3y^3 + 9xy^2\frac{dy}{dx}$$

Finally we differentiate $y^2 + 7$.

$$\frac{d}{dx}(y^2 + 7) = \boxed{}$$

$$2y\frac{dy}{dx}$$

Hence

$$2xy + x^2\frac{dy}{dx} - 3y^3 - 9xy^2\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Rearrangement yields

$$\frac{dy}{dx} = \frac{3y^3 - 2xy}{x^2 - 9xy^2 - 2y}$$

Example 3.6

Find $\frac{d^2y}{dx^2}$ if $5x + 3y^2 = 7$.

Solution

First, we calculate $\frac{dy}{dx}$. Differentiating each term of the equation w.r.t. x we find

$$5 + 6y\frac{dy}{dx} = 0$$

so
$$\frac{dy}{dx} = -\frac{5}{6y}$$

Now $\frac{d^2y}{dx^2}$ means $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ so we must differentiate $-\frac{5}{6y}$ with respect to x, that is $\frac{d}{dx}\left(-\frac{5}{6y}\right)$. Here we are differentiating a function of y w.r.t. x. Using the result

$$\frac{d}{dx}(f(y)) = \frac{df}{dy}\frac{dy}{dx}$$

we have

$$\frac{d}{dx}\left(-\frac{5}{6y}\right) = \frac{d}{dy}\left(-\frac{5}{6y}\right)\frac{dy}{dx}$$
$$= \frac{5}{6y^2}\frac{dy}{dx}$$
$$= \frac{5}{6y^2}\left(-\frac{5}{6y}\right) = \frac{-25}{36y^3}$$

Exercises

1 Find $\frac{dy}{dx}$ given
(a) $x^2 + y^2 = 1$ (b) $3x - y^3 = 10$
(c) $x^2 + 2y^2 + 3x - 7y = 9$
(d) $x^3 - y^3 = x + y$ (e) $\sqrt{x} + \sqrt{y} = 10$

2 Find $\frac{dx}{dt}$ given
(a) $4 + \sin x = t$ (b) $3\cos 2x - t^2 = 20$
(c) $\tan x = \frac{1}{t}$ (d) $4\sin t - \cos 3x = \cos t$
(e) $\tan 2t - x^2 = \sin 2x$

3 Find $\frac{dy}{dx}$ given
(a) $2e^x + 3e^y = 10y$ (b) $e^{2x} - e^{3y} = x + 2y$
(c) $e^x e^y = x^2y^3$ (d) $e^{2x+3y} + 2x^3 - y^5 = 0$
(e) $2e^{y-x} = 3e^x + y^2$

4 Find $\frac{d^2y}{dx^2}$ given

$$x^2 + y^2 = 1$$

5 Find y'' given

$$\sin x + \cos y = 1$$

Solutions to exercises

1 (a) $-\frac{x}{y}$ (b) $\frac{1}{y^2}$ (c) $\frac{2x+3}{7-4y}$
(d) $\frac{3x^2-1}{3y^2+1}$ (e) $-\sqrt{\frac{y}{x}}$

2 (a) $\frac{1}{\cos x}$ (b) $-\frac{t}{3\sin 2x}$ (c) $-\frac{\cos^2 x}{t^2}$
(d) $\frac{-4\cos t - \sin t}{3\sin 3x}$ (e) $\frac{\sec^2 2t}{x + \cos 2x}$

3 (a) $\dfrac{2e^x}{10-3e^y}$ (b) $\dfrac{2e^{2x}-1}{3e^{3y}+2}$ (c) $\dfrac{2xy^3-e^xe^y}{e^xe^y-3x^2y^2}$

(d) $\dfrac{6x^2+2e^{2x+3y}}{5y^4-3e^{2x+3y}}$ (e) $\dfrac{3e^x+2e^{y-x}}{2e^{y-x}-2y}$

4 $-\dfrac{1}{y^3}$

5 $-\dfrac{\sin^2 y\sin x+\cos^2 x\cos y}{\sin^3 y}$

End of block exercises

1 In which of the following equations is y expressed implicitly in terms of x?

(a) $\sin x+\dfrac{x}{y}=3$ (b) $\sin y+\dfrac{y}{x}=3$

(c) $x^2-y=10$ (d) $x^2y+y^2x=\sqrt{y}$

(e) $\dfrac{x-y}{x+y}=e^x$

2 Find $\dfrac{dy}{dx}$ given

(a) $x^3-y^4=1$

(b) $2x^2-3y^2+2x-7y=0$

(c) $x^3-2y^4=x$

(d) $2x^2+y^2+3x+2y+7=0$

(e) $x^3-y^3+3x^2-y=0$

3 A circle, centre the origin, radius 5, has equation

$$x^2+y^2=25$$

Find the equation of the tangent to the circle which passes through the point (3, 4).

4 Find $\dfrac{dx}{dt}$ given

(a) $2xt+x^2-xt^2=0$ (b) $x^2t^2-3x^2t=t^3$

(c) $4(x+t)(x-t)-3xt=0$

(d) $\dfrac{x^2}{t^3}-\dfrac{t^2}{x^2}+x+t=1$

(e) $\dfrac{x+t}{x-2t}-\dfrac{3x}{t^2}=0$

5 Find $\dfrac{dy}{dx}$:

(a) $e^x+e^y=x^3+y^4$ (b) $x\sin y=y\cos x$

(c) $xe^y-2ye^x=xy$ (d) $(x+2y)e^{-y}=x^2y$

(e) $\sin 2x\cos 3y-2\sin y\cos x=0$

6 Differentiate the following expressions w.r.t. x:

(a) $(x+y)^n$ (b) $(x^2+y)^n$ (c) $(x^2+y^2)^n$

Solutions to exercises

1 (b) and (d)

2 (a) $\dfrac{3x^2}{4y^3}$ (b) $\dfrac{4x+2}{6y+7}$ (c) $\dfrac{3x^2-1}{8y^3}$

(d) $-\dfrac{4x+3}{2y+2}$ (e) $\dfrac{3x^2+6x}{3y^2+1}$

3 $y=\dfrac{-3x+25}{4}$

4 (a) $\dfrac{2x(t-1)}{2t+2x-t^2}$ (b) $\dfrac{3t^2+3x^2-2x^2t}{2xt^2-6xt}$

(c) $\dfrac{3x+8t}{8x-3t}$ (d) $\dfrac{3x^5+2xt^5-x^3t^4}{2x^4t+2t^6+x^3t^4}$

(e) $\dfrac{2xt+3t^2+6x}{6x-t^2-6t}$

5 (a) $\dfrac{3x^2-e^x}{e^y-4y^3}$ (b) $\dfrac{y\sin x+\sin y}{\cos x-x\cos y}$

(c) $\dfrac{y-e^y+2ye^x}{xe^y-2e^x-x}$ (d) $\dfrac{2xye^y-1}{2-x-2y-x^2e^y}$

(e) $\dfrac{2\cos 2x\cos 3y+2\sin y\sin x}{3\sin 2x\sin 3y+2\cos y\cos x}$

6 (a) $n(x+y)^{n-1}\left(1+\dfrac{dy}{dx}\right)$

(b) $n(x^2+y)^{n-1}\left(2x+\dfrac{dy}{dx}\right)$

(c) $n(x^2+y^2)^{n-1}\left(2x+2y\dfrac{dy}{dx}\right)$