Mada Altiary

Infinite Series

Infinite sequence: $a_1, a_2, \ldots, a_n$ متتابعة لانهائية

Infinite Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ متسلسلة لانهائية

Remark: To determine if an infinite series converges or diverges we could consider the sequance of partial sum.

معنى الملاحظة هو انه كي نحدد ما اذا كانت المتسلسله اللا نهائيه متقاربة أو متباعدة فإننا ننظر الى متتابعة المجموع الجزئي اذا تقترب من عدد معين فإن المتسلسله تكون تقاربية وإذا لايوجد نمط محدد للاعداد فإن المتسلسله تكون تباعديه.

Example:

Determine wether the series $1-1+1-1+1-1+\cdots$ converges or diverges. If it converges find the sum.

Solution:-

نسمي S_1, S_2, S_3, S_4

Partial Sum [المجموع الجزئي].

نسمي المتوالية:

متتابعة المجموع الجزئي.

$S_1 = 1$

$S_2 = 1-1 = 0$

$S_3 = 1-1+1 = 1$

$S_4 = 1-1+1-1 = 0$

The sequance of partial sum:

$1, 0, 1, 0, \ldots$

converge or diverge? $\xrightarrow{\text{why}}$ لأنها لا تقترب من قيمة محددة

Types of series
أنواع المتسلسلات

1. Geometric Series:

Form: $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \quad (a \neq 0)$

convergent $|r| < 1$ divergent $|r| \geq 1$

Sum: $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

Example: In each part determine whether the series converges and if so find its sum

a) $\sum_{k=0} \frac{5}{4^k}$

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \cdots + \frac{5}{4^k} + \cdots$$

is a geometric series with $a = 5$, $r = \frac{1}{4}$

$\because |r| = |\frac{1}{4}| = \frac{1}{4} < 1$

$\therefore$ the series is convergent and the sum

$$\frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{5}{3/4} = 5 \cdot \frac{4}{3} = \frac{20}{3}$$

b)- $\sum_{k=1}^{\infty} 3^{2k} 5^{1-k}$

$$\sum_{k=1}^{\infty} 3^{2k} 5^{1-k} = \sum_{k=1}^{\infty} \frac{(3^2)^k}{5^{-(1-k)}}$$

$$= \sum_{k=1}^{\infty} \frac{9^k}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} \frac{9 \cdot 9^k \cdot 9^{-1}}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} 9 \cdot \frac{9^{k-1}}{5^{k-1}}$$

$$= \sum_{k=1}^{\infty} 9 \left(\frac{9}{5}\right)^{k-1}$$

geometric series : $a = 9$ and $r = 9/5$

$\because |r| = |9/5| > 1 \Rightarrow$ the given series is divergent.

2- Harmonic Series:

is a divergent series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

3- P- Series:

$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{k^p} + \cdots}$$

convergent $p > 1$ | divergent $p \leq 1$

Example 2: Determine the whether the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$ converges or divrges.

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

or:

$$\sum_{k=1}^{\infty} \frac{1}{(k)^{1/3}} = 1 + \frac{1}{(2)^{1/3}} + \frac{1}{(3)^{1/3}} + \cdots + \frac{1}{(k)^{1/3}} + \cdots$$

P series with $P = 1/3 < 1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}}$ divergent.

Testing for Convergence or Divergence of a series

1. The divergence test

- If $\lim_{k\to\infty} u_k \neq 0$ then $\sum u_k$ is divergent
- If $\lim_{k\to\infty} u_k = 0$, then $\sum u_k$ may either converge or diverge. (Fail)

Example: Determine wether the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$ converges or diverges.

$$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{k/k}{\frac{k}{k}+\frac{1}{k}} = \lim_{k\to\infty} \frac{1}{1+1/k}$$

$$= \frac{1}{1+\frac{1}{\infty}} = \frac{1}{1+0} = 1 \neq 0$$

$\therefore \sum_{k=1}^{\infty} \frac{k}{k+1}$ is divergent.

2. The Integral Test:

If $u_k = f(k)$ $\forall n$ an $f(x)$ is continuous, positive, and decreasing on $[1,\infty)$ then

- If $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} u_k$ converges.
- If $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty} u_k$ diverges.

Example : Use the integral test to determine whether the following series converge or diverge.

(a)- $\sum_{k=1}^{\infty} \frac{1}{k}$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b$$

$$= \lim_{b\to\infty} [\ln b - \ln 1]$$

$$= \ln \infty - 0$$

$$= \infty \rightarrow \text{diverges.}$$

$\therefore \int_1^{\infty} \frac{1}{x}\,dx$ diverges $\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

b) $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty} \Big[\frac{-1}{x}\Big]_1^b$$

$$= \lim_{b\to\infty} \Big[\frac{-1}{b} + 1\Big]$$

$$= \frac{-1}{\infty} + 1 = 0 + 1 = 1 \rightarrow \text{converges.}$$

$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2}$ convergent

More Exercise s on diverges or converges of series

Determine wether the following series converges or diverges

1). $\sum_{k=0}^{\infty} 2^k$

$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + \cdots + 2^k$

Type: Geometric , $a = 1$, $r = 2 > 1$

$\therefore \sum_{k=0}^{\infty} 2^k$ divergent

2. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \cdots$

Type: P-Series , $p = 2 > 1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{k^2}$ convergent

3. $\sum_{k=1}^{\infty} \frac{1}{k}$

Type: Harmonic

$\therefore \sum_{k=1}^{\infty} \frac{1}{k}$ divergent

4. $\sum_{k=0}^{\infty} \frac{3}{10^k}$

$\sum_{k=0}^{\infty} \frac{3}{10^k} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k} + \cdots$

Type: Geometric , $a = \frac{3}{10}$, $r = \frac{1}{10} < 1$

$\therefore \sum_{k=0}^{\infty} \frac{3}{10^k}$ convergent and $\displaystyle\sum_{k=0}^{\infty} \frac{3}{10^k} = \frac{3/10}{1-1/10} = \frac{3/10}{9/10} = 3/9$

5. $\displaystyle\sum_{k=1}^{\infty} \frac{4k-1}{7k+4}$

في النهايات إذا كانت درجة
البسط = درجة المقام
النهاية = المعاملات .

$\displaystyle\lim_{k\to\infty} \frac{4k-1}{7k+4} = \frac{4}{7} \neq 0$

$\Rightarrow \displaystyle\sum_{k=1}^{\infty} \frac{4k-1}{7k+4}$ divergent by divergent test

6. $\displaystyle\sum_{k=0}^{\infty} x^k$

$\displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots + x^k + \cdots$

Type : Geometric , $a = 1$, $r = x$ → متغير

- If $x < 1 \Rightarrow \displaystyle\sum_{k=0}^{\infty} x^k$ convergent and

  $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

- If $x > 1 \Rightarrow \displaystyle\sum_{k=0}^{\infty} x^k$ divergent.

Maclaurin and Taylor polynomials

1-Maclaurin polynomials

Linear Approximation:

$$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

at $x_0 = 0$

$$f(x) = f(0) + f'(x_0)\,x$$

Quadratic Approximation:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$$

at $x_0 = 0$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2$$

Example:- Find the local linear and quadratic approximation of e^x at $x = 0$

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$$

$$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$$

linear: $f(x) = f(0) + f'(0)\,x$

$$e^x = 1 + x$$

Quadratic: $f(x) \simeq f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2$

$$e^x \approx 1 + x + \frac{1}{2}x^2$$

nth Maclaurin polynomial

We define the nth Maclaurin Polynomial for f about $x=0$ to be:

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{n}(0)}{n!}x^n$$

Example: Find the Maclaurin polynomial P_0, P_1, P_2, P_3 and P_n for e^x

$f(x) = e^x$ $\qquad f(0) = e^0 = 1$

$f'(x) = e^x$ $\qquad f'(0) = e^0 = 1$

$f''(x) = e^x$ $\qquad f''(0) = e^0 = 1$

$f'''(x) = e^x$ $\qquad f'''(0) = e^0 = 1$

$\vdots$ $\qquad \vdots$

$\therefore$ $f^{(n)}(x) = e^x$ $\qquad f^{(n)}(0) = e^0 = 1$

$P_0(x) = f(0) = 1$

$P_1(x) = f(0) + f'(0)x = 1 + x$

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$

$= 1 + x + \frac{1}{2!}x^2$

$$P_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(x)}{3!}x^3$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

$$P_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(x)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

2-Tylor Polynomials

We define the nth Taylor Polynomial for f about $x = x_0$ to be:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

ملاحظه: كثيرة الحدود ماكلورين ماهي
الإحالة خاصه من تايلور

$x_0 = 0$ ← ماكلورين

$x_0 =$ أي نقطة أخرى ← تايلور

الحدود الاربعه الاولى

Example: Find the first four Taylor Polynomial for $\ln x$ about $x=2$

$f(x) = \ln x$ $\qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x}$ $\qquad f'(2) = \frac{1}{2}$

$f''(x) = -1/x^2$ $\qquad f''(2) = \frac{-1}{4}$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(2) = \frac{2}{8} = \frac{1}{4}$

Then:

$P_0(x) = f(2) = \ln 2.$

$P_1(x) = f(2) + f'(2)(x-2)$

$= \ln 2 + \frac{1}{2}(x-2)$

$P_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$

$= \ln 2 + \frac{1}{2}\ln(x-2) - \frac{1}{8}(x-2)^2$

$P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$

$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1/4}{6}(x-2)^3$

$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3.$

HW: Find the nth Maclaurin polynomials for

a) $\sin x$

b) $\cos x$

محاضرات في أساسيات التكامل

SCAN ME

Made Altiany

Indefinite Integral

The function $F(x)$ is called an antiderivative of $f(x)$ if $F'(x) = f(x)$. For example:

The antiderivative of x is $\frac{x^2}{2}$ because $\left(\frac{x^2}{2}\right)' = x$.

The antiderivative of $\cos x$ is $\sin x$ because $(\sin x)' = \cos x$.

Example: show that the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$

$$F'(x) = \frac{d}{dx}[F(x)]$$

$$= \frac{d}{dx}\left[\frac{1}{3}x^3\right]$$

$$= \frac{1}{3}(3x^2)$$

$$= x^2 = f(x).$$

Thus $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$.

Note: If $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ is also an antiderivative of $f(x)$ for any constant C.

$$\frac{d}{dx}[x^2] = 2x, \quad \frac{d}{dx}[x^2+2] = 2x+0, \quad \frac{d}{dx}[x^2+3] = 2x+0$$

we see that $2x$ is an antiderivative of x^2, x^2+2 and x^2+3.

$$\frac{d}{dx}[F(x)+C] = \frac{d}{dx}F(x) + 0 = f(x).$$

- Any two antiderivative of f differ by an additive constant, so every antiderivative of f can be written in the form $F(x)+C$.

Note: The process of finding antiderivative is called: Integration.

Definition: If the function $F(x)$ is an antiderivative of $f(x)$ then the expression $F(x)+C$, where C is an arbitrary constant is called the indefinite Integral of $f(x)$ with respect to x and denoted $\int f(x)\,dx$

By this definition:

Integral sign $\int \underbrace{f(x)}_{\text{integrand}} dx = F(x)+C$

Table of Basic Integrals

(1) $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1$

(2) $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$

(3) $\displaystyle\int e^x\,dx = e^x$

(4) $\displaystyle\int \ln x\,dx = x\ln x - x$

(5) $\displaystyle\int \sin x\,dx = -\cos x$

(6) $\displaystyle\int \cos x\,dx = \sin x$

(7) $\displaystyle\int \tan x\,dx = \ln|\sec x|$

(8) $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$

(9) $\displaystyle\int \sec^2 x\,dx = \tan x$

(10) $\displaystyle\int \sec x \tan x\,dx = \sec x$

(11) $\displaystyle\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\frac{x}{a}$

(12) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}$

(13) $\displaystyle\int \frac{a}{x\sqrt{x^2-a^2}}\,dx = \sec^{-1}\frac{x}{a}$

(14) $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\frac{x}{a} = \ln(x+\sqrt{x^2-a^2})$

(15) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a} = \ln(x+\sqrt{x^2+a^2})$

Note: In the lecture of integration by parts you will learn how to drive formula (4). Proved of formulas (7) and (8) will be in the lecture of integrating power of secants and tangents.

Example: Evaluate the following:

$\int x^2\,dx$	$\int x^3\,dx$	$\int \frac{1}{x^5}\,dx$	$\int \sqrt{x}\,dx$
$\int x^2 dx = \frac{x^3}{3} + C$	$\int x^3 dx = \frac{x^4}{4} + C$	$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx$ $= \frac{x^{-4}}{4} + C$	$\int \sqrt{x}\,dx = \int x^{1/2}\,dx$

Properties of the indefinite Integral:

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively and that C is a constant. Then

1. For any constant C: $\int c\,f(x)\,dx = c\int f(x)\,dx$

2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$

Example: Evaluate the following:

1. $\int 4\cos x\,dx = 4\int \cos x\,dx = 4\sin x + C$

2. $\int (x + x^2)\,dx = \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + C$

3. $\int \frac{\cos x}{\sin^2 x}\,dx$	4. $\int \frac{t^2 - 2t^4}{t^4}\,dt$	5. $\int \frac{x^2}{x^2+1}\,dx$	Remeber
$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\,dx$	$= \int \left(\frac{t^2}{t^4} - \frac{2t^4}{t^4}\right) dt$	$= \int \frac{x^2+1-1}{x^2+1}\,dx$	$\tan x = \frac{\sin x}{\cos x}$
$= \int \cot x \cdot \csc x\,dx$	$= \int \left(\frac{1}{t^2} - 2\right) dt$	$= \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right) dx$	$\cot x = \frac{\cos x}{\sin x}$
$= -\csc x + C$	$= \int t^{-2}\,dt - 2\int dt$	$= \int dx - \int \frac{1}{x^2+1}\,dx$	$\csc x = \frac{1}{\sin x}$
	$= \frac{-1}{t} - 2t + C$	$= x - \tan^{-1} x + C$	$\sec x = \frac{1}{\cos x}$

Integration by substitution

A composite function is a function of the form:

$$F(g(x))$$

outer function — inner or inside function

For example:–

If we let $u = g(x) = x^2$. Then

- The outer function is $\sin u$.
- The inner function is x^2

Recall the chain rule of differentiation says that:

$$\boxed{\frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x).} \qquad (1)$$

Example: Use the chain rule to find the derivative of the composite function $f(g(x)) = \sin(x^2)$

$$\frac{d}{dx} F(g(x)) = \frac{d}{dx} \sin(x^2) = \cos x^2 \cdot 2x = 2x \cos x^2 .$$

Now: What is the rule to integrate the composite function i.e What is the answer for $\int \sin^2 x \, dx$?!

Remark: There is no chain rule for integration of composite function. However, reversing rule (1) tell us that

$$\int F'(g(x)) \cdot g'(x) = F(g(x)) + c \qquad (2)$$

By our example we have:

$$\int 2x \cos x^2 \, dx = \sin x^2 + c$$

If F is an antiderivative of f then $F' = f$ and we can write (2) as:

$$\int f(g(x)) \cdot g'(x) \, dx = F(g(x)) + c$$

Substitution Rule for indefinite Integrals

u- Substitution is used when integral contains some function and it's derivative. That is for an integral of the form

$$\int f(g(x))\, g'(x)\, dx$$

Deriv. of inner

inner function

$$\int 2x \sin(\underbrace{x^2}_{\text{composite function}})\, dx$$

Let $u = x^2$ (the 'inside')

$du = 2x\, dx$

The integral becomes

$$\int \sin(u)\, du = -\cos(u) + c$$

$$= -\cos(x^2) + c$$

Deriv. of inner

composite function

$$\int x \cos(x^2)\, dx$$

inner function

Let $u = x^2$

$du = 2x\, dx \Rightarrow \frac{1}{2} du = x\, dx$

$$\int \cos(u) \cdot \frac{1}{2} du = \frac{1}{2} \int \cos(u)\, du$$

$$= \frac{1}{2} \sin(u) + c$$

$$= \frac{1}{2} \sin(x^2) + c$$

$$\int \frac{f'(x)}{f(x)} = f(x)^{-1} \cdot f'(x)\, dx$$

$$\int \frac{2x+1}{x^2+x}\, dx$$

Let $u = x^2 + x$

$du = 2x+1\, dx$

The integral becomes

$$\int \frac{du}{u} = \int \frac{1}{u} du = \ln(u) + c$$

$$= \ln|x^2+x| + c$$

$$\int \frac{2e^{2x}}{e^{2x}-1}\, dx$$

Let $u = e^{2x} - 1$

$du = 2e^{2x}$

$$\int \frac{2e^{2x}}{e^{2x}-1}\, dx = \int \frac{1}{u}\, du = \ln|u| + c$$

$$= \ln|e^{2x}-1| + c$$

General formula:

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c}$$

$$\int (x^2+2x+5)^3 (2x+2)dx.$$

Let $u = x^2+2x+5 \Rightarrow du = 2x+2$

The integral becomes:

$$\int u^3 du = \frac{u^4}{4} + C$$

$$= \frac{(x^2+2x+5)^4}{4} + C$$

$$\int \frac{(\frac{1}{x}+4)^5}{x^2} dx$$

$$\int \frac{(\frac{1}{x}+4)^5}{x^2} dx = \int (\frac{1}{x}+4)^5 . x^{-2}$$

Let $u = \frac{1}{x}+4 \Rightarrow du = -\frac{1}{x^2} dx$ (1)

The integral (1) becomes

$$\int -u^5 . du = -\frac{u^6}{6} + C$$

$$= -\frac{1}{6}\left(\frac{1}{x}+4\right)^6 + C$$

$$\int 2x(x^2+1)^3 dx$$

let $u = 2x \Rightarrow du = 2x dx$

The integral becomes:

$$\int u^3 du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{4} + C$$

General Formula:

$$\boxed{\int (f(x))^n . f'(x) dx = \frac{(f(x))^{n+1}}{n+1} + C}$$

$$\int 2x . e^{x^2} dx$$

let $u = x^2 \Rightarrow du = 2x$

The integral becomes:

$$\int e^u . du = e^u + C$$

$$= e^{x^2} + C$$

$$\int \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx$$

let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}} dx$

The integral becomes:

$$\int e^u . du = e^u + C$$

$$= e^{\sqrt{x}} + C$$

$$\int e^{3x+1} . 3 dx$$

Let $u = 3x+1 \Rightarrow du = 3 dx$

The integral becomes:

$$\int e^u . du = e^u + C = e^{3x+1} + C$$

General Formula:-

$$\int e^{f(x)} . f'(x) dx = e^{f(x)} + C$$

$$\int (x^2+1)^{50} \cdot 2x\,dx$$

let $u = x^2+1 \Rightarrow du = 2x\,dx$ ✓

Thus the integral becomes

$$\int (u)^{50} \cdot du = \frac{u^{51}}{51} + C$$

$$= \frac{(x^2+1)^{51}}{51} + C$$

$$\int (x-8)^{23}\,dx$$

Let $u = x-8 \Rightarrow du = dx$ ✓

Thus the integral becomes

$$\int u^{23}\,du = \frac{u^{24}}{24} + C$$

$$\int \frac{dx}{(\frac{1}{3}x-8)^5}$$

$$\int (\tfrac{1}{3}x-8)^{-5}\,dx$$

فقط ثابت ✓

Let $u = \frac{1}{3}x - 8 \Rightarrow du = \frac{1}{3}dx$

$\Rightarrow dx = 3\,du$. Thus the integral becomes

$$\int (u)^{-5} \cdot (3\,du)$$

$$3\int (u)^{-5} \cdot du$$

$$= 3\,\frac{u^{-4}}{-4} + C$$

$$= \frac{-3}{4}\left(\tfrac{1}{3}x-8\right)^{-4} + C$$

$$\int (3x^2+1)^3\,x\,dx$$

فقط الثابت

let $u = 3x^2+1 \Rightarrow du = 6x\,dx$ ✓

$\Rightarrow x\,dx = \frac{1}{6}\,du$

Thus the integral becomes

$$\int (u^3) \cdot \frac{1}{6}\,du = \frac{1}{6}\int u^3\,du$$

$$= \frac{1}{6}\,\frac{u^4}{4} + C$$

$$= \frac{1}{24}(3x^2+1)^4 + C$$

$$\int (x^2-3)^3\,dx$$

let $u = x^2-3 \Rightarrow du = 2x$ ثابت ومتغير ✗

* we can not solve this problem by u-substituation ?!!

* The reason is we have $du = 2x$ and we don't have an x in our integral. we can n't multiply by a variable to adjust our integral.

$$\int \cos(5x)\,dx$$

Let $u = 5x \Rightarrow du = 5\,dx$

$\Rightarrow dx = \frac{1}{5}\,du$

Thus the integral becomes

$$\int \cos u \cdot \left(\frac{1}{5}\right) du$$

$$= \frac{1}{5}\int \cos u\,du$$

$$= \frac{1}{5}\sin u + C$$

$$= \frac{1}{5}\sin(5x) + C$$

$$\int \sin(5x)\,dx$$

Let $u = 5x \Rightarrow du = 5\,dx$

$\Rightarrow dx = \frac{1}{5}\,du$

The integral becomes:

$$\frac{1}{5}\int \sin u \cdot du = \frac{-1}{5}\cos u + C$$

$$= \frac{-1}{5}\cos(5x) + C$$

General Formulas:-

1- $\int \cos ax\,dx = \frac{\sin ax}{a} + C$

2- $\int \sin ax\,dx = \frac{-\cos(ax)}{a} + C$

$$\int 2x - \sin 3x\,dx$$

$$= \int 2x\,dx - \int \sin 3x\,dx$$

تكامل مباشر

إما حله بالتعويض

أو نستخدم القاعدة ؟

$$= x^2 + \frac{\cos(3x)}{3} + C$$

$$\int \frac{1}{x} + \sec^2 \pi x\,dx$$

$$= \int \frac{1}{x} + \underbrace{\int \sec^2(\pi x)\,dx}_{(1)}$$

let $u = \pi x \Rightarrow du = \pi\,dx$

$\Rightarrow dx = \frac{1}{\pi}\,du.$

Thus the integral (1) becomes

$$\int \sec^2(\pi x) \cdot dx = \int \sec^2{}^{(u)} \cdot \left(\frac{1}{\pi}\right) du$$

$$= \frac{1}{\pi}\tan u + C$$

$$= \frac{1}{\pi}\tan(\pi x) + C$$

substitute the result back into equation (1) we get:

$$\int \frac{1}{x} + \sec^2 \pi x\,dx = \ln|x| + \frac{1}{\pi}\tan(\pi x) + C$$

$$\int \sin^2 x \cos x \, dx$$

Let $u = \sin x \Rightarrow du = \cos x \, dx$

The integral becomes:

$$\int u^2 \, du = \frac{u^3}{3} + C$$

$$= \frac{\sin^3 x}{3} + C$$

$$\int \cos^4 x \sin x \, dx$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

The integral becomes:

$$\int u^4 (-du) = -\frac{u^5}{5} + C$$

$$= -\frac{1}{5} \cos^5 x + C$$

$$\int \sin^5(x+1) . \cos(x+1) \, dx$$

Let $u = \sin(x+1) \Rightarrow du = \cos(x+1) \, dx$

The integral becomes

$$\int u^5 . du = \frac{1}{6} u^6 + C$$

$$= \frac{1}{6} \sin^6(x+1) + C$$

$$\int \sin x \cos x \, dx$$

let $u = \cos x \Rightarrow du = -\sin x \, dx$

The integral becomes :-

$$\int \cos x \sin x \, dx = \int -u \, du$$

$$= -\frac{u^2}{2} + C$$

$$= -\frac{1}{2} \cos^2 x + C$$

or :-

let $u = \sin x \Rightarrow du = \cos x \, dx$

The integral becomes:

$$\int u . du = \frac{u^2}{2} + C = \frac{1}{2} \sin^2 x + C$$

ملاحظات :

١- لكي نستخدم التكامل بالتعويض في النوع :

$$\int \sin^m x \cos^n x \, dx$$

لابد أن تكون إحدى الأسس تساوي الواحد أو كلاهما.

- في حال إحدى الأسس واحد نختار الـ u لتكون الدالة المرفوعة لأس غير الواحد. أما في حال جميع الأسس $= 1$ نستطيع أن نختار أياً منهم بـ u.

- في حال جميعهم مرفوعين لأسس مختلفة يكون حلهم بطرق أخرى ندرسها لاحقاً

$$\int x^4 \sqrt[3]{3-5x^5}\, dx$$

$$\int x^4 \sqrt[3]{3-5x^5}\, dx = \int x^4 (3-5x^5)^{1/3}\, dx$$

let $u = 3-5x^5 \Rightarrow du = -25x^4\, dx$

$\Rightarrow -\frac{1}{25} du = x^4\, dx$

The integral becomes:-

$$\int u^{1/3} \cdot \left(-\frac{1}{25}\right) du$$

$$= -\frac{1}{25} \int u^{1/3}\, du$$

$$= -\frac{1}{25} \frac{u^{4/3}}{4/3} + C$$

$$= -\frac{1}{25} \cdot \frac{3}{4} u^{4/3} + C$$

$$= \frac{-3}{100} (3-5x^5)^{4/3} + C$$

$$\int 3x^2 \sqrt{x^3+3}\, dx$$

let $u = x^3+3 \Rightarrow du = 3x^2\, dx$

The integral becomes:

$$\int \sqrt{u}\, du = \int u^{1/2}\, du$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \frac{2}{3} (x^3+3)^{\frac{3}{2}} + C$$

$$\int x\sqrt{x-1}\, dx$$

let $u = x-1 \Rightarrow du = dx$ ✓

$x = u-1$

The integral becomes:

$$\int (u+1)^2 \cdot u^{1/2}\, du$$

$$= \int (u^2+2u+1)\, u^{1/2}\, du$$

$$= \int (u^{2+1/2} + 2u^{1+\frac{1}{2}} + u^{1/2})\, du$$

$$= \int (u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + u^{1/2})\, du$$

$$= \frac{2}{7} u^{\frac{7}{2}} + \frac{4}{5} u^{\frac{5}{2}} + \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{7} (x-1)^{\frac{7}{2}} + \frac{4}{5} (x-1)^{\frac{5}{2}} + \frac{2}{3} (x-1)^{3/2} + C$$

Determine which of the following integrals can not be solved by u-substitution rule: ✓

1)- $x\sqrt{2x-1}$

2)- $2\sqrt{4-x^2}$

3)- $x^5\sqrt{4-x^2}$

$$\int \frac{dx}{1+3x^2} = \int \frac{dx}{1+(\sqrt{3}x)^2}$$

Let $u = \sqrt{3}x \Rightarrow du = \sqrt{3}\,dx$

$\Rightarrow dx = \frac{1}{\sqrt{3}}\,du.$

The integral becomes:

$$\int \frac{du/\sqrt{3}}{1+u^2} = \frac{1}{\sqrt{3}} \int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} u + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \sqrt{3}x + C$$

$$\int \frac{dx}{9+x^2} = \int \frac{dx}{9(1+\frac{x^2}{9})}$$

$$= \frac{1}{9} \int \frac{dx}{1+(\frac{x}{3})^2}$$

let $u = \frac{x}{3} \Rightarrow du = \frac{1}{3}\,dx$

$\Rightarrow 3\,du = dx$

The integral becomes

$$\frac{3}{9} \int \frac{du}{1+u^2} = \frac{1}{3} \tan^{-1} u + C$$

$$= \frac{1}{3} \tan^{-1} \frac{x}{3} + C$$

$$\int \frac{dx}{\sqrt{2-x^2}} = \int \frac{dx}{\sqrt{2(1-\frac{x^2}{2})}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{1-(\frac{x}{\sqrt{2}})^2}} \quad (1)$$

let $u = \frac{x}{\sqrt{2}} \Rightarrow du = \frac{1}{\sqrt{2}}\,dx$

$\Rightarrow dx = \sqrt{2}\,du$

The integral (1) becomes:

$$\frac{1}{\cancel{\sqrt{2}}} \int \frac{\cancel{\sqrt{2}}\,du}{\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1} \frac{x}{\sqrt{2}} + C$$

General Formulas:-

$$\boxed{\begin{array}{l} 1.\ \int \frac{du}{a^2+u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C \\ 2.\ \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1} \frac{u}{a} + C \end{array}}$$

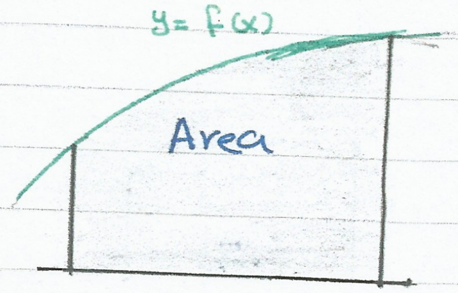
Remarks:-

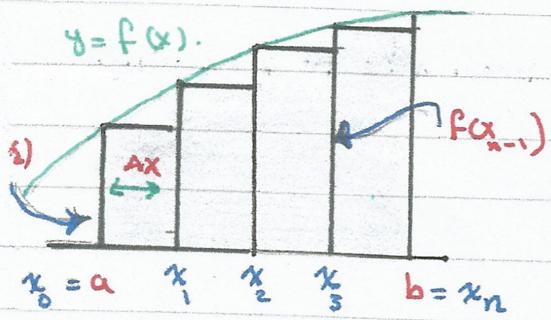
All these kind of integral can be solved by "Trigonometric Substitution" technique. We will study this technique later on.

Definite Integral and Area.

The Area under a curve:

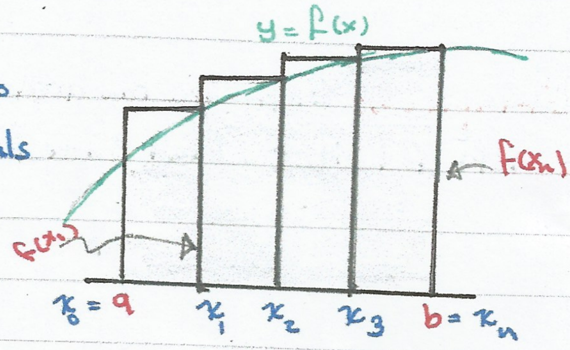
Let $f(x)$ be non-negative on $[a,b]$.
Find the area of the region lying beneath the curve $y = f(x)$ and above the x-axis from $x = a$ to $x = b$.





Divide $[a,b]$ in to n equal subintervals of width Δx

$\Delta x = \dfrac{b-a}{n}$



Left- Hand sum

$= f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$

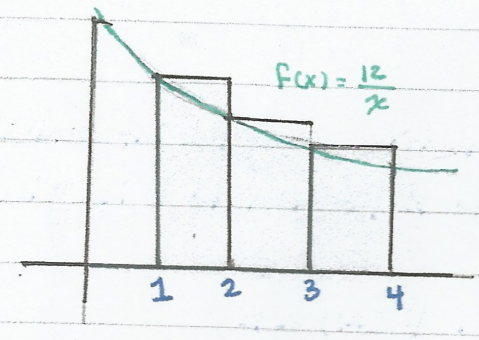
$= \sum_{i=0}^{n-1} f(x_i)\Delta x$

Right - Hand sum

$= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

$= \sum_{i=1}^{n} f(x_i)\Delta x$

* **Example (1):** Estimate the area under $f(x) = \dfrac{12}{x}$ from $x = 1$ to $x = 4$ with left and right hand sums for $n = 3$.



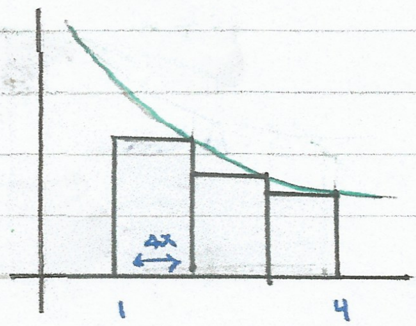
$\Delta x = \dfrac{4-1}{3} = 1$

$L.H.S = f(1)\Delta x + f(2)\Delta x + f(3).\Delta x$

$= \left[\dfrac{12}{1} + \dfrac{12}{2} + \dfrac{12}{3}\right] \quad \Delta x = 1$

$= 12 + 6 + 4$

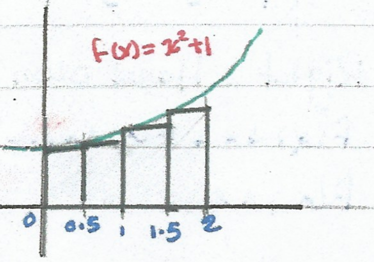
$= 22$



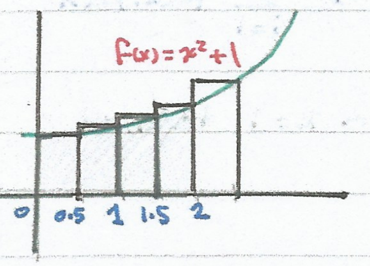
$$R.H.S = f(2)\Delta x + f(3)\Delta x + f(4)\Delta x$$
$$= \left[\frac{12}{2} + \frac{12}{3} + \frac{12}{4}\right] \cdot 1$$
$$= 6 + 4 + 3$$
$$= 13$$

$$13 < \text{Area} < 22$$

Example (2): Estimate the area under $f(x) = x^2+1$ from $x=0$ to $x=2$ with left and right - hand sum for $n=4$



$$\Delta x = \frac{2-0}{4} = 0.5$$
$$L.H.S = f(0)\Delta x + f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x$$
$$= [1 + 1.25 + 2 + 3.25](0.5)$$
$$= (2.5)(0.5) = 3.75$$



$$\Delta x = 0.5$$
$$R.H.S = f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x$$
$$= [1.25 + 2 + 3.25 + 5](0.5)$$
$$= [11.5](0.5) = 5.75$$

$$3.75 < \text{Area} < 5.75$$

Definition: Assume that f is a continuous function along the interval $[a,b]$ Then the definite integral of f with respect to x from a to b is given by

$$\int_a^b f(x)\,dx = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x}_{\text{Riemann Sum}}$$

where n represent of approximating rectangels with a uniform width equal to $\Delta x = b-a/n$ and where x_i represents some point in the ith

Theorem: If a function f is continuous on an interval $[a,b]$, then f is integrable on $[a,b]$, and the net signed A between of f and the interval $[a,b]$ is

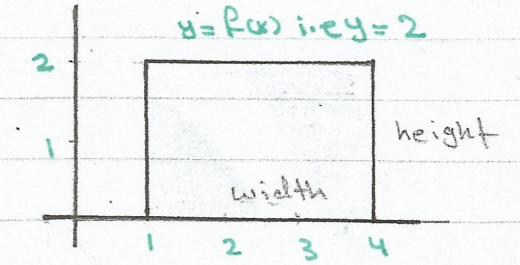
$$A = \int_a^b f(x)\,dx.$$

Ex: sketch the region whose area is represented by the definite integral and evaluate the integral using appropriate formula from geometry

1. $\int_1^4 2 \, dx$

sol:

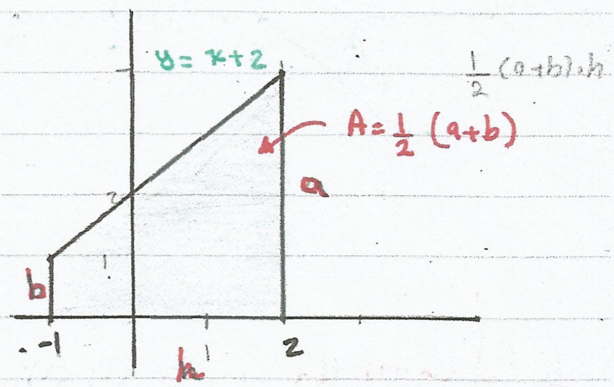
$$A = \int_1^2 2\,dx = (\text{area of rectangle})$$
$$= 2 \cdot 3 = 6$$



2. $\int_{-1}^{2} (x+2)\,dx$

sol:

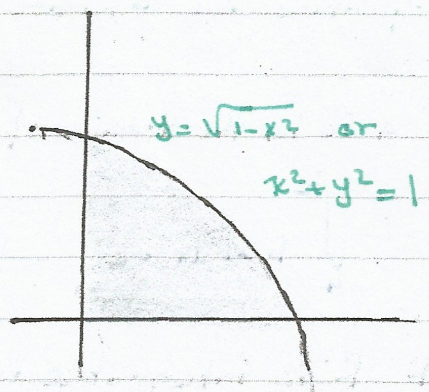
$$A = \int_{-1}^{2} (x+2)\,dx = (\text{area of trapezoid})$$
$$= \frac{1}{2}(1+4) \cdot 3$$
$$= \frac{15}{2}$$



3. $\int_0^1 \sqrt{1-x^2}\,dx$

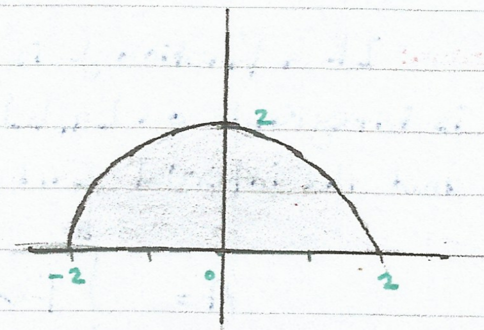
$$A = \int_0^1 \sqrt{1-x^2} = (\text{area of quarter circle})$$

$$= \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$



$x^2+y^2=4$

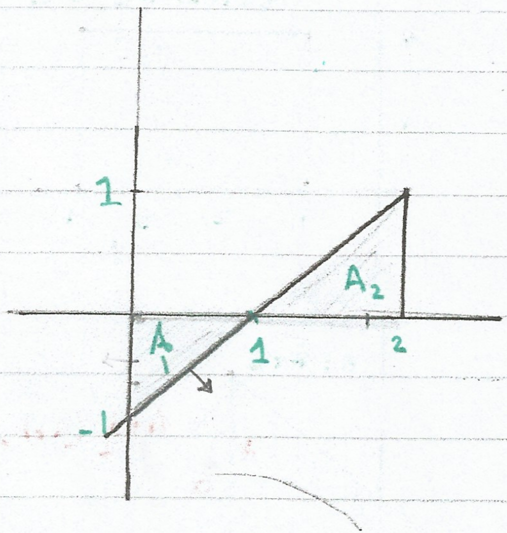
4. $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\, dx$



$$A = \int_{-2}^{2} \sqrt{4-x^2}\, dx = (\text{area of half circle})$$
$$= \frac{1}{2}\pi \cdot 2^2 = 2\pi$$

Evaluate:

ⓐ $\displaystyle\int_{0}^{2} (x-1)\, dx$ ⓑ $\displaystyle\int_{0}^{1} (x-1)\, dx$



$f(x) = x-1$

x	0	1	2
$f(x)$	-1	0	1

A_1 = Area of tringle
$= -\frac{1}{2}(1)(1) = -\frac{1}{2}$

$A_2 = \frac{1}{2}(1)(1) = \frac{1}{2}$

Thus
$$\int_{0}^{2} (x-1)\, dx = A_1 + A_2 = -\frac{1}{2} + \frac{1}{2} = 0$$

ⓑ $\displaystyle\int_{0}^{1} (x-1)\, dx = -\frac{1}{2}(1)(1) = -\frac{1}{2}$

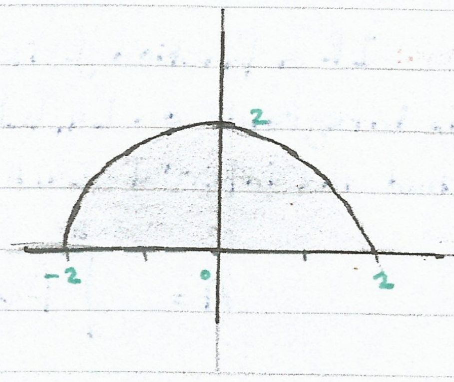
* Properties of definite integral:

ⓐ If a is in the domain of f, we define
$$\int_{a}^{a} f(x)\, dx = 0$$

ⓑ If f is integrable on [a,b], then we define
$$\int_{a}^{b} f(x)\, dx = -\int_{b}^{a} f(x)\, dx.$$

$x^2+y^2=4$

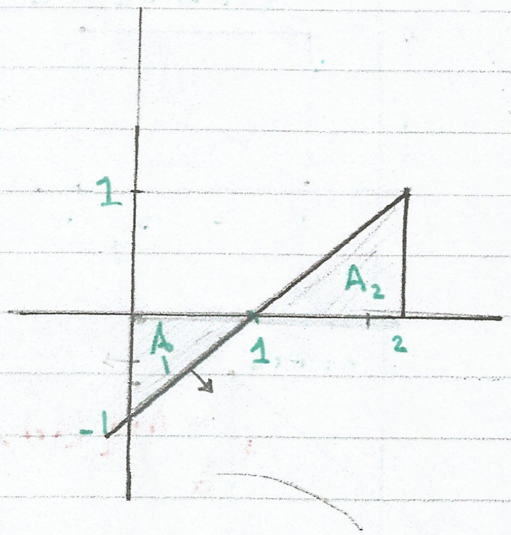
4. $\int_{-2}^{2} \sqrt{4-x^2}\, dx$



$$A = \int_{-2}^{2} \sqrt{4-x^2}\, dx = (\text{area of half circle})$$
$$= \frac{1}{2}\pi \cdot 2^2 = 2\pi$$

Evaluate:

ⓐ $\int_{0}^{2} (x-1)\, dx$ ⓑ $\int_{0}^{1} (x-1)\, dx$



$f(x) = x-1$

x	0	1	2
$f(x)$	-1	0	1

A_1 = Area of tringle
$= -\frac{1}{2}(1)(1) = -\frac{1}{2}$

$A_2 = \frac{1}{2}(1)(1) = \frac{1}{2}$

Thus
$$\int_{0}^{2} (x-1)\, dx = A_1 + A_2 = -\frac{1}{2} + \frac{1}{2} = 0$$

ⓑ $\int_{0}^{1} (x-1)\, dx = -\frac{1}{2}(1)(1) = -\frac{1}{2}$

* Properties of definite integral:

ⓐ If a is in the domain of f, we define

$$\int_{a}^{a} f(x)\, dx = 0$$

ⓑ If f is integrable on $[a, b]$, then we define

$$\int_{a}^{b} f(x)\, dx = -\int_{b}^{a} f(x)\, dx.$$

3. $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c is any number.

Examples:-

Evaluate the following:

1. $\int_1^1 x^2\,dx = 0$

2. $\int_4^0 x\,dx = -\int_0^4 x\,dx = -\left.\frac{x^2}{2}\right|_0^4 = -\left(\frac{4^2}{2} - 0\right) = -8.$

3. $\int_{-\frac{1}{2}}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

$= \left(\frac{\pi}{6}\right) - \left(-\frac{\pi}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3}$

Example: Evaluate $\int_0^6 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ 3x - 2 & \text{if} \quad x \geq 2 \end{cases}.$$

$\int_0^6 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx$

$= \left.\frac{x^3}{3}\right|_0^2 + \left.\frac{3}{2}x^2 - 2x\right|_2^6$

$= \left(\frac{2^3}{3} - 0\right) + (42 - 2)$

The Fundamental Theorems of Calculus

The first fundamental theorem of calculus:

If $f(x)$ is continuous on $[a,b]$ and F is any antiderivative of $f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The second fundamental theorem of calculus.

If $f(x)$ is continuous on an interval and a is any number in this interval then the function

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of $f(x)$. In the other words,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

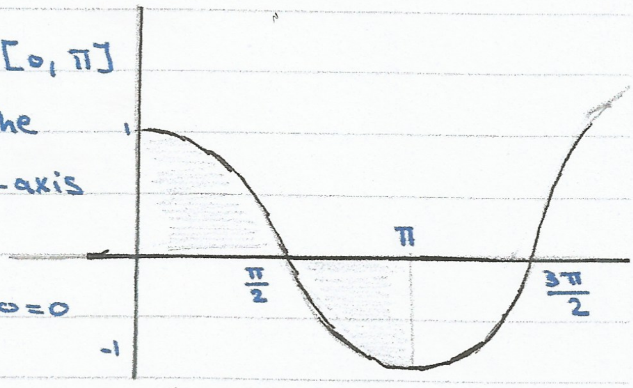
Ex:1. Find the area under the curve $y = \cos x$ over the interval $[0, \frac{\pi}{2}]$

2. Make a conjecture about the value of the integral $\int_0^{\pi} \cos x\,dx$.

1. $A = \int_0^{\pi/2} \cos x\,dx = \sin x \Big|_0^{\frac{\pi}{2}} = \sin(\frac{\pi}{2}) - \sin(0) = 1 - 0 = 1.$

2- $\int_0^{\pi} \cos x\,dx = 0$ because, on the interval $[0, \pi]$ the portion of area above the x-axis is the same as the portion of area below the x-axis

$$\int_0^{\pi} \cos x\,dx = \sin x \Big|_0^{\pi} = \sin(\pi) - \sin(0) = 0 - 0 = 0$$



Ex (2):- Evaluate: $\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$, $\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t}\,dt\right]$

By SFT:

$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$ and $\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t}\,dt\right] = \frac{\sin x}{x}$

$$\int_4^9 x^2\sqrt{x}\,dx = \int_4^9 x^2 \cdot x^{1/2}\,dx$$

$$= \int_4^9 x^{5/2}\,dx$$

$$= \frac{2}{7}x^{7/2}\Big|_4^9$$

$$= \frac{2}{7}\left[9^{7/2} - 4^{7/2}\right]$$

$$= \frac{2}{7}\left[3^7 - 2^7\right]$$

$$= \frac{4118}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\,dx = \frac{-\cos x}{5}\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{5}\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$= -\frac{1}{5}(0-1) = \frac{1}{5}$$

$$\int_0^{\ln(3)} 5e^x\,dx = 5\int_0^{\ln 3} e^x\,dx$$

$$= 5e^x\Big|_0^{\ln 3}$$

$$= 5\left(e^{\ln 3} - e^0\right)$$

$$= 5(3-1) = 10$$

$$\int_0^{\pi/3} \sec^2 x\,dx = \tan x\Big|_0^{\pi/3}$$

$$= \tan\left(\frac{\pi}{3}\right) - \tan(0)$$

$$= \sqrt{3} - 0 = \sqrt{3}$$

$$\int_{-e}^{-1} \frac{1}{x}\,dx = \ln|x|\Big|_{-e}^{-1}$$

$$= \ln|-1| - \ln|-e|$$

$$= \overset{0}{\ln(1)} - \overset{1}{\ln(e)} = -1$$

$$\int_1^9 \sqrt{x}\,dx = \int_1^9 x^{1/2}\,dx$$

$$= \frac{2}{3}x^{3/2}\Big|_1^9$$

$$= \frac{2}{3}\left((\sqrt{9})^3 - (\sqrt{1})^3\right)$$

$$= \frac{2}{3}(27-1) = \frac{52}{3}$$

$$\int_0^3 (9-x^2)\,dx = \int_0^3 9\,dx - \int_0^3 x^2\,dx$$

$$= 9x - \frac{x^3}{3}\Big|_0^3$$

$$= \left(9(3) - \frac{3^3}{3}\right) - (0-0)$$

$$= 27 - 9 = 18.$$

Evaluating definite integral by substituation

There is two options for calculating a definite integral using substitution:

Method 1: change the limits

Let $u = g(x) \Rightarrow du = g'(x)$

If $x = a \Rightarrow u = g(a)$
If $x = b \Rightarrow u = g(b)$ } new limits

Then:

$$\int_a^b f(g(x)).g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example:

$$\int_0^2 x(x^2+1)^3\,dx$$

let $u = x^2+1 \Rightarrow du = 2x\,dx$

$\Rightarrow x\,dx = \frac{1}{2}du.$

If $x = 0 \Rightarrow u = 1$

$x = 2 \Rightarrow u = 5$

Then

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du$$

$$= \frac{1}{2}\frac{u^4}{4}\Big|_1^5$$

$$= \frac{1}{8}(5^4 - 1^4)$$

$$= 78.$$

Method 2: Solve an indefinite integral first

First evaluate the indefinite integral

$$\int f(g(x)).g'(x)\,dx$$

by substitution in terms of x and use the original limits of integration to evaluate the definite integral

Example:-

$$\int_0^2 x(x^2+1)^3\,dx$$

let $u = x^2+1 \Rightarrow du = 2x\,dx$

$\Rightarrow x\,dx = \frac{1}{2}du$

The integral becomes:

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du$$

$$= \frac{1}{2}\frac{u^4}{4} + C$$

$$= \frac{1}{8}(x^2+1)^4 + C$$

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{8}(x^2+1)^4 + C\Big|_0^2$$

$$= \frac{1}{8}.5^4 + \cancel{C} - \frac{1}{8} - \cancel{C}$$

$$= 78.$$

$$\int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x \, dx$$

let $u = \sin 2x \Rightarrow du = 2\cos 2x \, dx$

$\Rightarrow \frac{1}{2} du = \cos 2x \, dx$

If $x = 0 \Rightarrow u = \sin 2(0) = \sin(0) = 0$

If $x = \frac{\pi}{8} \Rightarrow u = \sin 2\left(\frac{\pi}{8}\right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Then

$$\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx = \int_0^{1/\sqrt{2}} u^5 . \left(\tfrac{1}{2} du\right)$$

$$= \frac{1}{2} \frac{u^6}{6} \Big|_0^{1/\sqrt{2}}$$

$$= \frac{1}{2}\left[\left(\frac{1}{\sqrt{2}}\right)^6 - 0\right] = \frac{1}{96}$$

$$\int_2^5 (2x-5)(x-3)\, dx$$

let $u = x - 3 \Rightarrow du = dx$

$x = u + 3$

If $x = 2 \Rightarrow u = 2 - 3 = -1$

If $x = 5 \Rightarrow u = 5 - 3 = 2$

The integral becomes:

$$\int_{-1}^{2} (2(u+3) - 5)\, u^9 \, du$$

$$= \int_{-1}^{2} (2u^{10} + u^9)\, du$$

$$= 2 . \frac{u^{11}}{11} + \frac{u^{10}}{10} \Bigg]_{-1}^{2}$$

$$= \frac{52233}{110}$$

$$\int_0^{3/4} \frac{dx}{1-x}$$

let $u = 1 - x \Rightarrow du = -dx \Rightarrow dx = -du$

If $x = 0 \Rightarrow u = 1$

$x = \frac{3}{4} \Rightarrow u = \frac{1}{4}$

The integral becomes:

$$\int_1^{1/4} \frac{-du}{u} = -\ln|u| \Bigg]_1^{1/4}$$

$$= -\ln\left|\tfrac{1}{4}\right| + \underbrace{\ln|1|}_{=0}$$

$$= -\ln \frac{1}{4}$$

$$\int_0^{\ln 3} e^x (1 + e^x)^{1/2} \, dx$$

Let $u = 1 + e^x \Rightarrow du = e^x dx$

If $x = 0 \Rightarrow u = 1 + e^0 = 1 + 1 = 2$

$x = \ln 3 \Rightarrow u = 1 + e^{\ln 3} = 1 + 3 = 4$

The integral becomes:

$$\int_2^4 u^{1/2} du = \frac{u^{3/2}}{3/2} \Bigg|_2^4$$

$$= \frac{2}{3}\left[4^{3/4} - 2^{3/2}\right]$$

$$= \frac{2}{3}\left[2^3 - (\sqrt{2})^3\right]$$

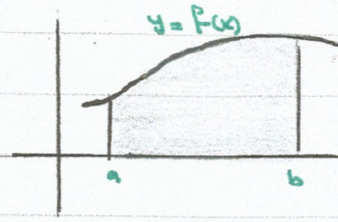
$$= \frac{2}{3}\left[8 - 2\sqrt{2}\right]$$

$$= \frac{16 - 4\sqrt{2}}{3}$$

Application of Definite integrals
" Area between curves "

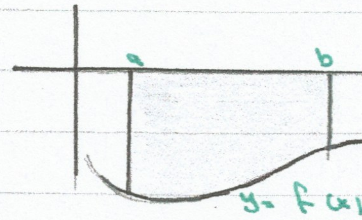
1- Area under a curve:

Area under a curve and above the x-axis.



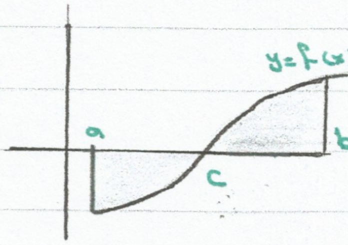
$$\text{Area} = \int_a^b f(x)\,dx$$

Area under a curve and below the x-axis



$$\text{Area} = \left| \int_a^b f(x)\,dx \right|$$

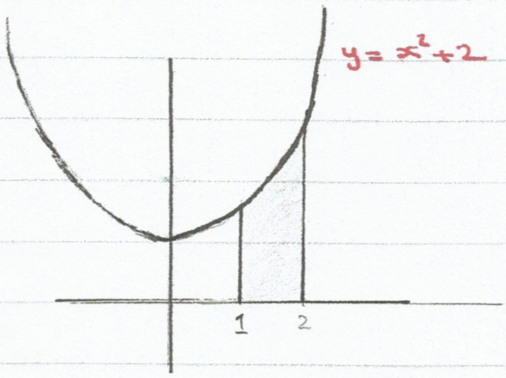
part of the curve is below the x-axis, part of it is the above the x-axis.



$$\text{Total Area} = \left| \int_a^c f(x)\,dx \right| + \int_c^b f(x)\,dx$$

Examples:~

1)- Find the area underneath the curve $y = x^2 + 2$ from $x = 1$ to $x = 2$.



$$\text{Area} = \int_a^b f(x)\,dx = \int_1^2 (x^2 + 2)\,dx$$

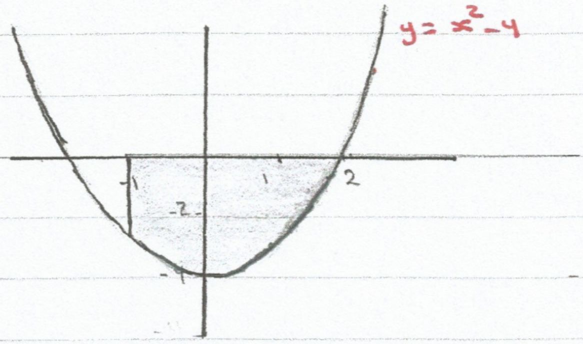
$$= \frac{x^3}{3} + 2x \Big|_1^2$$

$$= \left[\left(\frac{8}{3} + 4\right) - \left(\frac{1}{3} + 2\right) \right]$$

$$= \frac{13}{3} \text{ units}^2$$

2. Find the area bounded by $y = x^2 - 4$, the x-axis and the line $x = -1$ and $x = 2$

$$\text{Area} = \left| \int_a^b f(x)\,dx \right| = \int_{-1}^{2} (x^2 - 4)\,dx$$



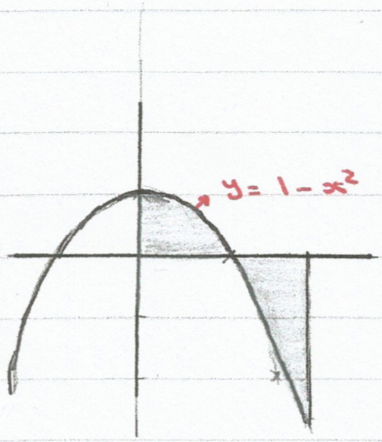
$$= \left| \left[\frac{x^3}{3} - 4x \right]_{-1}^{2} \right|$$

$$= \left| \left[\left(\frac{8}{3} - 8 \right) - \left(-\frac{1}{3} + 4 \right) \right] \right|$$

$$= |-9| = 9 \text{ units}^2$$

3. Find the area between the curve $y = 1 - x^2$ and the x-axis over the interval $[0, 2]$

$$\text{Total Area} = \int_0^1 (1 - x^2)\,dx + \int_1^2 |1 - x^2|\,dx$$



$$= \left[x - \frac{x^3}{3} \right]_0^1 + \left| \left[x - \frac{x^3}{3} \right]_1^2 \right|$$

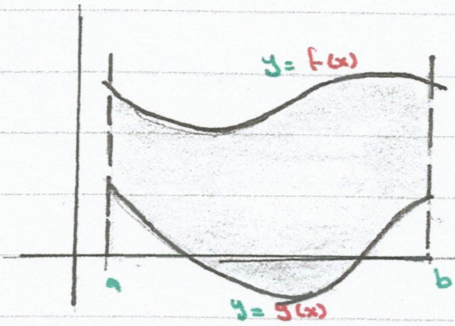
$$= \left(1 - \frac{1}{3}\right) + \left| \left(2 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) \right|$$

$$= \frac{2}{3} + \left| -\frac{2}{3} - \frac{2}{3} \right|$$

$$= \frac{2}{3} + \left| \frac{-4}{3} \right|$$

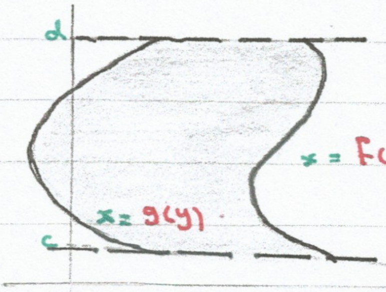
$$= \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2 \text{ units}^2.$$

2. Area between two curves:



$$\text{Area} = \int_a^b f(x) - g(x)\, dx$$

$$= \int_a^b \begin{pmatrix}\text{upper} \\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{lower} \\ \text{function}\end{pmatrix} dx$$

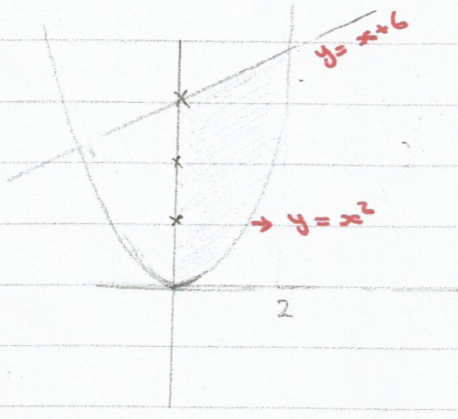


$$\text{Area} = \int_c^d f(y) - g(y)\, dy$$

$$= \int_c^d \begin{pmatrix}\text{right} \\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{left} \\ \text{function}\end{pmatrix} dy$$

Examples:-

1. Find the area of the region bounded above by $y = x +$, bounded below by $y = x^2$, and bounded of the sides by the lines $x = 0$ and $x = 2$.



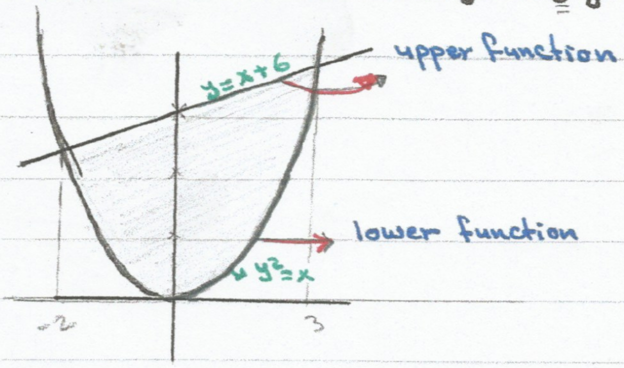
$$A = \int_0^2 (x+6) - (x^2)\, dx$$

$$= \int_0^2 x\, dx - \int_0^2 6\, dx - \int_0^2 x^2\, dx$$

$$= \frac{x^2}{2}\Big|_0^2 - 6x\Big|_0^2 - \frac{x^3}{3}\Big|_0^2$$

$$= \frac{34}{3}.$$

2. Find the area of the region that is enclosed between the curves $y = x^2$ & (and) $y = x + 6$



First we have to find the limits of integration:

$y = x^2 \rightarrow (1)$

$y = x + 6 \rightarrow (2)$

By substitute (1) in (2), we get

$x^2 = x + 6 \Rightarrow x^2 - x - 6 = 0$

$\therefore \Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x = 3 \text{ or } x = -2$

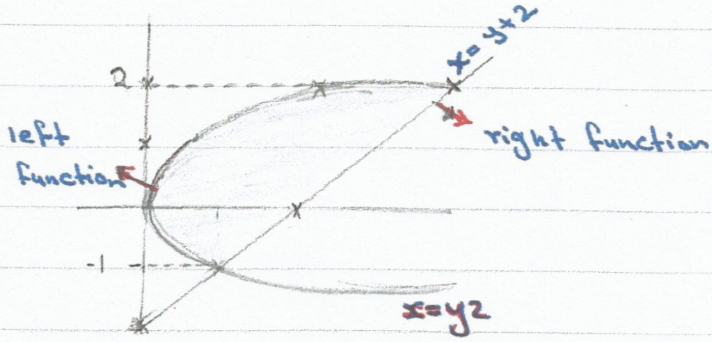
Thus:

$$A = \int_{-2}^{3} (x+6) - (x^2)\, dx$$

$$= \frac{x^2}{2} + 6x - \frac{x^3}{3} \Big|_{-2}^{3}$$

$$= \frac{125}{6}$$

3. Find the area of the region enclosed by $x = y^2$ and $y = x - 2$ integrating with respect to y.



First we have to find the limits of integration.

$y = x - 2 \rightarrow (1)$

$y^2 = x \rightarrow (2)$

by substitute (2) in (1), we get

$y = y^2 - 2 \Rightarrow y^2 - y - 2 = 0$

$\Rightarrow (y-2)(y+1) = 0$

$\Rightarrow y = 2 \text{ or } y = -1$

Thus:

$$A = \int_{-1}^{2} \left(\begin{matrix}\text{right} \\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{left} \\ \text{function}\end{matrix}\right) dy$$

$$= \int_{-1}^{2} (y+2) - y^2 \, dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_{-1}^{2}$$

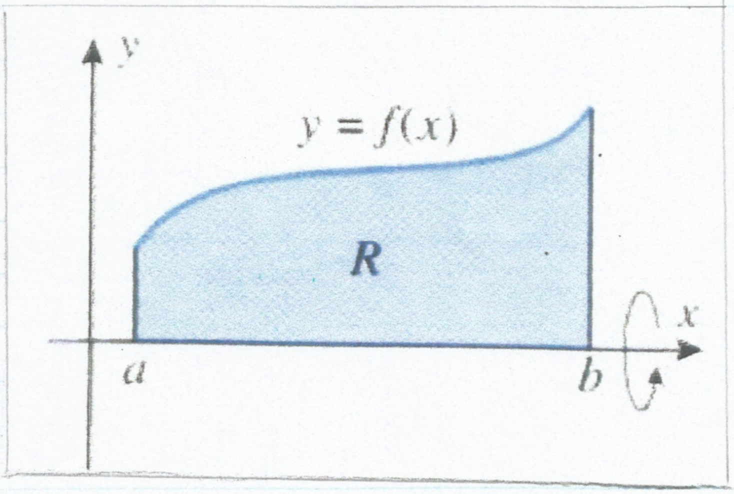
$$= \frac{9}{2}$$

Volumes by Integration.

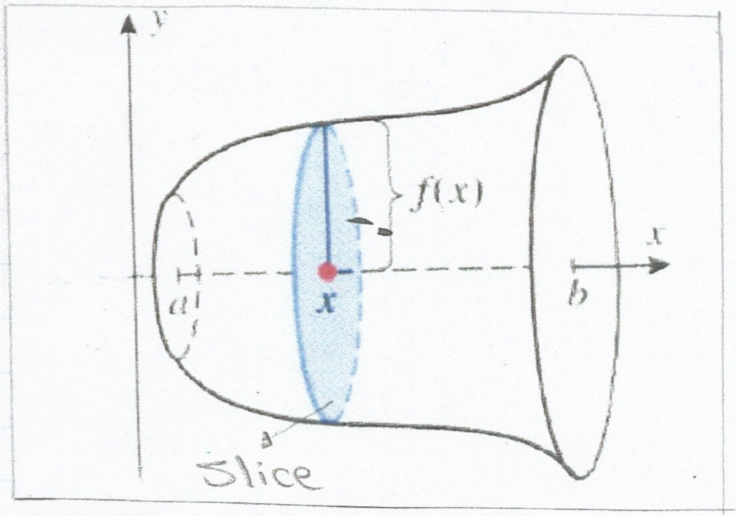
1- Finding volume of a solid of revolution using a disk method.

2- Finding volume of a solid of revolution using a washer method

Introduction:



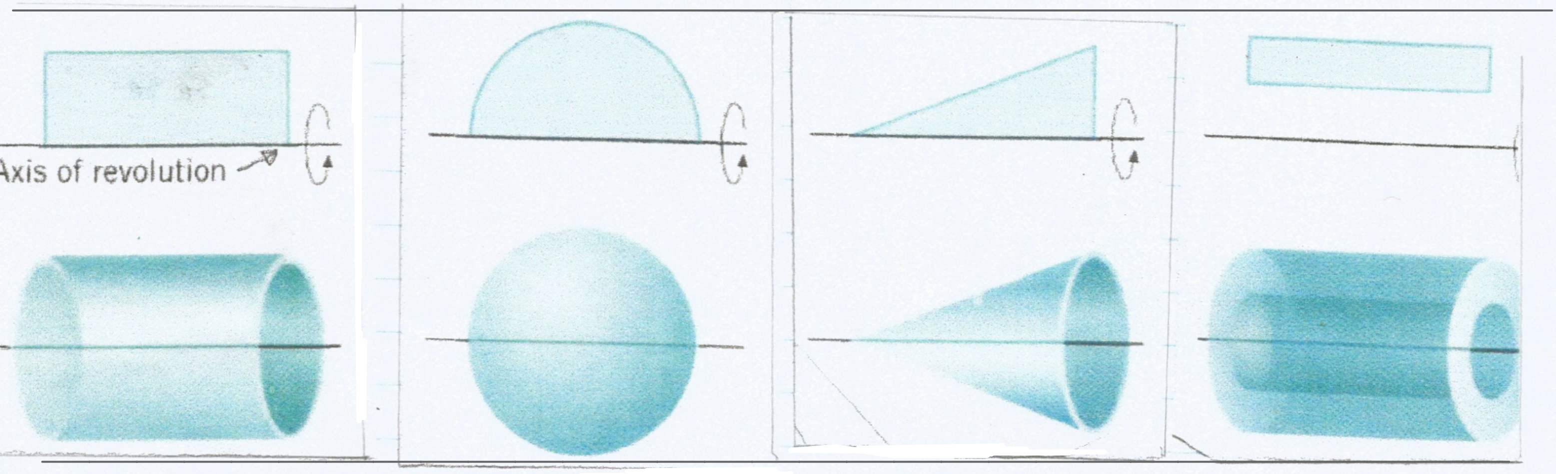
plane Region to be revolved about x-axis



Solid of Revolution

A solid of revolution is a solid that is generated by revolving a plan region a bout a line that lies in the same plan as the region. The line is called the axis of revolution.

Some familiar solids:



Volumes by disks prependicular To

The x-axis

$$V = \int_a^b \pi \, [f(x)]^2 \, dx$$

The y-axis

$$V = \int_a^b \pi \, [u(y)]^2 \, dy$$

Examples:

Example 1: Determine the volume of the solid of revolution formed by revolving the area enclosed by $2x^2 + 3$; the x-axis , $x=0$ and $x = 2$.

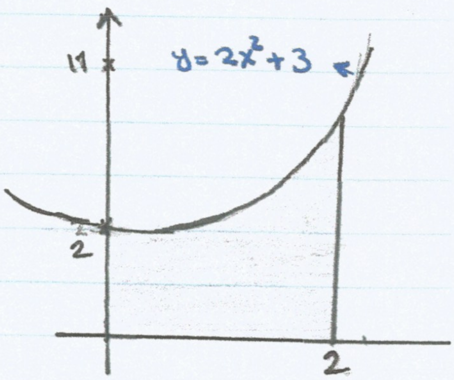
$$V = \int_0^2 \pi (2x^2+3)^2 \, dx$$

$$= \pi \int_0^2 (4x^4 + 12x^2 + 9) \, dx$$

$$= \pi \left[\frac{4x^5}{5} + \frac{12x^3}{3} + 9x \right]_0^2$$

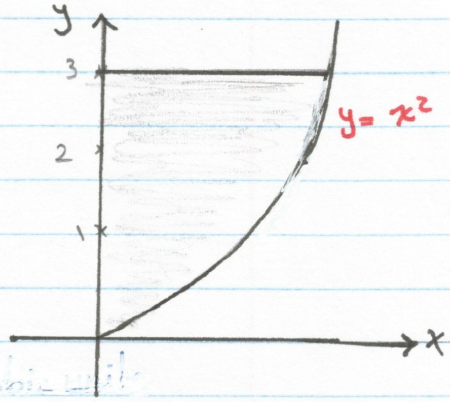
$$= \pi \left[\left(\frac{128}{5} + 32 + 18 \right) - (0) \right]$$

$$= \pi (25.6 + 32 + 18) = 75.6\pi \text{ cubic units.}$$



Example 2: compute the volume of the solid generated by revolving the plan. region bounded by $\boxed{y = x^2}$, $y = 1$, $y = 3$ about the y-axis.

$$V = \int_1^3 \pi \, [f(y)]^2 \, dy$$



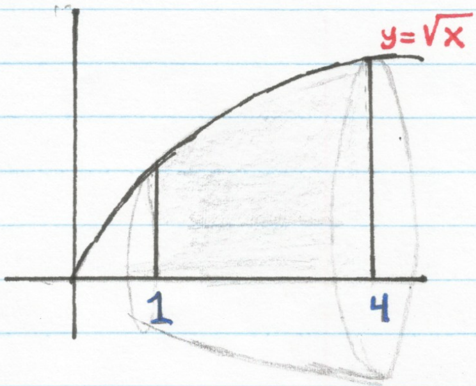
Since $y = x^2$, then $x = \sqrt{y}$

$$\text{Hence, } V = \int_1^3 \pi \, (\sqrt{y})^2 \, dy$$

$$= \pi \left[\frac{y^2}{2} \right]_1^3 = \pi \, [(4.5) - (0.5)]$$

$$= 4\pi \text{ cubic units}$$

Example 3: Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis



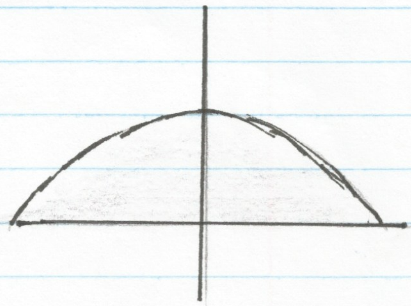
$$V = \int_1^4 \pi \, [f(x)]^2 \, dx$$

$$= \int_1^4 \pi \, (\sqrt{x})^2 \, dx = \int_1^4 \pi x \, dx$$

$$= \pi \int_1^4 x \, dx = \pi \left[\frac{x^2}{2} \right]_1^4$$

Example 4: Drive the formula of the volume of a siphere of radius r.

$$x^2 + y^2 = r^2 \Rightarrow y = \sqrt{r^2 - x^2}$$



$$V = \int_{-r}^{r} \pi \, (\sqrt{r^2 - x^2})^2 \, dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2) \, dx$$

$$= \pi \left[r^2 \int_{-r}^{r} dx - \int_{-r}^{r} x^2 \, dx \right]$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$$

$$= \pi \left[(r^2 \cdot r) - (r^2 \cdot (-r)) - \left(\frac{r^3}{3} - \frac{(-r)^3}{3} \right) \right]$$

$$= \pi \left[r^3 + r^3 - \frac{r^3}{3} - \frac{r^3}{3} \right]$$

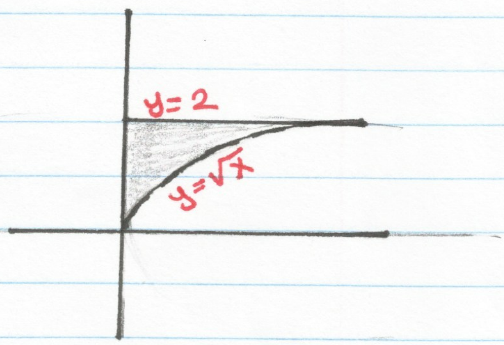
$$= \pi \left[2r^3 - \frac{2r^3}{3} \right]$$

$$= \pi \left[\frac{6r^3 - 2r^3}{3} \right]$$

$$= \pi \frac{4}{3} r^3 = \frac{4}{3} \pi r^3 .$$

Example 5: Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$ and $x = 0$ is revolved about the y- axis.

لابد الدالة المعطاة ع السؤال تكون على الشكل $x = f(y)$



$$y = \sqrt{x} \Rightarrow x = y^2$$

$$V = \int_0^2 \pi [u(y)]^2 \, dy$$

$$= \int_0^2 \pi y^4 \, dy$$

$$= \frac{\pi y^5}{5} \Big|_0^2 = \frac{32}{5} \pi$$

Volume by washer perpendicular to

The x-axis | The y-axis

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy$$

▶ **Example 1** Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

Solution:

$$V = \int_a^b \pi\,[f(x)]^2 - [g(x)]^2\,dx$$

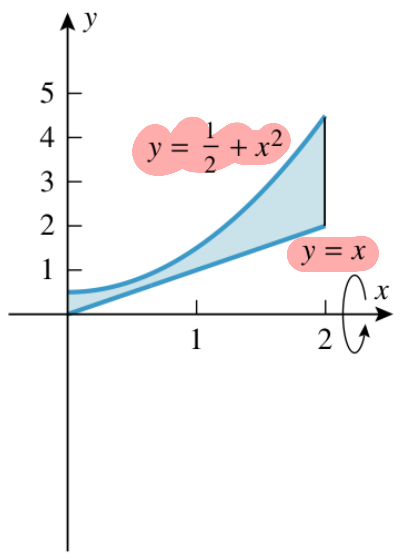
$$= \int_0^2 \pi\left[\tfrac{1}{2} + x^2\right]^2 - x^2\,dx$$

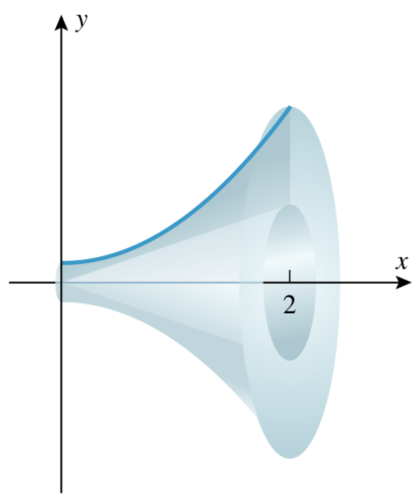
$$= \pi \int_0^2 \left(\tfrac{1}{4} + 2 \cdot \tfrac{1}{2} x^2 + x^4 - x^2\right) dx$$

$$= \pi \int_0^2 \left(\tfrac{1}{4} + \cancel{x^2} + x^4 - \cancel{x^2}\right) dx$$

$$= \pi \int_0^2 \left(\tfrac{1}{4} + x^4\right) dx$$

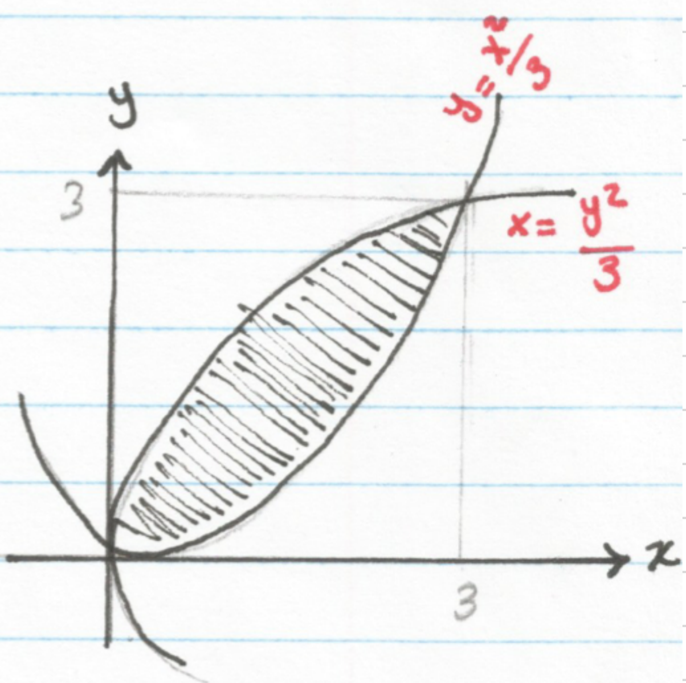
$$= \pi \left[\tfrac{1}{4}x + \frac{x^5}{5}\right]_0^2 = \frac{69}{10}\pi$$





Example 2: The area enclosed between the two curves $x^2 = 3y$ and $y^2 = 3x$ is rotated about the $\boxed{x\text{-axis}}$. Determine the volume of the solid formed.

Since $x^2 = 3y$, then $y = \frac{x^2}{3}$, and
$y^2 = 3x$, then $y = \sqrt{3x}$



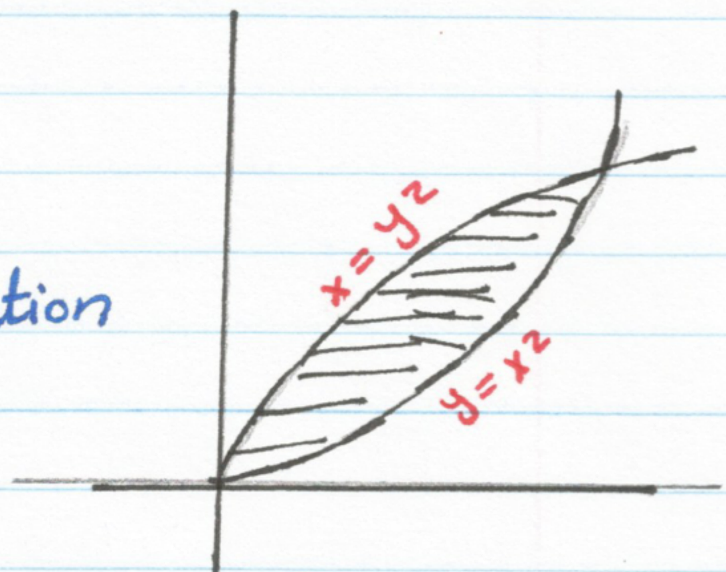
$$V = \int_0^3 \pi \left([f(x)]^2 - [g(x)]^2\right) dx$$

$$= \pi \int_0^3 \left(3x - \frac{x^4}{9}\right) dx = \pi \left[\frac{3x^2}{2} - \frac{x^5}{45}\right]_0^3$$

$$= \pi \left[\left(\frac{27}{2} - \frac{243}{45}\right) - (0)\right]$$

$$= \pi [13.5 - 5.4] = 8.1\pi$$

Example 3: Find the volume of the solid that results when the region enclosed by the given curve is revolved about the y-axis. $y = x^2$, $x = y^2$.



Since $y = x^2$, then $x = \sqrt{y}$

To find the limits, we solve the equation

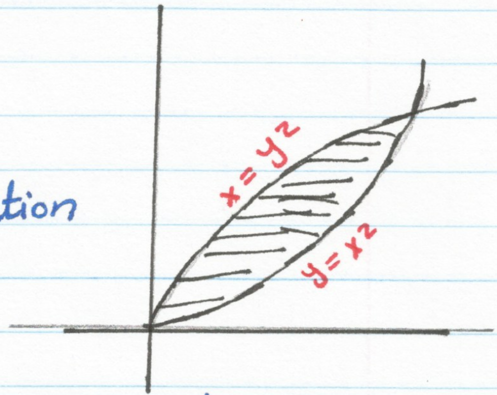
$$y^2 = \sqrt{y} \quad \text{or} \quad y^4 = y$$

The only solutions of this are $y = 0$ or $y = 1$, wich must therefor be the limits.

$$V = \int_0^1 \pi \left[[f(x)]^2 - [g(x)]^2\right] dy$$

$$= \pi \int_0^1 y - y^4 \, dy = \left[\frac{y^2}{2} - \frac{y^5}{5}\right]_0^1 = \frac{3\pi}{10}$$

Example 3: Find the volume of the solid that results when the region enclosed by the given curve is revolved about the y-axis. $y = x^2$, $x = y^2$.



Since $y = x^2$, then $x = \sqrt{y}$

To find the limits, we solve the equation

$$y^2 = \sqrt{y} \quad \text{or} \quad y^4 = y$$

The only solutions of this are $y=0$ or $y=1$, wich must therefor be the limits.

$$V = \int_0^1 \pi \left[[f(x)]^2 - [g(x)]^2\right] dy$$

$$= \pi \int_0^1 y - y^4 \, dy = \left[\frac{y^2}{2} - \frac{y^5}{5}\right]_0^1 = \frac{3\pi}{10}$$

Length of Plane Curve

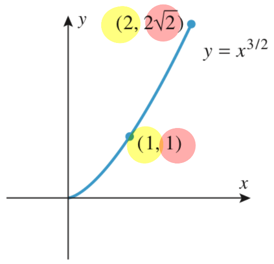
6.4.2 DEFINITION If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (3)$$

Moreover, for a curve expressed in the form $x = g(y)$, where g' is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ can be expressed as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad (5)$$

Example: Find the arc length of the curve $y = x^{3/2}$ from $(1,1)$ to $(2, 2\sqrt{2})$.



Method 1: $y = x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2} x^{1/2}$

$$L = \int_1^2 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{9}{4} x}\, dx$$

$$= \int_{13/4}^{11/2} \sqrt{u} \cdot \frac{4}{9}\, du$$

$$= \frac{4}{9} \int_{13/4}^{11/2} u^{1/2}\, du$$

$u = 1 + \frac{9}{4} x$

$du = \frac{9}{4} dx \Rightarrow dx = \frac{4}{9} du$

if $x = 1 \Rightarrow u = \frac{13}{4}$

$x = 2 \Rightarrow u = \frac{11}{2}$

$$= \frac{4}{9} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_{\frac{13}{4}}^{\frac{11}{2}}$$

$$= \frac{4}{9} \cdot \frac{2}{3} \left[\left(\frac{11}{2}\right)^{3/2} - \left(\frac{13}{4}\right)^{\frac{3}{2}} \right]$$

$$= \frac{8}{27} \left[\left(\sqrt{\frac{11}{2}}\right)^3 - \left(\frac{\sqrt{13}}{2}\right)^3 \right] \simeq 2.09.$$

Method 2:

$$y = x^{\frac{3}{2}} \Rightarrow x^{\frac{3}{2} \cdot \frac{2}{3}} = y^{\frac{2}{3}}$$

$$\Rightarrow x = y^{\frac{2}{3}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{2}{3} y^{-\frac{1}{3}}$$

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + \left(\frac{2}{3} y^{-1/3}\right)^2} \, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9} y^{-2/3}} \, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9 y^{2/3}}} \, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{\frac{9y^{2/3}+4}{9y^{2/3}}}\, dy$$

$$= \int_1^{2\sqrt{2}} \frac{\sqrt{9y^{2/3}+4}}{\sqrt{9\,y^{2/3}}}\, dy$$

$$= \int_1^{2\sqrt{2}} \frac{1}{3\,y^{1/3}} \sqrt{9\,y^{2/3}+4}\, dy$$

$$= \frac{1}{3} \int_1^{2\sqrt{2}} y^{-1/3} \sqrt{9y^{2/3}+4}\, dy$$

$$= \frac{1}{3} \int_{13}^{22} \sqrt{u} \cdot \frac{1}{6}\, dy$$

$$= \frac{1}{3} \cdot \frac{1}{6} \int_{13}^{22} u^{1/2}\, dy$$

$$= \frac{1}{18} \cdot \frac{2}{3} \left[(22)^{3/2} - (13)^{3/2} \right]$$

$=$

$$u = 9y^{2/3} + 4$$

$$du = 9 \cdot \frac{2}{3}\, y^{-1/3}\, dy$$

$$\Rightarrow y^{-1/3}\, dy = \frac{1}{6}\, du$$

if $y = 1 \Rightarrow u = 9(1)^{2/3} + 4 = 13$

$y = 2\sqrt{2} \Rightarrow u = 9(2\sqrt{2})^{2/3} + 4 = 22$

Area of a surface of revolution

6.5.2 DEFINITION If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as

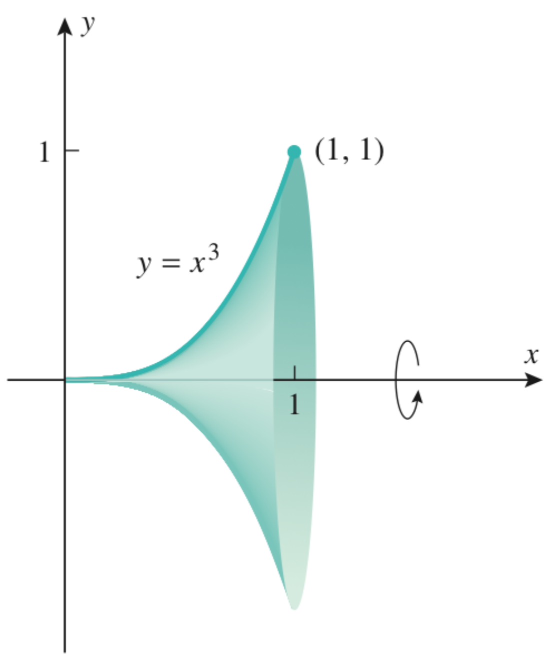
$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Moreover, if g is nonnegative and $x = g(y)$ is a smooth curve on the interval $[c, d]$, then the area of the surface that is generated by revolving the portion of a curve $x = g(y)$ between $y = c$ and $y = d$ about the y-axis can be expressed as

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

▶ **Example 1** Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.



$$S = \int_0^1 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 2\pi x^3\sqrt{1 + (3x^2)^2}\, dx$$

$$= 2\pi \int_0^1 x^3(1 + 9x^4)^{1/2}\, dx$$

$$= \frac{2\pi}{36}\int_1^{10} u^{1/2}\, du \qquad \boxed{\begin{aligned} u &= 1 + 9x^4 \\ du &= 36x^3\, dx \end{aligned}}$$

$$= \frac{2\pi}{36} \cdot \frac{2}{3}u^{3/2}\Big]_{u=1}^{10} = \frac{\pi}{27}(10^{3/2} - 1) \approx 3.56 ◀$$

• If $x = 0$
$u = 1 + 9(0)^4 = 1$
• If $x = 1$
$u = 1 + 9(1)^4 = 10$

▶ **Example 2** Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.

Solution: $y = x^2 \longrightarrow x = \sqrt{y}$

$x = 1 \rightarrow y = 1 \quad , \quad x = 2 \rightarrow y = 4$

$$\therefore S = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy .$$

$\because x = \sqrt{y} \quad \therefore \frac{dy}{dx} = \frac{1}{2\sqrt{y}}$

$$\therefore S = \int_1^4 2\pi \sqrt{y} \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}$$

Hyperbolic Function

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{Even}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{Odd}}$$

6.9.1 DEFINITION

Hyperbolic sine	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
Hyperbolic cosine	$\cosh x = \dfrac{e^x + e^{-x}}{2}$
Hyperbolic tangent	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
Hyperbolic cotangent	$\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
Hyperbolic secant	$\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$
Hyperbolic cosecant	$\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

Example:

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

$$\sinh 2 = \frac{e^2 - e^{-2}}{2} \approx 3.6269 \blacktriangleleft$$

Hyperbolic Identities

.9.2 THEOREM

$\cosh x + \sinh x = e^x$	$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
$\cosh x - \sinh x = e^{-x}$	$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
$\cosh^2 x - \sinh^2 x = 1$	$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\sinh 2x = 2 \sinh x \cosh x$
$\cosh(-x) = \cosh x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\sinh(-x) = -\sinh x$	$\cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$

Derivative and integral formulas for Hyperbolic functions

).3 **THEOREM**

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

► **Example 2**

$$\frac{d}{dx}[\cosh(x^3)] = \sinh(x^3) \cdot \frac{d}{dx}[x^3] = 3x^2 \sinh(x^3)$$

$$\frac{d}{dx}[\ln(\tanh x)] = \frac{1}{\tanh x} \cdot \frac{d}{dx}[\tanh x] = \frac{\operatorname{sech}^2 x}{\tanh x}$$ ◄

► **Example 3**

$$\int \sinh^5 x \cosh x\, dx = \tfrac{1}{6} \sinh^6 x + C \qquad \boxed{\begin{aligned} u &= \sinh x \\ du &= \cosh x\, dx \end{aligned}}$$

$$\begin{aligned} \int \tanh x\, dx &= \int \frac{\sinh x}{\cosh x} dx \\ &= \ln|\cosh x| + C \qquad \boxed{\begin{aligned} u &= \cosh x \\ du &= \sinh x\, dx \end{aligned}} \\ &= \ln(\cosh x) + C \end{aligned}$$

LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad \cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

▶ Example 5

$$\sinh^{-1} 1 = \ln(1 + \sqrt{1^2+1}) = \ln(1+\sqrt{2}) \approx 0.8814$$

$$\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\ln 3 \approx 0.5493 \quad ◀$$

■ DERIVATIVES AND INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \qquad \frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1 \qquad \frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1 \qquad \frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

6.9.6 THEOREM *If $a > 0$, then*

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \;\; \textit{or} \;\; \ln(u + \sqrt{u^2+a^2}) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \;\; \textit{or} \;\; \ln(u + \sqrt{u^2-a^2}) + C, \quad u > a$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & |u| < a \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & |u| > a \end{cases} \;\; \textit{or} \;\; \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C, \quad |u| \neq a$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C \;\; \textit{or} \;\; -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right) + C, \quad 0 < |u| < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \;\; \textit{or} \;\; -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right) + C, \quad u \neq 0$$

▶ **Example 6** Evaluate $\int \frac{dx}{\sqrt{4x^2-9}}, x > \frac{3}{2}$.

Solution:

$$\int \frac{dx}{\sqrt{4(x^2-9/4)}}$$

$$= \int \frac{dx}{\sqrt{4}\sqrt{x^2-(3/2)^2}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2-(\frac{3}{2})^2}}$$

$$= \frac{1}{2}\cosh^{-1}\frac{x}{3/2} + C$$

$$= \frac{1}{2}\cosh^{-1}\frac{2x}{3} + C$$

Integration by parts

Integration by parts is useful for:

[1] Integrating the product of more than one type of function. For example: $x \ln x$, $e^x \cos x$, $x^2 \sin x$, $\frac{\ln x}{x^5}$, ...

[2] Integrating some class of function such as:
$\ln x$, $\tan^{-1} x$, $\sin^{-1} x$, power of sec, ...

Formula:

$$\int u\,dv = u.v - \int v\,du$$

* Guidelines for selecting u and dv

L-I-A-T-E: choose u to be the function that comes first in the list:

L: Logrithmic function.
I: Inverse Trigonometric function.
A: Algebraic function
T: Trig. Function
E: Exponential function.

Examples: product of more than one type of function

$$\int x \cos x\,dx$$

Let $u = x$ $\quad dv = \cos x\,dx$
$du = dx$ $\quad v = \sin x$

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx$$

$$= x \sin x + \cos x + C$$

or:

Diff		Int
x	$+$	$\cos x$
1	$-$	$\sin x$
0		$-\cos x$

$$\int \cos x\,dx = x \sin x + \cos x + C$$

Comes first in the list

$\int x^2 e^{-x}\,dx$

$u = x^2$ $\quad dv = e^{-x}\,dx$

$du = 2x\,dx$ $\quad v = -e^{-x}$

$-x^2 e^{-x} + 2\int \underbrace{x e^{-x}}\,dx$ (1)

$u = x$ $\quad dv = e^{-x}\,dx$

$du = dx$ $\quad v = -e^{x}$

$= -xe^{-x} + \int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + C$ (2)

by substitution 2] in [1],

$= -x^2 e^{-x} + 2\left[-xe^{-x} - e^{-x} + C\right]$

$= -e^{-x}(x^2 + 2x + 2) + C$

or:

x^2	+	e^{-x}
$2x$	−	$-e^{-x}$
2	+	e^{-x}
0		e^{-x}

$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$= -e^{-x}(x^2 + 2x + 2) + C$

$\int t^2 \cos t\,dt$

$u = t^2$ $\quad dv = \cos t$

$du = 2t$ $\quad v = \sin t$

$= t^2 \sin t - 2\int \underbrace{t \sin t}\,dt$ (1)

$u = t$ $\quad dv = \sin t$

$du = dt$ $\quad v = -\cos t$

$= -t\cos t + \int \cos t\,dt$

$= -t\cos t + \sin t$ (2)

by substitution (2) in (1), we get

$t^2 \sin t + 2t\cos t - 2\sin t + C$

or:

t^2		$\cos t$
$2t$	+	$\sin t$
2	−	$-\cos t$
0	+	$-\sin t$

$t^2 \sin t - 2t\cos t - 2\sin t + C$

$$\int x e^{x} \, dx$$

Let $u = x$ $\quad dv = e^{x} dx$

$du = dx$ $\quad v = e^{x}$

$$\int x e^{x} dx = x \cdot e^{x} - \int e^{x} dx$$

$$= x \cdot e^{x} - e^{x} + C$$

Diff.		Int.
x	$+$	e^{x}
1	$-$	e^{x}
0		e^{x}

$$\int x e^{x} dx = x \cdot e^{x} - e^{x}$$

$$\int x^2 \sqrt{x-1} \, dx$$

في هذا المثال جميع الدالتين Algebraic
في هذه الحالة نختار الدوال كالآتي.
dv: تكون الدالة المعقدة.
و u: الدالة الأسهل

Let $u = x^2$ $\quad dv \; (x-1)^{1/2} dx$

$du = 2dx$ $\quad v = \frac{2}{3} (x-1)^{3/2}$

$$\int x^2 \sqrt{x-1} \, dx = \frac{2}{3} x^2 (x-1)^{3/2} - \frac{4}{3} \int \underbrace{x (x-1)^{3/2} dx}_{(1)}$$

* ملاحظة: هذا المثال تم حله في المحاضرة السابقة باستخدام التعويض.

$u = x$ $\quad dv = (x-1)^{3/2} dx$

$du = dx$ $\quad v = \frac{2}{5} (x-1)^{5/2}$

$$\int x (x-1)^{3/2} dx = \frac{2}{5} x (x-1)^{5/2} - \frac{2}{5} \int (x-1)^{5/2} dx$$

$$= \frac{2}{5} x (x-1)^{5/2} - \frac{2}{5} \times \frac{7}{2} (x-1)^{7/2} + C \quad (2)$$

by substitution (2) in (1). we get

$$\int x^2 \sqrt{x-1} \, dx = \frac{2}{3} x^2 (x-1)^{3/2} - \frac{4}{3} \left[\frac{2}{5} x (x-1)^{5/2} - \frac{4}{35} (x-1)^{7/2} \right] + C$$

$$= \frac{2}{3} x^2 (x-1)^{3/2} - \frac{8}{15} x (x-1)^{5/2} + \frac{16}{105} (x-1)^{7/2} + C$$

Diff		Int
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$2/3 \, (x-1)^{3/2}$
2	$+$	$4/15 \, (x-1)^{5/2}$
0		$8/105 \, (x-1)^{7/2}$

$$\int x^2 \sqrt{x-1} \, dx = \frac{2}{3} x^2 (x-1)^{3/2} - \frac{8}{15} x (x-1)^{5/2} + \frac{16}{105} (x-1)^{7/2} + C$$

$\int e^x \cos x \, dx$

$u = \cos x \qquad dv = e^x dx$

$du = -\sin x \, dx \qquad v = e^x$

$\int e^x \cos x \, dx = e^x \cos x + \underbrace{\int e^x \sin x \, dx}_{(1)}$

$u = \sin x \qquad dv = e^x dx$

$du = \cos x \, dx \qquad v = e^x$

$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$ (2)

By substitution (1) in (2) we get

$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$

$= 2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$

$= \int e^x \cos x \, dx = \frac{e^x \cos x}{2} + \frac{e^x \sin x}{2} + c$

$\int x^2 \ln x \, dx$

$u = \ln x \qquad dv = x^2 \, dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^3}{3}$

$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{\cancel{x^3}}{3} \cdot \frac{1}{\cancel{x}} \, dx$

$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \, dx$

$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C$

2. Eamples (special type of function):

$\int \sin^{-1} x \, dx$

$u = \sin^{-1} x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}} dx \qquad v = x$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \underbrace{\int \frac{x}{\sqrt{1-x^2}} dx}_{} \quad (1)$$

let $u = 1 - x^2 \qquad du = -2x \, dx$

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\frac{1}{2} \int u^{-1/2} du = -\sqrt{u} + C = -\sqrt{1-x^2} + C$$

plug this result back into the equation (1) to get

$$\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$$

$\int \ln x \, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

$$\int \ln x \, dx = \ln x . x - \int x . \frac{1}{x} dx$$
$$= x \ln x - x + C$$

$\int \ln^2 x \, dx$

$u = \ln^2 x \qquad dv = dx$

$du = \frac{2}{x} \ln x \, dx \qquad v = x$

$$\int \ln^2 x \, dx = x \ln^2 x - \int \cancel{x} \left(\frac{2}{\cancel{x}} \ln x \right) dx$$

$$= x \ln^2 x - 2 [x \ln x - x] + C$$

$$= x \ln^2 x - 2x \ln x - 2x + C.$$

$$\int_0^1 \tan^{-1} x \, dx$$

$u = \tan^{-1} x$ $\qquad$ $dv = dx$

$du = \dfrac{dx}{1+x^2}$ $\qquad$ $v = x$

$$= x\tan^{-1} \Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= x\tan^{-1} x \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= x\tan^{-1} x \Big|_0^1 - \frac{1}{2}\Big[\ln|1+x^2|\Big]_0^1$$

$$= \Big[1\left(\frac{\pi}{4}\right) - 0\Big] - \frac{1}{2}\Big[\ln|2| - \underbrace{\ln|1|}_{0}\Big]$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} - \ln 2^{1/2}$$

$$= \frac{\pi}{4} - \ln\sqrt{2}.$$

Ex: 8.2

3, 7, 11, 19, 55.

Integrating Power of Sines and Cosines

Reduction formula:

$$\boxed{\begin{aligned} \int \sin^n x\,dx &= -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n}\int \sin^{n-2} x\,dx \\ \int \cos^n x\,dx &= \frac{1}{n}\sin x \cos^{n-1} x + \frac{(n-1)}{n}\int \cos^{n-2} x\,dx \end{aligned}}$$

Evaluate $\int \cos^4 x\,dx$

we use the reduction formula (2). With $n=4$ in this formula we have

$$\int \cos^4 x\,dx = \frac{1}{4}\sin x \cos^3 x + \frac{3}{4}\int \cos^2 x\,dx$$

Then use the reduction formula again with $n=2$:

$$\int \cos^4 x\,dx = \frac{1}{4}\sin x \cos^3 x + \frac{3}{4}\left[\frac{1}{2}\sin x \cos x + \frac{1}{2}\int 1\,dx\right] \quad \cos^0 x = 1$$

$$= \frac{1}{4}\sin x \cos^3 x + \frac{3}{8}\sin x \cos x + \frac{3}{8}x + C$$

Evaluate $\int \sin^2 x\,dx$

we use the reduction formula (1). With $n=2$ in this formula we have

$$\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1\,dx \quad \text{since } \sin^0 x = 1$$

$$= -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$$

Ex: Find the volume V of the solid that is obtained when the region under the curve $y = \sin^2 x$ over the interval $[0, \pi]$ is revolved about the x-axis

$$V = \int \pi \sin^4 x\,dx = \pi \int \sin^4 dx$$

$$= \pi\left[\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right]_0^{\pi} = \frac{3}{8}\pi^2$$

Prove How!!

From reduction formula

Integrating products of powers of sines and cosines

1- Guide line of integrals of the form:

$$\int \sin^m x \cos^n x \, dx$$

$\int \sin^m x \cos^n x \, dx$	Procedure
m, n one odd / one even	- factor out one power from the trig. function that has odd pows - use $\cos^2 x + \sin^2 x = 1$ to transform the remaining even power of the trig. function - use u-substitution to finish the problem. Let u= other trig. function.
m, n Both odd	- choose one of the trig. function and factor out one power - continue as the first procedure
m, n Both even	- replace all even powers using the half angle identities: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$
m, n one or Both = 1	- Just use u-substitition. Let u= the other trig. function with power $\neq 1$. If: both = 1, choose ether.

Remark:

The methods in the above table can be applied if m=0 or n=0 to integrate the powers of sine and cosine.

m, n: one odd / one even

$$\int \sin^4 x \cos^5 x \, dx$$

$$= \int \sin^4 x \cos^4 x \cos x \, dx$$

$$= \int \sin^4 x \, (1-\sin^2 x)^4 \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$$= \int u^4 (1-u^2)^2 \, du$$

$$= \int u^4 (1 - 2u^2 + u^4 \, du)$$

$$= \int u^4 - 2u^6 + u^8 \, du$$

$$= \frac{1}{5} u^5 - \frac{2u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{1}{5} \sin^5 x - \frac{2}{7} \sin^7 x + \frac{1}{9} \sin^9 x + C$$

m, n: Both odd

$$\int \cos^3 x \sin^3 x \, dx$$

$$= \int \cos^3 x \sin^2 x \sin x \, dx$$

$$= \int \cos^3 x \, (1-\cos^2 x)^2 \sin x \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$

$$= -\int u^3 (1-u^2) \, du$$

$$= \int u^3 (u^2 - 1) \, du = \int u^5 - u^3 \, du$$

$$= \int u^5 \, du - \int u^3 \, du$$

$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$

$$= \frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x + C$$

m, n: Both even

$$\int \sin^4 x \cos^4 x \, dx$$

$$= \int (\sin^2 x)^2 (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1}{2}[1-\cos 2x]\right)^2 \left(\frac{1}{2}[1+\cos 2x]\right)^2 dx$$

$$= \frac{1}{16} \int (1-\cos 2x)^2 (1+\cos 2x)^2 \, dx$$

$$= \frac{1}{16} \int (1-\cos 2x)^2 \, dx = \frac{1}{16} \int \sin^4 (2x) \, dx$$

Let $u = 2x$

$du = 2x \, dx$

$\Rightarrow dx = \frac{du}{2}$

$$= \frac{1}{16} \int \sin^4 u \cdot \frac{du}{2} = \frac{1}{32} \int \sin^4 u \, du$$

$$= \frac{1}{32} \left(\frac{3}{8} u - \frac{1}{4} \sin 2u + \frac{1}{32} \sin 4u \right) + C$$

$$= \frac{1}{32} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x + C$$

m, n: one or both = 1

$$\int \cos^{10} x \sin x \, dx$$

Let $u = \cos x \Rightarrow du = -\sin x \, dx$

$$-\int u^{10} \, du = -\frac{1}{11} [u^{11}] + C$$

$$= -\frac{\cos x}{11} + C$$

Integration product of powers of Secant and Tangent.

* Guidelines of integral of the form:

$$\int \tan^m x \sec^n x \, dx$$

$\int \tan^m x \sec^n x \, dx$	General
m: the power of $\tan x$ odd	- Factor out the power of sec and one power of $\tan x$. - use $\tan^2 x = \sec^2 - 1$ to transform the remaning even power of $\tan x$. - use u-substitution to finish the problem. Let $u = \sec x \Rightarrow du = \sec x \tan x$.
n: The power of $\sec x$ even	- factor out $\sec^2 x$ - use $\sec^2 x = 1 + \tan^2 x$ to transform the remaning power of $\sec x$ to be in term of $\tan x$ - use u-substitution to finish the problem. Let $u = \tan x \Rightarrow du = \sec^2 x \, dx$
m: even n: odd	- use the relevant identities to convert $\tan x$ to $\sec x$. - use the reduction frmolu for power of $\sec x$.

m: (power of tan x) odd

$\int \tan^3 x \sec^3 x \, dx$

$= \int \tan^2 x \sec^2 x (\tan x \sec x) \, dx$

$= \int (\sec^2 x - 1) \sec^2 x (\tan x \sec x) \, dx$

$u = \sec x \Rightarrow du = \sec x \tan x \, dx$

$= \int (u^2 - 1) u^2 \, du$

$= \int u^4 - u^2 \, du$

$= \frac{u^5}{5} - \frac{u^3}{3} + C$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

m: (power of tan) even / n: (power of sec) odd.

$\int \tan^2 x \sec x \, dx$

$= \int (\sec^2 x - 1) \sec x \, dx$

$= \int \sec^3 x \, dx - \int \sec x \, dx$

$= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x$

$- \ln |\sec x + \tan x| + C$

$= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan$

$+ C.$

n: (power of sec) even

$\int \tan^2 x \sec^4 x \, dx$

$= \int \tan^2 x \sec^2 x \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$= \int \tan^2 x (\tan^2 x + 1) \sec^2 x \, dx$

$= \int u^2 (u^2 + 1) \, du$

$= \int u^4 + u^2 \, du$

$= \frac{u^5}{5} + \frac{u^3}{3} + C$

$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C$

For all other cases, there is no set method

$\int \tan x \, dx$

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

$u = \cos x \qquad du = -\sin x \, dx$. Thus

$\int \tan x \, dx = \int \frac{-1}{u} \, du$

$= -\ln |u| + c$

$= -\ln |\cos x| + c$

$= \ln |\cos x|^{-1} + c \qquad \frac{1}{\cos x} = \sec x$

$= \ln |\sec x| + c$

Evaluate: $\int \cos^3 x \, dx$

Since $n=3$ (odd), we can apply the second procedure in the table. Then we have

$$\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx$$

$$= \int (1-\sin^2 x) \cos x \, dx$$

Let $u = \sin x$ $\quad du = \cos x \, dx$. Thus

$$\int \cos^3 x \, dx = \int (1-u^2) \cdot du = u - \frac{1}{3}u^3 + C$$

$$= \sin x - \frac{1}{3}\sin^3 x + C.$$

Evaluate: $\int \cos^4 x \, dx$.

Since $n=4$ (even), we apply the third procedure in the table. Then we have:

$$\int \underbrace{\cos^4 x}_{(\cos^2 x)^2} \, dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2 dx \qquad \boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$$

$$= \int \left(\frac{1+2\cos(2x)+\cos^2(2x)}{4}\right) dx$$

$$= \frac{1}{4}\int \left(1+2\cos(2x)+\cos^2(2x)\right) dx \qquad \boxed{\cos^2 2x = \frac{1+\cos 4x}{2}}$$

$$= \frac{1}{4}\int \left(1+2\cos(2x)+\frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{4}\int \left(1+2\cos(2x)+\frac{1}{2}+\frac{\cos(4x)}{2}\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2}+2\cos(2x)+\frac{1}{2}\cos(4x)\right) dx \qquad \boxed{\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)+C}$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \cancel{2}\cdot\frac{1}{\cancel{2}}\sin 2x + \frac{1}{2}\cdot\frac{1}{4}\sin 4x\right) + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Integrating power of tangents and secants

Reduction formula:

$$\boxed{\begin{aligned} &\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx, \quad n \geq 2 \\ &\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\,\tan x + \frac{n-2}{n-1}\int \sec^{n-2}dx, \quad n \geq 2 \end{aligned}}$$

Example: Evaluate $\int \tan^3 x\,dx$

by formula (1) with $n=3$ we get:

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \int \tan x\,dx.$$

since: $\boxed{\int \tan x\,dx = \ln|\sec x| + c}$, we will have

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \ln|\sec x| + c$$

Evaluate $\int \sec^3 x\,dx$

by formula (2) with $n=3$ we get

$$\int \sec^3 dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\int \sec x\,dx$$

Since: $\boxed{\int \sec x\,dx = \ln|\sec x + \tan x| + C}$. we have

$$\int \sec^3 dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + c$$

3- Trigonometric identities that are useful for computing integrals:

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha-\beta) - \sin(\alpha+\beta)\right]$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right]$$

Example:

Evaluate: $\int \sin 7x \cos 3x$

$$= \frac{1}{2}\int \left(\sin(4x) + \sin(10x)\right) dx$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + C$$

$$\int \sin 3x \cos 5x \, dx$$

$$= \frac{1}{2}\int \left(\sin(-2x) + \sin(8x)\right) dx$$

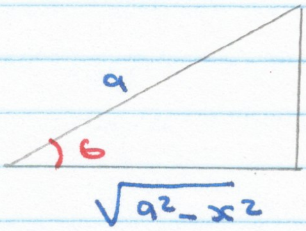
$$= \frac{1}{2}\int \left(\sin(8x) - \sin(2x)\right) dx$$

$$= \frac{1}{2}\left[-\frac{1}{8}\cos(8x) + \frac{1}{2}\cos(2x)\right] + C$$

$$= -\frac{1}{16}\cos(8x) + \frac{1}{4}\cos(2x) + C$$

Trigonometric Substituation

1. Sine Substitution: $\sqrt{a^2 - x^2}$



$x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$

Ex: Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$. Thus

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}$$

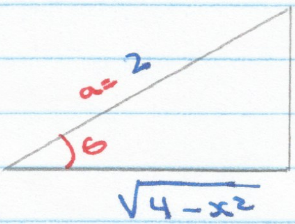
$$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4(1-\sin^2\theta)}} \quad \cos^2\theta$$

$$= \int \frac{\cancel{2\cos\theta}\, d\theta}{\sin^2\theta \cdot \cancel{2\cos\theta}}$$

$$= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

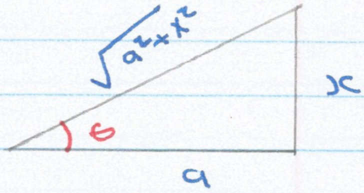
$\cot\theta = \frac{\sqrt{4-x^2}}{x}$. Thus we have



$\cot\theta =$ المجاور / المقابل

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$

2- Tangent substitution: $\sqrt{a^2+x^2}$



$x = a\tan\theta \Rightarrow dx = a\sec^2\theta \, d\theta.$

Ex: Find the arc length of the curve $y = \frac{x^2}{2}$ from $x=0$ to $x=1$

$$L = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1+x^2}\, dx$$

$x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta.$

if $x=0 \Rightarrow \theta = \tan^{-1}x = \tan^{-1}(0) = 0$

if $x=1 \Rightarrow \theta = \tan^{-1}(1) = \pi/4$

$$L = \int_0^1 \sqrt{1+x^2}\, dx = \int_0^{\pi/4} \sqrt{\underbrace{1+\tan^2\theta}_{=\sec^2\theta}}\ \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sqrt{\sec^2\theta}\cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} |\sec\theta| \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sec^3\theta\, d\theta$$

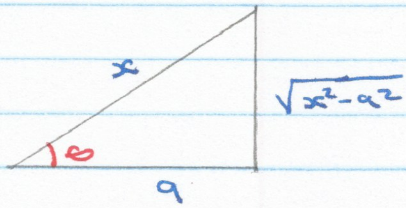
$\int \sec^3 x\, dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x$

From

$$= \frac{1}{2}\left[\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln(\sqrt{2}+1)\right] = 1.148.$$

3- Secant substitution: $\sqrt{x^2-a^2}$



$x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\, d\theta$

Ex: Evaluate: $\int \frac{\sqrt{x^2-25}}{x}\,dx$

$x = 5\sec\theta \Rightarrow dx = 5\sec\theta\tan\theta\, d\theta$. Thus

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}(5\sec\theta\tan\theta)\,d\theta$$

$$= \int \frac{5\,|\tan\theta|}{5\cancel{\sec\theta}}(5\cancel{\sec\theta}\tan\theta)\,d\theta$$

$$= 5\int \tan^2\theta\, d\theta$$

$$= 5\int (\sec^2\theta - 1)\,d\theta$$

$$= 5\tan\theta - 5\theta + C$$

$\tan\theta = \frac{\sqrt{x^2-25}}{5}$. and $x = 5\sec\theta \Rightarrow \sec\theta = \frac{x}{5} \Rightarrow \theta = \sec^{-1}\left(\frac{x}{5}\right)$

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = 5\,\frac{\sqrt{x^2-25}}{5} - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$

$$= \sqrt{x^2-25} - 5$$

Exc 8.4: 1, 7, 5, 17

Integrating rational functions by partial functions

- A rational function has the form :

$$f(x) = \frac{P(x)}{Q(x)}$$

where $p(x)$ and $Q(x)$ are polynomials. For example $f(x) = \dfrac{x^2-3}{x^4+3}$

Note:

If $\deg p(x) < \deg Q(x) \Rightarrow$ proper rational function

If $\deg p(x) \geq \deg Q(x) \Rightarrow$ Improper rational function.

A- **Integrating proper rational function**

Method:

Partial fraction decomposition (P.D.F).

How the method of P.F.D can be done?

we factor the denomenator $Q(x)$ as completly as possible. Then for each factor in $Q(x)$, we can use the following table to determine the term(s) we pick-up in the P.D.F.

	Term in $Q(x)$	From where	Term in P.F.D	Integration rule.
linear fa. factor rule	$(ax+b)$	real roots	$\dfrac{A}{ax+b}$	$\int \frac{1}{ax+b}dx = \frac{1}{a}.\ln\lvert ax+b\rvert + C$
	$(ax+b)^2$	double real roots	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \dots + \dfrac{A_n}{(ax+b)^n}$	$\int \frac{1}{(ax+b)^2}dx = -\frac{1}{a}.\frac{1}{ax+b} + C$
Quadratic factor rule	ax^2+bx+c	Im. roots	$\dfrac{Ax+B}{ax^2+bx+c}$	$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ $\int \frac{2ax+b}{ax^2+bx+c} = \ln\lvert ax^2+bx+c\rvert + C$

Case 1: $Q(x)$ is a quadratic with distinct real roots:

Evaluate: $\displaystyle\int \frac{dx}{x^2+x-2}$

$x^2+x-2=(x+2)(x-1) \Rightarrow$ by the linear facor rule (1)

$$\frac{1}{x^2+x+2} = \frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$= \frac{1}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$\Rightarrow 1 = A(x-1) + B(x+2)$$

Let $x=1 \Rightarrow 1 = 3B \Rightarrow B = \frac{1}{3}$

$x=-2 \Rightarrow 1 = -3A \Rightarrow A = -\frac{1}{3}$.

Thus

$$\int \frac{dx}{x^2+x+2} = -\frac{1}{3}\int \frac{dx}{x+2} + \frac{1}{3}\int \frac{dx}{x-1}$$

$$= -\frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

$$= \ln|x-1|^{1/3} - \ln|x+2|^{1/3} + C$$

$$= \ln\left|\frac{x-1}{x+2}\right|^{1/3} + C$$

Case 2: $Q(x)$ is a quadratic with double roots:

Evaluate: $\displaystyle\int \frac{3x-4}{x^2-4x+4}\,dx$

$x^4-4x+4=(x-2)(x-2) \Rightarrow$ by the Linear factor rule (2)

$$\frac{3x-4}{x^2-4x+4}=\frac{3x-4}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}$$

$$\Rightarrow 3x-4=A(x-2)+B$$

if we let $x=2 \Rightarrow 2=B$

if we let $x=3 \Rightarrow 5=A+B \Rightarrow 5=A+2 \Rightarrow A=3$

Thus

$$\int \frac{3x-4}{x^2-4x+4}\,dx = 3\int \frac{1}{x-2}\,dx + 2\int \frac{1}{(x-2)^2}\,dx$$

$$= 3\ln|x-2| + \frac{2}{x-2} + C$$

Evaluate: $\displaystyle\int \frac{4}{(x-2)^2}\,dx$

$$\frac{4x}{(x-2)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2} \Rightarrow \text{by the Linear factor rule (2)}$$

$$4x=A(x-2)+B$$

Let $x=2 \Rightarrow 8=B$

$x=0 \Rightarrow 0=-2A+B \Rightarrow 0=-2A+8 \Rightarrow A=4$

Thus

$$\int \frac{4x}{(x-2)^2}\,dx = 4\int \frac{1}{x-2}\,dx + 8\int \frac{1}{(x-2)^2}\,dx$$

$$= 4\ln|x-2| + \frac{8}{x-2} + C$$

Case 3: $Q(x)$ is a quadratic with imagenary roots:

Evaluate: $\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx$

$$3x^3-x^2+3x-1 = x^2(3x-1)+(3x-1)$$

$$= (3x-1)(x^2+1) \Rightarrow$$ Linear factor rule (1) + quadratic factor

$$\frac{x^2+x-2}{3x^3-x^2+3x+1} = \frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow x^2+x-2 = A(x^2+1) + (Bx+C)\underset{x-1/3}{(3x-1)}$$

Let $x = \frac{1}{3} \Rightarrow \frac{1}{9}+\frac{1}{3}-2 = A\left(\frac{1}{9}+1\right) \Rightarrow \frac{-14}{9} = \frac{10}{9}A \Rightarrow \boxed{A = -7/5.}$

x^2: $1 = A+3B \Rightarrow 1 = -\frac{7}{5}+3B \Rightarrow B = \frac{4}{5}$

x^0: $-2 = A-C \Rightarrow -2 = -7/5 - C \Rightarrow C = \frac{3}{5}.$

Thus:

$$\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx = \frac{-7}{5}\int\frac{dx}{3x-1} + \frac{4}{5}\int\frac{x}{x^2+1}\,dx + \frac{3}{5}\int\frac{dx}{x^2+1}$$

$$= -\frac{7}{5}\cdot\frac{1}{3}\int\frac{3\,dx}{3x-1} + \frac{4}{5}\cdot\frac{1}{2}\int\frac{2x\,dx}{x^2+1} + \frac{3}{5}\int\frac{dx}{x^2+1}$$

$$= \frac{-7}{15}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + C$$

B- **Integrating Improper rational function:**
Method: long division.

If $f(x) = \frac{P(x)}{Q(x)}$ is a rational function where $P(x)$ and $Q(x)$ are polynomial functions and deg $P(x)$ is greater than or equal to the deg of $Q(x)$, then by long division:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$ proper rational function

Ex: Evaluate: $\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\,dx$

$$f(x) = \frac{3x^4 + 3x^3 - 5x^2 + x + 1}{x^2 + x - 2} = (3x^2 + 1) + \frac{1}{x^2 + x - 2}$$

Ex1

Thus: $\int f(x)\,dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$

$$\int f(x)\,dx = x^3 + x + \underbrace{\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C}_{\text{result from Ex1}}$$

$$3x^2 + 1 = S(x)$$

$$\begin{array}{r|l} x^2+x-2 & 3x^4+3x^3-5x^2+x-1 \\ & \underline{3x^4+3x^3-6x^2} \\ & x^2+x-1 \\ & \underline{x^2+x-2} \\ & 1 = R(x) \end{array}$$

Ex: Evaluate $\int \frac{2x^2 + 4x - 7}{x^2 + x - 6}\,dx$

$$\frac{2x^2 + 4x - 7}{x^2 + x - 6} = 2 + \frac{2x + 5}{(x+3)(x-2)}\,dx$$

Thus:

$$\int f(x)\,dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx$$

$$= \int f(x)\,dx = \int 2\,dx + \boxed{\int \frac{2x+5}{(x+3)(x-2)}}\,dx$$

$$\frac{2x+5}{(x+3)(x-2)} = \frac{A}{(x+3)} + \frac{B}{(x-2)}$$

$$2 = S(x)$$

$$\begin{array}{r|l} Q(x) = x^2+x-6 & 2x^2+4x-7 \\ & \underline{-\ 2x^2 \mp 2x \pm 12} \\ R(x) & = \boxed{2x+5} \end{array}$$

$$2x+5 = A(x-2) + B(x+3)$$

If $x = 2 \Rightarrow 9 = 5B \Rightarrow B = \frac{9}{5}$

If $x = -3 \Rightarrow -1 = -5A \Rightarrow A = \frac{1}{5}$

Thus:

$$\int \frac{2x+5}{(x+3)(x-2)} = \int \frac{A}{(x+3)} dx + \int \frac{B}{(x-2)} dx$$

$$= \frac{1}{5} \int \frac{1}{x+3} dx + \frac{9}{5} \int \frac{1}{x-2} dx$$

$$= \frac{1}{5} \ln|x+3| + \frac{9}{5} \ln|x-2| + C$$

Hence

$$\int f(x)\,dx = 2x + \frac{1}{5} \ln|x+3| + \frac{9}{5} \ln|x-2| + C$$

Remarks:

There are some cases in which the method of P.F.D is inappropriate. For example,

$$\int \frac{2x-1}{x^2-x-6} dx$$

The numerator is the derivative of the denomenator. we can solve it by u-substituation

let $u = x^2 - x - 6 \Rightarrow du = 2x-1\,dx$

Thus:

$$\int \frac{2x-1}{x^2-x-6} dx = \int \frac{1}{u} du = \ln|u| + C$$

$$= \ln|x^2 - x - 6| + C$$

$$\int \frac{x-1}{x^2+1} dx$$

$$= \int \frac{x}{x^2+1} dx - \int \frac{1}{x^2+1} dx$$

$$= \frac{1}{2} \int \frac{2x}{x^2+1} dx - \int \frac{1}{x^2+1} dx$$

$$= \ln|x^2+1| - \tan^{-1}(x) + C$$

H.W: Ex: 8.5

5, 13, 25, 3, 9, 17

Improper Integral.

1)- Type A or Infinity:

$$\int_a^{\infty} f(x)\,dx \quad , \quad \int_{-\infty}^{b} f(x)\,dx \quad , \quad \int_{-\infty}^{\infty} f(x)\,dx$$

Definition:

1- The improper integrard $\int_a^{\infty} f(x)\,dx$ is defined as

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx$$

2- The improper integral $\int_{-\infty}^{b} f(x)\,dx$ is defined as

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^{b} f(x)\,dx$$

3- The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Remarks:

1- The improper integrals $\int_a^{\infty} f(x)\,dx$ and $\int_{-\infty}^{b} f(x)\,dx$ are called convergent if the limit exsit and divergent if the limit does n't exsist.

2- We call the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ defined in (3) convergent if both terms converge and diverge if either term diverges.

Examples:

1. Evaluate $\int_1^{\infty} \frac{1}{x}\,dx$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} (\ln t - \underbrace{\ln 1}_{=0})$$

$$= \lim_{t\to\infty} \ln t = \infty$$

The limit does not exsit so the improper integral $\int_1^{\infty} \frac{1}{x}\,dx$ is divergent

2. Evaluate $\int_1^{\infty} \frac{1}{x^3}\,dx$

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^{t} x^{-3}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^t = -\frac{1}{2} \lim_{t\to\infty} \left[\frac{1}{x^2}\right]_1^t$$

$$= -\frac{1}{2} \lim_{t\to\infty} \left[\frac{1}{t^2} - 1\right]$$

$$= -\frac{1}{2}[0-1] = \frac{1}{2}$$

The limit exsit so the improper integral $\int_1^{\infty} \frac{1}{x^3}\,dx$ is convergent.

3. Evaluate $\int_{-\infty}^{0} e^{x}\,dx$

$$\int_{-\infty}^{0} e^{x}\,dx = \lim_{t\to-\infty} \int_t^0 e^{x}\,dx$$

$$= \lim_{t\to-\infty} \left[e^{x}\right]_t^0$$

$e^{-\infty} = 0$

$$= \lim_{t\to-\infty} (e^{0} - e^{t})$$

$$= (1-0) = 1 \qquad \text{(convergent)}$$

4. Evaluate: $\int_0^{\infty} (1-x)\, e^{-x}\, dx$

$$\int_0^{\infty} (1-x)\, e^{-x}\, dx = \lim_{t\to\infty} \int_0^{t} \underbrace{(1-x)\, e^{-x}}_{\text{حاصل ضرب دالتین}}\, dx$$

By LIATE we choose

$u = 1-x \qquad dv = e^{-x}dx$

$du = -dx \qquad v = -e^{-x}$

Thus
$$\int (1-x)\, e^{-x}\, dx = (1-x)(-e^{-x}) - \int (-e^{-x})(-dx)$$
$$= (x-1)(e^{-x}) + e^{-x} + C$$
$$= x e^{-x} - e^{-x} + e^{-x} + C$$
$$= x e^{-x} + C$$

Note that:
$$\lim_{x\to\infty} (x e^{-x}) = \lim_{x\to\infty} \left(\frac{x}{e^{x}}\right) = \frac{\infty}{e^{\infty}} = \frac{\infty}{\infty}$$ کمیه غیرمعرفه

Thus by L' Hospital's Rule we have
$$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^{x}}$$
$$= \lim_{x\to\infty} \frac{x'}{(e^{x})'}$$
$$= \lim_{x\to\infty} \frac{1}{e^{x}} = 0$$

$$\int_0^{\infty} (1-x)\, e^{-x}\, dx = \lim_{t\to\infty} \int_0^{t} (1-x)\, e^{-x}\, dx = \lim_{t\to\infty} x e^{-x} \Big]_0^{t}$$
$$= \lim_{t\to\infty} (t e^{-t} - 0 \cdot e^{0})$$
$$= 0 - 0 = 0 \quad \text{(convergent)}$$

5. Evaluate: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{c} \frac{dx}{1+x^2} + \int_{c}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{t \to -\infty} \int_{t}^{0} \frac{dx}{1+x^2} + \lim_{t \to \infty} \int_{0}^{t} \frac{dx}{1+x^2}$$

$$= \lim_{t \to -\infty} \tan^{-1} x \Big|_{t}^{0} + \lim_{t \to \infty} \tan^{-1} x \Big|_{0}^{t}$$

$$= \lim_{t \to -\infty} [\tan^{-1}(0) - \tan^{-1}(t)] + \lim_{t \to \infty} [\tan^{-1}(t) - \tan^{-1}(0)]$$

$$= \lim_{t \to} [0 - \tan^{-1}(t)] + \lim_{t \to \infty} [\tan^{-1}(t) - 0]$$

$$= - \lim_{t \to -\infty} [\tan^{-1}(t)] + \lim_{t \to \infty} [\tan^{-1}(t)]$$

$$= - [\tan^{-1}(-\infty)] + [\tan^{-1}(\infty)]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = 2\frac{\pi}{2} = \pi \quad \text{(convergent).}$$

2]- Type B or interval a, b.

$\int_{a}^{b} f(x)\, dx$: discontinuous at a or b.

$\int_{a}^{b} f(x)\, dx$: dis continuity some where between a and b

Definition:

1)- If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_{a}^{b} f(x)\,dx = \lim_{t \to b^-} \int_{a}^{t} f(x)\, dx$$

2)- If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_{a}^{b} f(x)\, dx = \lim_{t \to a^+} \int_{t}^{b} f(x)\, dx.$$

3- If f has a discontinuity at c, where $a<c<b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Examples:

1- Evaluate $\int_1^2 \frac{dx}{1-x}$

This function is continuous on $(1, 2]$ and is discontinuous at $a=1$

Then:

$$\int_1^2 \frac{dx}{1-x} = \lim_{t\to 1^+} \int_t^2 \frac{dx}{1-x}$$

$$= \lim_{t\to 1^+} \left[-\ln|1-x|\right]_t^2$$

$$= -\lim_{t\to 1^+} \left[\overset{\ln|-1|}{\ln|1-2|} - \ln|1-t|\right]$$

$$= -\lim_{t\to 1^+} \left[\ln|1| - \ln|1-t|\right]$$

$$= -(\ln|1| - \ln(0))$$

$$= -(0-(-\infty)) = -\infty \quad \text{(divergent).}$$

2- Evaluate: $\int_0^1 \frac{dx}{1-x}$

This function is continuous on $[0,1)$ and is discontinuous at $b=1$

Thus

$$\int_0^1 \frac{dx}{1-x} = \lim_{t\to 1^-} \int_0^t \frac{dx}{1-x}$$

$$= -\lim_{t\to 1^-} \left[\ln|1-x|\right]_0^t$$

$$= -\lim_{t\to 1^-} \left[\ln|1-t| - \overset{\text{zero}}{\cancel{\ln|1|}}\right]$$

$$= -[\ln(0)] = -(-\infty) = \infty \quad \text{(divergent).}$$

3- Evaluate: $\int_{-2}^{2} \frac{1}{x^{2/3}}\, dx$

This function is discontinuous at 0, where $-2 < 0 < 2$. Thus

$$\int_{-2}^{2} \frac{1}{x^{2/3}}\, dx = \lim_{t \to 0} \int_{-2}^{t} \frac{1}{x^{2/3}}\, dx + \lim_{t \to 2} \int_{t}^{2} \frac{1}{x^{3/2}}\, dx$$

$$= \lim_{t \to 0} 3x^{1/3}\Big]_{-2}^{t} + \lim_{t \to 0} 3x^{1/3}\Big]_{t}^{2}$$

$$= \lim_{t \to 0} \left[3t^{1/3} - 3(-2)^{1/3}\right] + \lim_{t \to 0} \left[3(2)^{1/3} - 3t^{1/3}\right]$$

$$= 0 - 3(-2)^{1/3} + 3(2)^{1/3} - 0$$

$$= 3(2)^{1/3} + 3(2)^{1/3} = 6.(2)^{1/3} \qquad \text{(convergent)}$$

Infinite series.

Sequences: An infinite sequance, or more simply sequance is an unending succession of numbers, called terms. Sequance can be written as:

$a_1, a_2, \dots, a_n, \dots = \{a_n\}_{n=1}^{\infty}$

$\underset{\text{first term}}{a_1}$ $\underset{\text{n th term (general term)}}{a_n}$

n is called the index for the sequance.

For example:

$1, 2, 3, 4, \dots$

$2, 4, 6, 8, \dots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

Example (1): In each part, find the general term of the sequance:

a)- $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

$$a_n = \frac{n}{n+1}$$

b)- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}$

$$a_n = \frac{1}{2^n}$$

c)- $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}$

$$a_n = (-1)^{n+1}\frac{n}{n+1}$$

d) $1, 3, 5, 7, \dots$

$a_n = 2n-1$

Remark:-

It is not essential to start the index at 1, sometimes it is more convenient to start it at 0. For example

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots = \left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2^n}\right\}_{n=0}^{\infty}$$

Definition: (limit of a sequance). Let $\{a_n\}$ be a sequance of real numbers. We say that the sequance $\{a_n\}$ converges to the real number a or tends to a and we write

$$a = \lim_{n\to\infty} a_n \quad \text{or simply} \quad a = \lim a_n$$

If for every $\epsilon > 0$, there is an integer N s.t

$$|a_n - a| < \epsilon \quad \text{whenever } n \geq N.$$

In this case we call the number a a limit of the sequance $\{a_n\}$.

- We say that the sequance $\{a_n\}$ converges if it converges to some number a. A sequance divergent if it does not converge to any number.

Theorem:-

Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ and k is some constant.

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n = kL$$

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = LM$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}, \text{ if } M \text{ is not } 0$$

■

Example: In each part, Determine whether the sequance convergence or diverges. If it converges find the limit

a) $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{n/n}{\frac{2n}{n}+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2+\frac{1}{\infty}} = \frac{1}{2+0} = \frac{1}{2}$$

Thus the sequance convergent to $\frac{1}{2}$.

b). $\left\{(-1)^{n+1} \frac{n}{2n+1}\right\}_{n=1}^{\infty}$

n odd

$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$

from above example

we got This result from the above example

n even

$\lim_{n\to\infty} (-1) \frac{n}{2n+1}$

$= \lim_{n\to\infty} (-1) \frac{n/n}{2n/n + 1/n}$

$= \lim_{n\to\infty} (-1) \frac{1}{2+\frac{1}{n}} = -\frac{1}{2}$

$$\therefore \lim_{n\to\infty} \frac{n}{2n+1} \neq \lim_{n\to\infty} (-1)\frac{n}{2n+1} \quad \text{(diverges)}$$

c). $\left\{(-1)^{n+1} \frac{1}{n}\right\}_{n=1}^{\infty}$

n odd

$\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$

n even

$\lim_{n\to\infty} (-1) \frac{1}{n} = -1 \cdot \frac{1}{\infty} = -1 \cdot 0 = 0$

Thus the sequance converges to 0.

d)- $\{8-2n\}_{n=1}^{\infty}$

$\lim\limits_{n\to\infty}(8-2n) = 8-2(\infty) = 8-\infty = -\infty$. The sequance diverges.

e) $\{1, \frac{1}{2}, \frac{1}{2^2}, \cdots \frac{1}{2^n}, \ldots\}$

$\lim\limits_{n\to\infty} \frac{1}{2^n} = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$. The sequance convergent to zero.

f)- $\{1, 2, 2^2, \cdots 2^n\}$

$\lim\limits_{n\to\infty} 2^n = 2^\infty = \infty$. The sequance diverges.

Example: Find the limit of the sequance $\{\frac{n}{e^n}\}_{n=1}^{\infty}$

$\lim\limits_{n\to\infty} \frac{n}{e^n} = \frac{\infty}{e^\infty} = \frac{\infty}{\infty}$ كمية غير معرفة

Using LHopital Rule:

$\lim\limits_{n\to\infty} \frac{n}{e^n} = \lim\limits_{n\to\infty} \frac{1}{e^n} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

Example: show that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

$\lim\limits_{n\to\infty} \sqrt[n]{n} = \lim\limits_{n\to\infty} (n)^{1/n} = (\infty)^{1/\infty} = (\infty)^0$

Using LHopital Rule:

$\lim\limits_{n\to\infty} \sqrt[n]{n} = \lim\limits_{n\to\infty} (n)^{1/n} = \lim\limits_{n\to\infty} e^{\ln n^{1/n}}$

$= \lim\limits_{n\to\infty} e^{\frac{1}{n}\ln n} \to$ خصائص الـ ln

$= \lim\limits_{n\to\infty} e^{\frac{\ln n}{n}}$

مشتقة

$= \lim\limits_{n\to\infty} e^{\frac{1/n}{1}}$

$= e^{1/\infty} = e^0 = 1$

Theorem: A sequance converges to a limit L iff the sequance of even numbered term and odd numbered term both converge to L.

Determine whether the sequence converges or diverges. If it converges . find the limit.

1)- $\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots$

odd numbered terms

$\lim_{n\to\infty} \left(\frac{1}{2^n}\right) = \frac{1}{2^\infty} = \frac{1}{\infty} = 0$

even numbered terms

$\lim_{n\to\infty} \left(\frac{1}{3^n}\right) = \frac{1}{3^\infty} = \frac{1}{\infty} = 0$

Thus the given sequance converges to zero.

2)- $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}$

odd numbered terms

$\lim_{n\to\infty} \{1\} = 1$

$\neq$

even-numbered terms

$\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$

The given sequance diverges.

Theorems-

If $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

Ex : $\left\{(-1)^n \frac{1}{2^n}\right\}_{n=0}^{\infty}$

$\left\{(-1)^n \frac{1}{2^n}\right\}_{n=0}^{\infty} = 1, -\frac{1}{2}, \frac{1}{2^2}, \ldots (-1)^n \frac{1}{2^n}, \ldots$

If we take the absolute value of each term, we obtain the sequance

$1, \frac{1}{2}, \frac{1}{2^2}, \ldots \frac{1}{2^n}$

Then:

$$\lim_{n\to\infty} \left| (-1)^n \frac{1}{2^n} \right| = \lim_{n\to\infty} \frac{1}{2^n} = \frac{1}{\infty} = 0$$

Therefore:

$$\lim_{n\to\infty} \left\{ (-1)^n \frac{1}{2^n} \right\} = 0 .$$

H.w {10.1 , 7, 23 , P.633 , 634}

Montone Sequances

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is called:

strictly increasing if : $a_1 < a_2 < a_3 < \dots < a_n < \dots$

increasing if : $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots$

strictly decreasing if : $a_1 > a_2 > a_3 > \dots > a_n > \dots$

decreasing if : $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$

- A sequance that is either increasing or decreasing is said to be monotone.

- A sequance that is either strictly increasing or strictly decreasing is said to be strictly monotone.

Ex:

Sequances	Description
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \frac{n}{n+1}, \dots$	strictly increasing
$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$	strictly decreasing
$1, 1, 2, 2, 3, 3, \dots$	increasing
$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \dots$	decreasing
$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots (-1)^{n+1}\frac{1}{n}, \dots$	Neither increasing nor decreasing

Testing for monotonicity: $a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$

Difference test	Ratio test	Derivative test	Description
$a_{n+1} - a_n > 0$	$\frac{a_{n+1}}{a_n} > 1$	$f'(x) > 0$	strictly increasing
$a_{n+1} - a_n \geq 0$	$\frac{a_{n+1}}{a_n} \geq 1$	$f'(x) \geq 0$	increasing
$a_{n+1} - a_n < 0$	$\frac{a_{n+1}}{a_n} < 1$	$f'(x) < 0$	strictly decreasing
$a_{n+1} - a_n \leq 0$	$\frac{a_{n+1}}{a_n} \leq 1$	$f'(x) \leq 0$	decreasing

Example 1 Show that the sequance: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ is a strictly increasing.

Sol:

$$a_n = \frac{n}{n+1}, \quad a_{n+1} = \frac{n+1}{n+2}$$

1- $a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0$

2- $\frac{a_{n+1}}{a_n} = \frac{n+1/n+2}{n/n+1} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n} > 1$

3. $f(x) = \frac{x}{x+1}$, and

$$f'(x) = \frac{1.(x+1) - x(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

Thus by (1, 2 and 3), the given sequance is strictly increasing.

Eample 2:-

Show that $\left\{ \frac{n^2}{e^n} \right\}$ is strictly decreasing

$a_n = \frac{n^2}{e^n}$, $a_{n+1} = \frac{(n+1)^2}{e^{n+1}}$

1. Difference test: $a_{n+1} - a_n = \frac{(n+1)^2}{e^{n+1}} - \frac{n^2}{e^n}$

$$= \frac{(n+1)^2 - n^2 e}{e^{n+1}} < 0 \text{ for } n \geq 2$$

so the given sequance is strictly decreasing.

2. Ratio Test: $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 / e^{n+1}}{n^2 / e^n} = \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}$

$$= \frac{(n+1)^2 e^n}{n^2 (e^{n+1})} = \frac{(n+1)^2 \cancel{e^n}}{\cancel{e^n} e n^2} = \frac{(n+1)^2}{n^2 e}$$

$$= \frac{(n+1/n)^2}{e} = \frac{(1+\frac{1}{n})^2}{e} < 1 \text{ for } n \geq 2$$

The sequance $\left\{ \frac{n^2}{e^n} \right\}$, $n \geq 2$, is strictly decreasing.

3. Derivative Test:

$f(x) = \frac{x^2}{e^x}$, $x \geq 1$ because $e^0 = 1$

$$f'(x) = \frac{e^x \cdot 2x - x^2 \cdot e^x}{(e^x)^2} = \frac{\cancel{e^x} x (2-x)}{e^{2x}}$$

$$= \frac{x(2-x)}{e^x} < 0 \text{ if } x > 2$$

The sequance $\left\{ \frac{n^2}{e^n} \right\}$, $n \geq 2$ is strictly decreasing.

* Properties that hold eventually:

Ex: The sequence $9, -8, -17, 12, 1, 2, 3, 4, \ldots$ is strictly increasing from the fifth term, then it is eventually strictly increasing.

Eample 2:-

show that $\left\{ \frac{n^2}{e^n} \right\}$ is strictly decreasing

$a_n = \frac{n^2}{e^n}$, $a_{n+1} = \frac{(n+1)^2}{e^{n+1}}$

1. Difference test: $a_{n+1} - a_n = \frac{(n+1)^2}{e^{n+1}} - \frac{n^2}{e^n}$

$= \frac{(n+1)^2 - n^2 e}{e^{n+1}} < 0$ for $n \geq 2$

so the given sequance is strictly decreasing.

2. Ratio Test: $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 / e^{n+1}}{n^2 / e^n} = \frac{(n+1)^2}{e^{n+1}} \cdot \frac{e^n}{n^2}$

$= \frac{(n+1)^2 e^n}{n^2 (e^{n+1})} = \frac{(n+1)^2 \cdot \cancel{e^n}}{\cancel{e^n} e n^2} = \frac{(n+1)^2}{n^2 e}$

$= \frac{(n+1/n)^2}{e} = \frac{(1+\frac{1}{n})^2}{e} < 1$ for $n \geq 2$

The sequance $\left\{ \frac{n^2}{e^n} \right\}$, $n \geq 2$, is strictly decreasing.

3. Derivative Test:

$f(x) = \frac{x^2}{e^x}$, $x \geq 1$ because $e^0 = 1$

$f'(x) = \frac{e^x \cdot 2x - x^2 \cdot e^x}{(e^x)^2} = \frac{e^x x (2-x)}{e^{2x}}$

$= \frac{x(2-x)}{e^x} < 0$ if $x > 2$

The sequance $\left\{ \frac{n^2}{e^n} \right\}$, $n \geq 2$ is strictly decreasing.

* Properties that hold eventually:

Ex: The sequence 9, -8, -17, 12, 1, 2, 3, 4, ...
is strictly increasing from the fifth term, then
it is eventually strictly increasing.

Ex: show that the sequance $\left\{\frac{10^n}{n!}\right\}_{n=1}^{\infty}$ is eventually strictly decreasing

$$a_n = \frac{10^n}{n!} \quad , \quad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

Using Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \frac{\cancel{10^n} \cdot 10}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{10^n}}$$

$$= \frac{10}{n+1} < 1 \qquad \text{for } n \geq 10$$

Thus the given sequence eventually strictly decreasing

* Convergance of monotone sequances:

Theorem:

If a sequance is increasing it satisfies one of the convergence properties below:

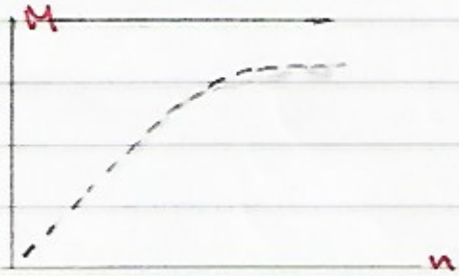
1- It increases with out bound and is therfore divergent

The $\lim_{n\to\infty} a_n = \infty$

2- It is bounded above by M ($a_n \leq M$ for all $n \geq 1$) and therefore $\lim_{n\to\infty} a_n = L \leq M$. The sequance converges to a number less than or equal to M.

Increases without bound

Diverges.



Bounded above by M

converges to $L \leq M$

Similarly, a monoton decreasing sequance satisfies one of the following properties:

- It decrease without bound and is therfore divergent. The $\lim_{n\to\infty} a_n = -\infty$

- It is bounded below by m ($a_n \geq m$ for all $n \geq 1$) and therefore $\lim_{n\to\infty} a_n = L \geq m$. The sequance converges to a number grater than or equal to m.

Show that the sequance $\left\{\frac{10^n}{n!}\right\}_{n=1}^{\infty}$ converges and find its limit.

from ex 3: $\frac{a_{n+1}}{a_n} = \frac{10}{n+1} \Rightarrow a_{n+1} = \frac{10}{n+1} a_n$

$$\lim_{n\to+\infty} a_{n+1} = \lim_{n\to\infty} \frac{10}{n+1} \cdot a_n$$

$$= \lim_{n\to\infty} \frac{10}{n+1} \cdot \lim_{n\to\infty} a_n$$

$$= \frac{10}{\infty} \cdot L$$

$$= 0 \cdot L = 0$$

Thus $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{10^n}{n!} = 0$

H.W: {10.2: 7, 19, p. 641 + 642}