Math 101

Nada
Altiary

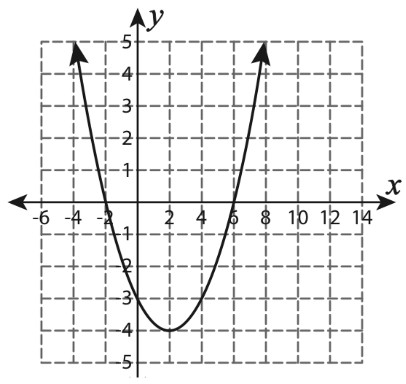
Math 101

رياضيات ٢

ا/ مدى وصل الله الطياري

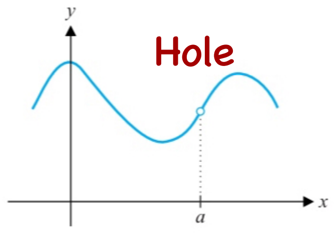
Continuity

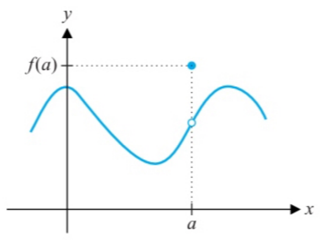
Continuity at point

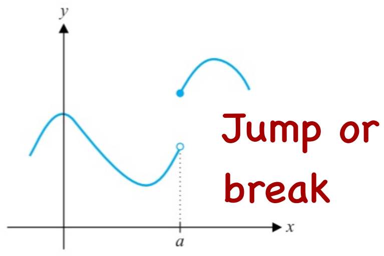


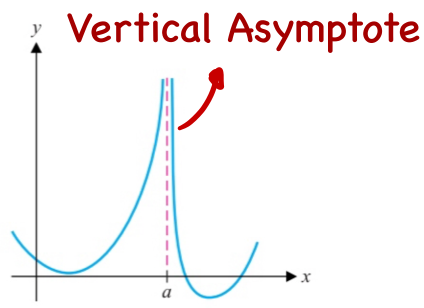
Graphically : A function f is continuous at a if its graph has no hole or break at a. Otherwise, we say that f is discontinuous.

- We will classify such discontinuities as : point, Jump and infinite.









f (a) undefined

Point discontinuity

- f(a) is defined.
- $\lim_{x \to a} f(x)$ exsit.

but

$\lim_{x \to a} f(x) \neq f(a)$

Point discontinuity

removable discontinuity

$\lim_{x \to a} f(x)$ DNE

$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

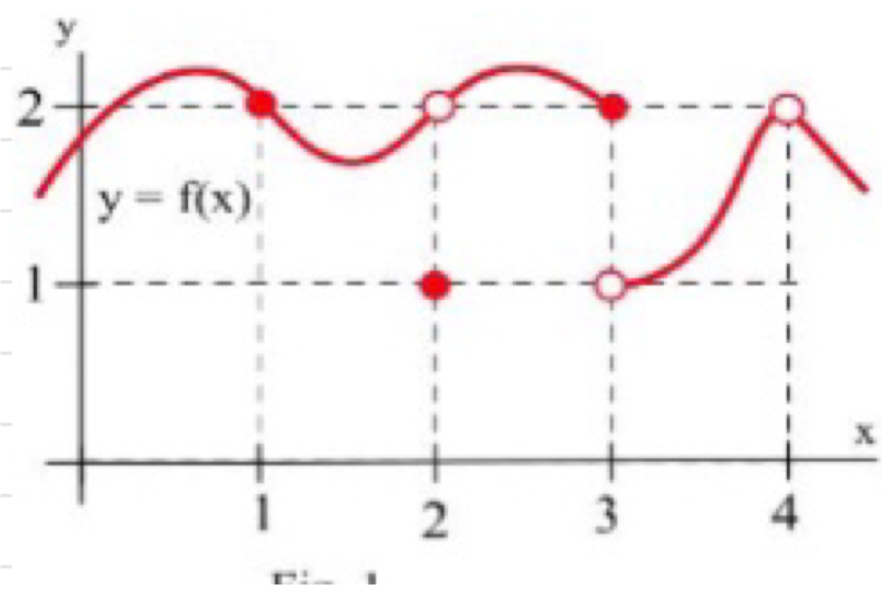
Jump discontinuity

- f(a) undefined
- $\lim_{x \to a} f(x)$ DNE

Infinite discontinuity

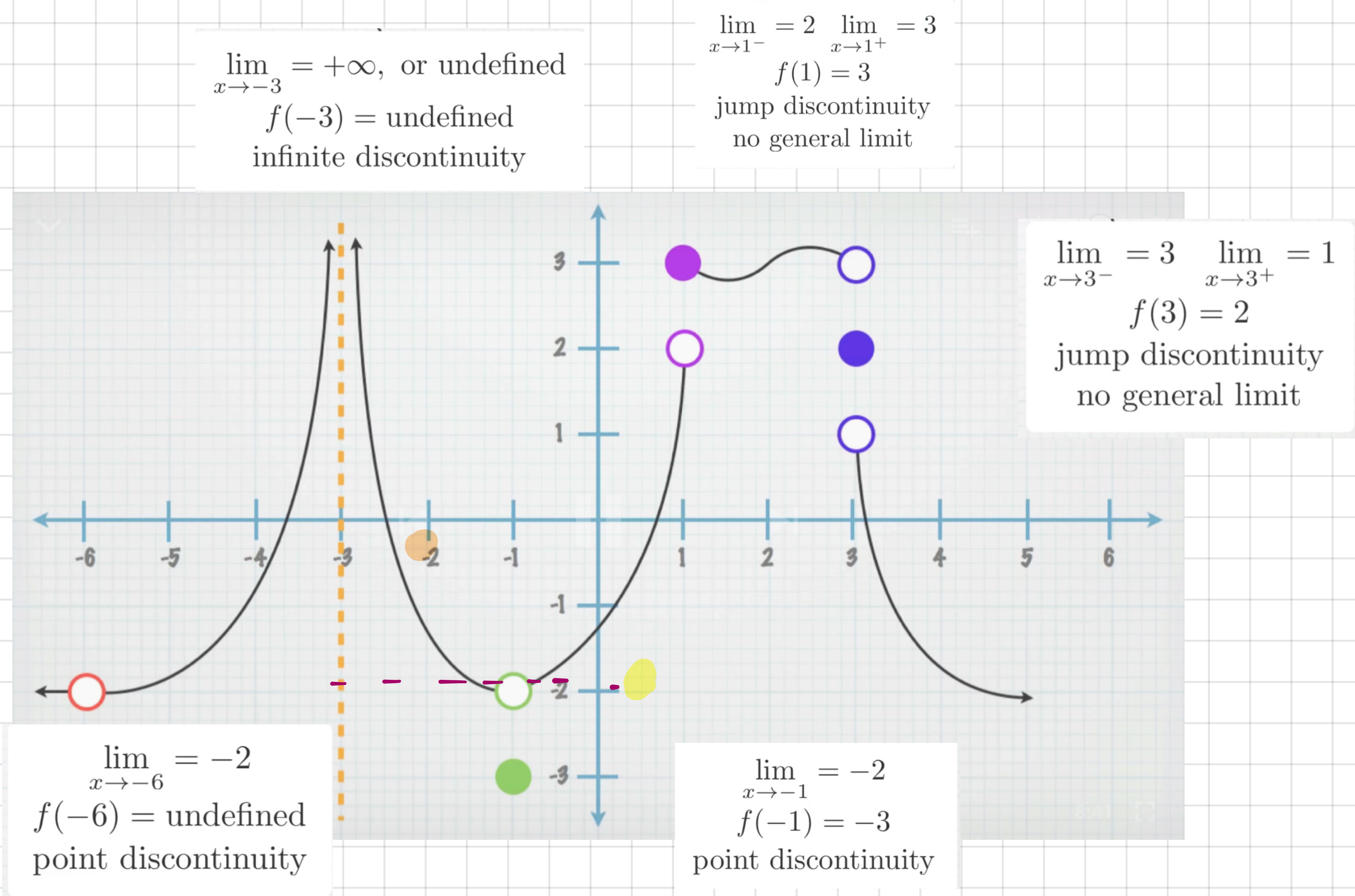
Non-removable discontinuity

Example 1: Use the following graph to fill the table



a	f(a)	$\lim_{x \to a} f(x)$
1	2	2
2	1	2
3	2	does not exist
4	undefined	2

Example 2:



Algebraically A function f is continuous at a iff

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exist
3. $\lim_{x\to a} f(x) = f(a)$

Discontinuity:

A function f fails to be continuous at a if one or more of the following conditions holds:

إنفصال قابل للإزالة **1.** If $f(a)$ is not defined.

إنفصال غير قابل للإزالة **2.** If $\lim_{x\to a} f(x)$ does not exist

إنفصال قفزي • $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$ or

إنفصال لانهائي • $\lim_{x\to a} f(x) = \pm\infty$

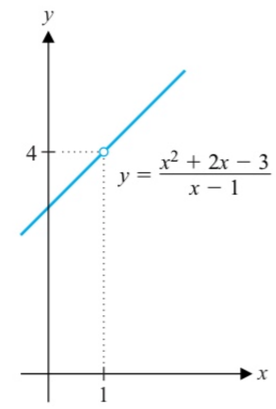
إنفصال قابل للإزالة **3.** If $\lim_{x\to a} f(x) \neq f(a)$

Example 1:

Determine whether the given functions are continuous at $a=1$

$$f(x)=\frac{x^2+2x-3}{x-1}$$

$$f(1)=\frac{(1)^2+2(1)-3}{1-1}=\frac{0}{0} \quad \text{undefined.}$$



f is discontinuous at $a=1$

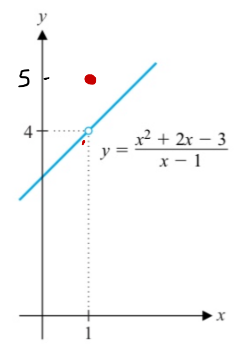
f has removable discontinuity at $a=1$

$$g(x)=\begin{cases}\dfrac{x^2+2x-3}{x-1}, & x\neq 1\\ 5 & x=1\end{cases}$$

1)- $f(1)=5$

2)- $\lim\limits_{x\to 1} f(x)=\lim\limits_{x\to 1}\dfrac{x^2+2x-3}{x-1}$

$$=\lim_{x\to 1}\frac{\cancel{(x-1)}(x+3)}{\cancel{x-1}}$$



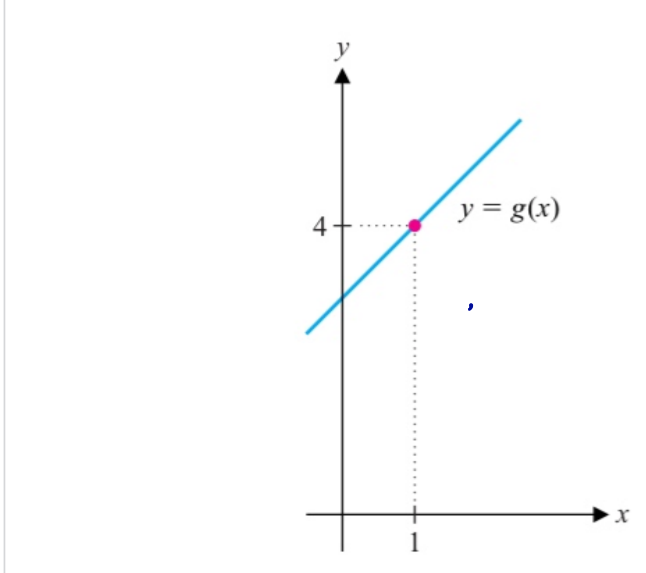
$$=\lim_{x\to 1} x+3=4$$

3)- $\lim\limits_{x\to 1} f(x)\neq f(1)$

$\therefore$ f is discontinuous at $a=1$

f has removable discontinuity at $a=1$

$$h(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1} & , x \neq 1 \\ 4 & x = 1 \end{cases}$$



1)- $F(1) = 4$

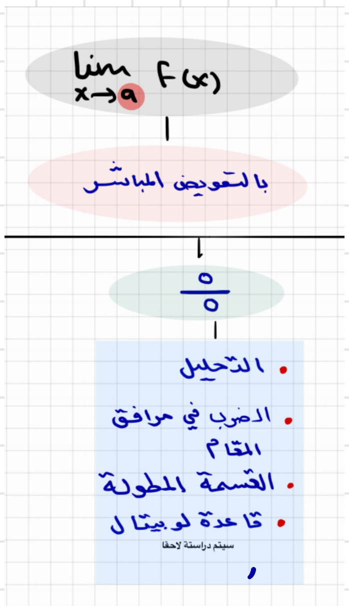
2)- $\lim_{x\to 1} F(x) = \lim_{x\to 1} \dfrac{x^2+2x-3}{x-1}$

$= \lim_{x\to 1} \dfrac{(\cancel{x-1})(x+3)}{\cancel{x-1}}$

$= \lim_{x\to 1} x+3 = 4$

3)- $\lim_{x\to 1} F(x) = F(1)$

$\therefore$ F is continuous at $x = 1$



$$h(x) = \begin{cases} \dfrac{x^3-1}{x-1} & x < 1 \\ -x^2+2x+2 & x \geq 1 \end{cases}$$

$h(1) = (-1)^2 + 2(1) + 2 = 3$

$\lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} -x^2 + 2(1) + 2 = 3$.

$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} \dfrac{x^3-1}{x-1} \quad [\frac{0}{0}]$

$= \lim_{x\to 1^-} \dfrac{(x^2+x+1)(\cancel{x-1})}{(\cancel{x-1})}$

$= \lim_{x\to 1^-} x^2+x+1 = 3$

$\Rightarrow \lim_{x\to 1} h(x) = 3$

$\because h(1) = \lim_{x\to 1} h(x)$

$\therefore$ F is continuous at $x = 1$

case 0/0 حل
بالقسمة المطولة

$$\begin{array}{r} x^2 + x + 1 \\ x-1 \enclose{longdiv}{x^3 + 1} \\ -x^3 \pm x^2 \\ \hline x^2 + 1 \\ -x^2 \pm x \\ \hline x + 1 \\ -x \mp 1 \\ \hline 0 \quad 0 \end{array}$$

$\therefore x^3+1 = (x^2+x+1)(x-1)$

Try to use
$(x^3 - a^3) =$
$(x-a)(x^2+ax+a^2)$

or :- حل case 0/0 بطريقه أخرى تعرف بطريقة لوبيتال سوف ندرسها لاحقا

تفاضل البسط

$\lim_{x\to 1^-} h(x) = \lim_{x\to 1} \dfrac{x^3-1}{x-1} = \lim_{x\to 1} \dfrac{3x^2}{1} = 3(1)^2 = 3$

تفاضل المقام

$\frac{d}{dx}[x^n] = nx^{n-1}$

Example 2:

Discuss the continuity of the following function at the given number

$$g(x) = \begin{cases} x+1 & , x < 2 \\ 2x-1 & , x > 2 \end{cases}$$

$f(2)$ undefined.

$\therefore$ f is discontinuous at $x=2$.

f has removable discontinuity

$$h(x) = \begin{cases} x+1 & , x < 2 \\ 2x-1 & , x \geq 2 \end{cases}$$

1) $F(2) = 2(2)-1 = 3$

2) $\lim\limits_{x \to 2^-} F(x) = \lim\limits_{x \to 2^-} x+1 = 3$

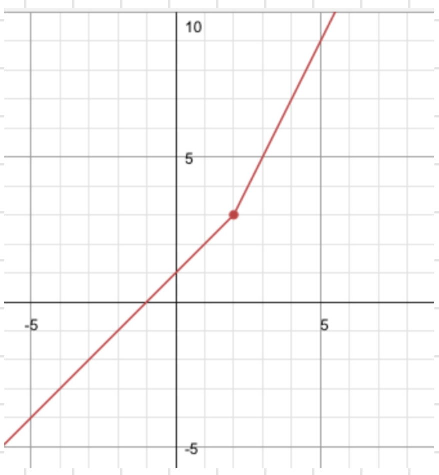
Exist and equal

$\lim\limits_{x \to 2^+} F(x) = \lim\limits_{x \to 2^+} 2x-1 = 3$

$\therefore \lim\limits_{x \to 2} F(x) = 3$

3) $\lim\limits_{x \to 2} F(x) = 3 = F(2)$

$\therefore$ F is continuous at $a=2$



$$f(x) = \begin{cases} x+1 & , x < 2 \\ 2x+1 & , x \geq 2 \end{cases}$$

1) $F(2) = 2(2)-1 = 3$

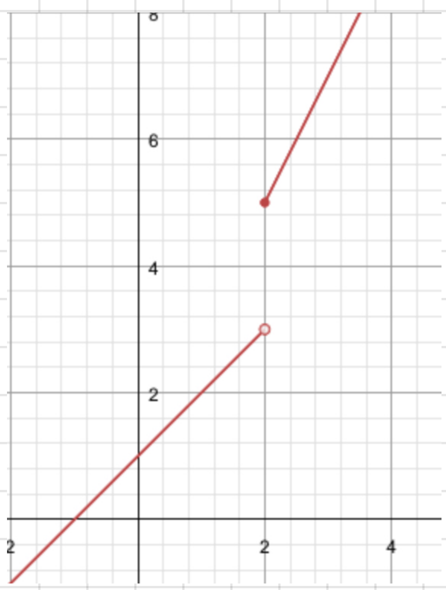
2) $\lim\limits_{x \to 2^-} F(x) = \lim\limits_{x \to 2^-} x+1 = 3$

Exist but not equal

$\lim\limits_{x \to 2^+} F(x) = \lim\limits_{x \to 2^+} 2x+1 = 5$

$\therefore \lim\limits_{x \to 2} F(x)$ DNE

$\therefore$ F has Jump discontinuity



Example 3: Find the value of a that makes the given function continuous

1)- $h(x) = \begin{cases} \dfrac{x^2-a^2}{x-a} & , x \neq a \\ 6 & , x = a \end{cases}$

$$h(a) = 6 = \lim_{x \to a} h(x)$$

$$= \lim_{x \to a} \frac{x^2-a^2}{x-a} \qquad \left[\tfrac{0}{0} \text{ , factor"}\right]$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{x-a}}$$

$$= \lim_{x \to a} x+a = 2a$$

$$\Rightarrow 6 = 2a \Rightarrow a = 3$$

1)- $f(x) = \begin{cases} 5x-2 & , x \geq 2 \\ ax^2+2 & , x < 2 \end{cases}$

- f is continuous at $x=2$ iff $\lim_{x \to 2} f(x) = f(a)$

1)- $f(2) = 5(2)-2 = 8$

2)- $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x-2) = 5(2)-2 = 8$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} ax^2+2 = a(2)^2+2 = 4a+2$

- $\lim_{x \to 2} f(x)$ exist iff $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

$$4a+2 = 8 \Leftrightarrow 4a = 6 \Leftrightarrow a = \frac{6}{4} = \frac{3}{2}$$

Continuity on open interval

Definition:

A function f is continuous on an open interval (a,b) if it is continuous at each point in the interval.

Remark:– A function f that is continuous on the entire line $(-\infty,\infty)$ is everywhere continuous.

Theorem: The following types of function are continuous at every point in their domains.

Functions الدوال	Forms الشكل	Domain المجال
Polynomial functions	$p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$	D=R
Rational functions	$r(x)=\frac{p(x)}{q(x)}$, $p(x)$ and ,$q(x)$ are polynomials.	D= R-{أصفار المقام}
Radical functions	$f(x)=\sqrt[n]{x}$ n: even	D=ماتحت الجذر≥0
	n: odd	D=R
Trigonometric functions	$\sin x$, $\cos x$, $\tan x$, $\sec x$, $\csc x$, $\cot x$	D=R
Exponential functions	e^x, a^x $a>0$	D= R
Logarithmic functions	$\ln x$, $\log_a x$	D= مابداخل الدالة>0

Example 1: Find the intervals in which each the following function is continuous.

1)- $F(x) = x^2 + 2x + 1$

$F(x)$ is continuous on $R = (-\infty, \infty)$

2)- $F(x) = \dfrac{x}{x^2 - 6x + 9}$

$F(x)$ is continuous on R-{ أصفار المقام }

$\therefore \quad x^2 - 6x + 9 = 0$

$(x-3)(x-3) = 0$

$\Rightarrow \quad x = 3$

$\therefore$ $f(x)$ is continuous on $R - \{3\} = (-\infty, 3) \cup (3, \infty)$.

3)- $f(x) = \sqrt[5]{x+2}$

$F(x)$ is continuous on R.

4)- $F(x) = \sqrt{x(x-1)}$

F is continuous iff $x(x-1) \geq 0$

$\Rightarrow x \geq 0$ or $x - 1 \geq 0$

$\Rightarrow x \geq 1$

5)- $F(x) = \ln(x+4)$

F is continuous iff $x + 4 > 0$

$x > -4$

$\therefore F$ is continuous on $(-4, \infty)$

6)- $f(x) = \sqrt[4]{x+7}$

f is continuous iff $x+7 \geq 0$

$x \geq -7$

$\therefore f$ is continuous iff $[-7, \infty)$

7)- $f(x) = \dfrac{1}{x^2+1}$

F is continuous on $\mathbb{R}$. Why?!

طالما x مربعة هذا يعني
اي رقم سالب با لتربيع
راح يصبح عدد موجب
وهذا العدد مجموع عليه
عدد آخر بالتالي تصبح
النتيجة دائماً موجبة

8)- $f(x) = \dfrac{x^2+x-12}{x^2-3x}$

F is continuous iff $x^2-3x \neq 0$

$x(x-3) \neq 0$

$\Rightarrow x \neq 0$ or $x-3 \neq 0$

$x = 3$

$\therefore F$ is continuous on $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$

9)- $f(x) = \sqrt{x^2+25}$

f is continuous iff $x^2+25 \geq 0$

$\therefore F$ is continuous on $\mathbb{R}$.

ما علاقة ذلك بمجال الدالة الجذرية؟
الدالة الجذرية تكون معرفة إذا
كان ما تحت الجذر ≥ 0 (صفر
أو عدد موجب) يعني
نستبعد الأعداد السالبة
وطالما أن x^2+25 مستحيل
تكون سالبة إذاً لا يوجد أعداد
كي نستبعدها بالتالي يكون المجال
كل الأعداد الحقيقية $\mathbb{R}$.

10) $f(x) = \sin(x^2-4)$

F is continuous on $\mathbb{R}$.

Continuity on a closed interval

A function f is continuous on the closed interval $[a,b]$ iff

1)- f continuous on (a,b).

2) $\lim\limits_{x \to a^+} f(x) = f(a)$

3)- $\lim\limits_{x \to b^-} f(x) = f(b)$

Example 1: Discuss the continuity of

a)- $f(x) = \sqrt{25 - x^2}$

Remember:

$x^2 + y^2 = r^2$

$y = \sqrt{25 - x^2}$

$y^2 = 25 - x^2$

$x^2 + y^2 = 25$

f is continuous if $25 - x^2 \geq 0$

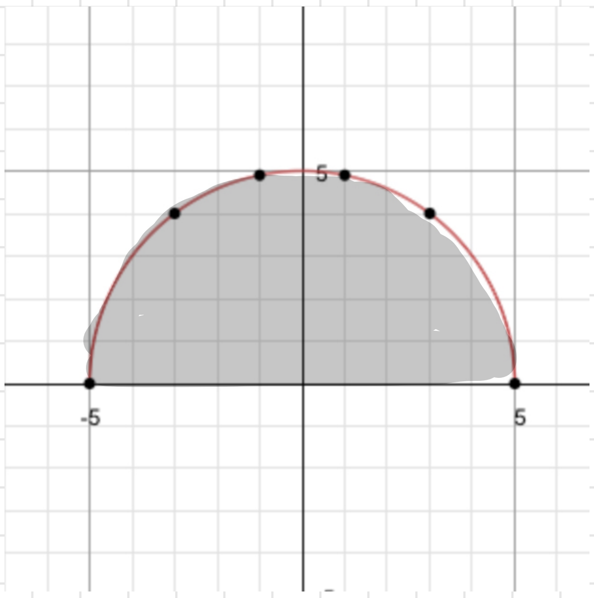
$(5-x)(5+x) \geq 0$

$\Rightarrow 5 - x \geqslant 0$ or $5 + x \geqslant 0$

$\Rightarrow 5 \geqslant x$ or $x \geqslant -5$

$\Rightarrow x \leqslant 5$ or $x \geqslant -5$

$\Rightarrow -5 \leqslant x \leqslant 5$



$\therefore D(f) = [-5, 5]$

1)- f is continuous on $(-5, 5)$

2)- $\lim\limits_{x \to 5^-} f(x) = \lim\limits_{x \to 5^-} \sqrt{25 - x^2} = \sqrt{25 - x^2} = 0 = f(5)$

3)- $\lim\limits_{x \to -5^+} f(x) = \lim\limits_{x \to -5^+} \sqrt{25 - x^2} = \sqrt{25 - 25} = 0 = f(-5)$

$\therefore f$ is continuous on $[-5, 5]$

b)- $g(x) = \sqrt{1 - x^2}$ H.W

One sided continuity

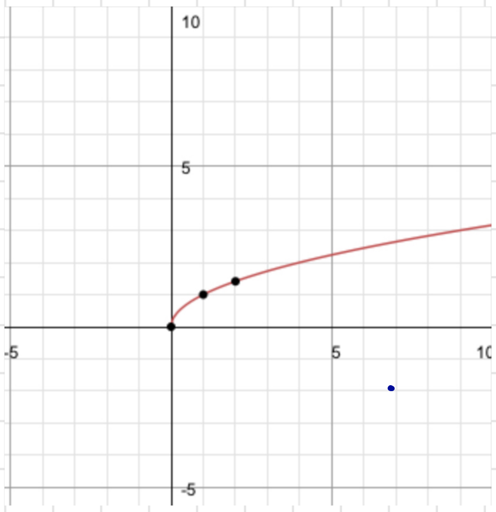
Right and left continuity:

- A function f is continuous from the right at a if
$$\lim_{x \to a^+} f(x) = f(a)$$

- A function f is continuous from the left at a if
$$\lim_{x \to a^-} f(x) = f(a)$$

Example 1: Discuss the continuity of the following function

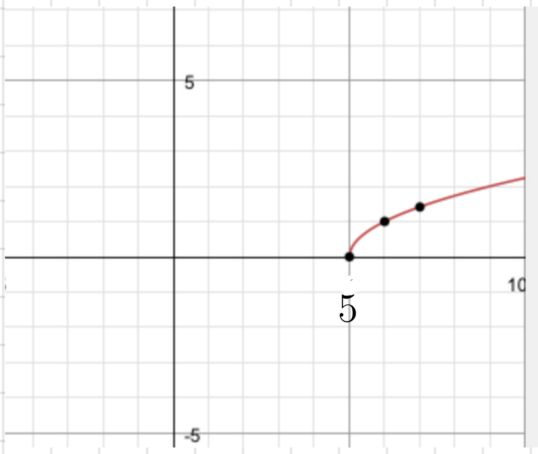
$f(x) = \sqrt{x}$



f is continuous iff $x \geq 0$

$\therefore f$ is continuous on $[0, \infty)$

$g(x) = \sqrt{x-5}$



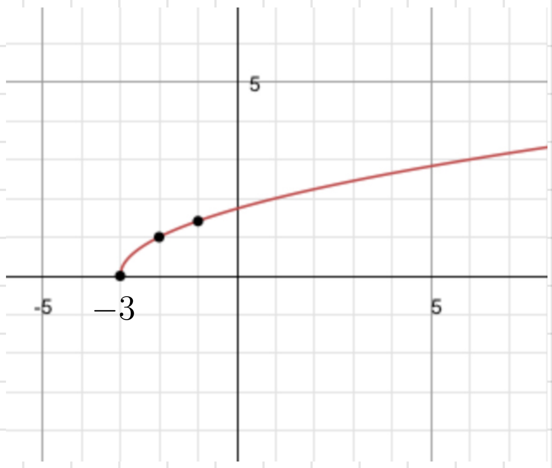
f is continuous iff $x - 5 \geq 0$

$\Rightarrow x \geq 5$

$\therefore f$ is continuous on $[5, \infty)$

Note

$h(x) = \sqrt{x+3}$



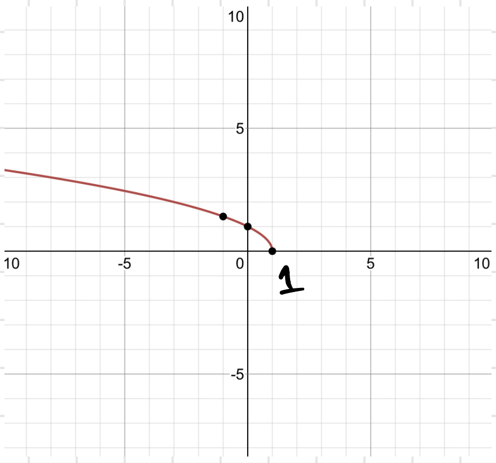
f is continuous iff $x + 3 \geq 0$

$x \geq -3$

$\therefore f$ is continuous on $[-3, \infty)$

Example 2: Discuss the continuity of the following function

$f(x) = \sqrt{1-x}$

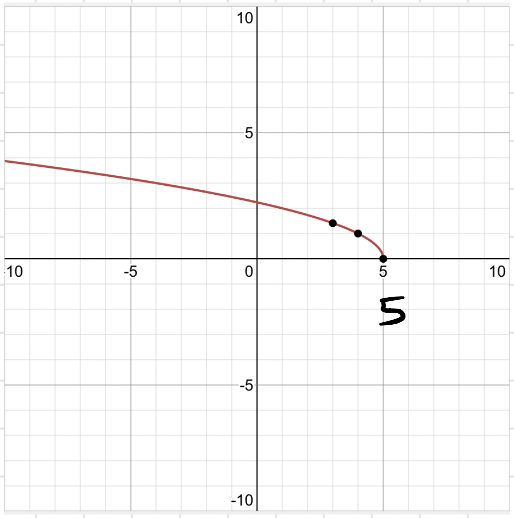


f is continuous iff $1 - x \geq 0$

$\Rightarrow 1 \geq x$

$\therefore f$ is continuous on $(-\infty, 1]$

$g(x) = \sqrt{5-x}$



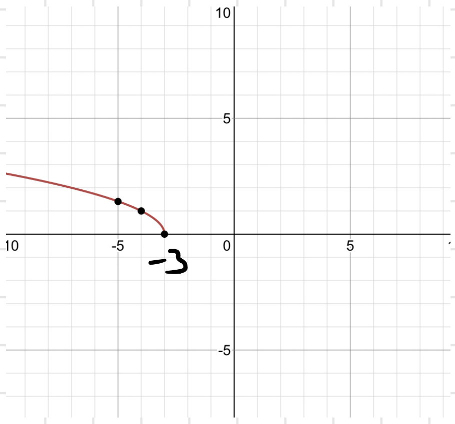
f is continuous iff $5 - x \geq 0$

$\Rightarrow 5 \geq x$ or $x \leq 5$

$\therefore f$ is continuous on $(-\infty, 5]$

Note

$h(x) = \sqrt{-3-x}$



f is continuous iff $-3 - x \geq 0$

$\Rightarrow -3 \geq x$ or $x \leq -3$

$\therefore f$ is continuous on $(-\infty, -3]$

Example 3: Discuss the continuity of the following function at the given number

$f(x) = \sqrt{x}$ at $a = 0$

1)_ $f(0) = \sqrt{0} = 0$

2)_ $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \sqrt{x}$

$= \sqrt{0} = 0$

3)_ $\because \lim_{x \to 0^+} f(x) = f(0)$

$\therefore f$ is continuous from the right

Notes

Example 4: Greatest integer function

The greatest integer function $[x]$ is the largest integer less than or equal to x.

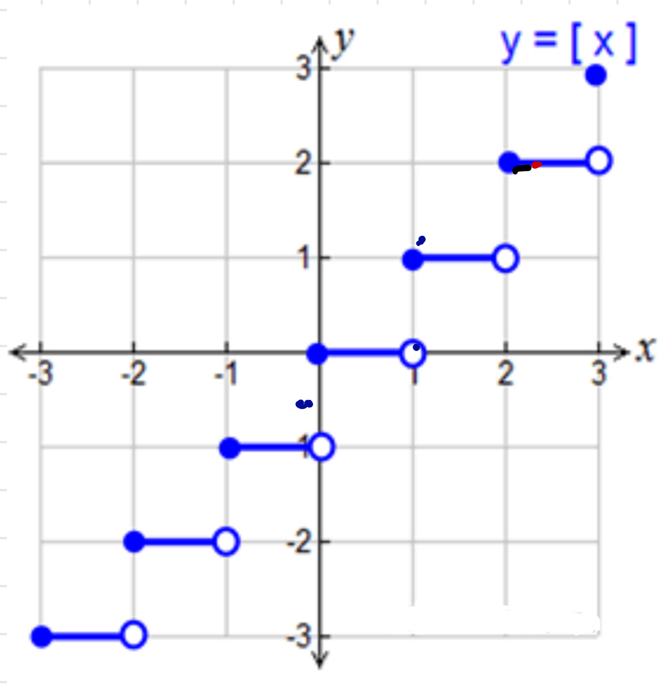
$$[x] = n \iff n \le x < n+1$$

$[2.9] = 2$ $\quad [0] = 0$ $\quad [1.4] = 1$ $\quad [3] = 3$

$[-2.51] = -3$ $\quad [-0.5] = -1$ $\quad [-1.01] = -2$ $\quad [-2] = -2$

$\lim_{x\to 1^-} [x] = 0$	$\lim_{x\to 1} [x] =$ DNE	$\lim_{x\to 1^+} [x] = 1$
$\lim_{x\to 2^-} [x] = 1$	$\lim_{x\to 2} [x]$ DNE	$\lim_{x\to 2^+} [x] = 2$
$\lim_{x\to 3^-} [x] = 2$	$\lim_{x\to 3} [x]$ DNE	$\lim_{x\to 3^+} [x] = 3$
$\lim_{x\to n^-} [x] = n-1$	$\lim_{x\to n} [x]$ DNE	$\lim_{x\to n^+} [x] = n$

Discuss the continuity of $g(x) = [x]$ at $a = n$ عدد صحيح



$g(n) = [n] = n$

$$\left.\begin{array}{l}\lim_{x\to n^+} [x] = n \\ \lim_{x\to n^-} [x] = n-1\end{array}\right\}$$ Exist but not equal

$\therefore$ g is discontinuous at $a = n$ or g is continuous from the right

$\therefore$ g has Jump discontinuity.

Remark:-

1)- There is a jump at each integer and so

$$\lim_{x\to n^+} [x] \neq \lim_{x\to n^-} [x]$$

2)- What about if a is not integer i.e $a = 1.5$

💡 Does $g(x) = [x]$ is continuous at $a = 1.5$

Theorem 2.4.1:* *[Properties of Continuity]

If f and g are continuous function at a and k is any real number, then the following functions are continuous at a.

1. Sum and Difference: $f \pm g$
2. Product: fg
3. Quotient: $\dfrac{f}{g}$ provided $g(a) \neq 0$
4. Constant multiple: kf.

Continuity of Composite of Function

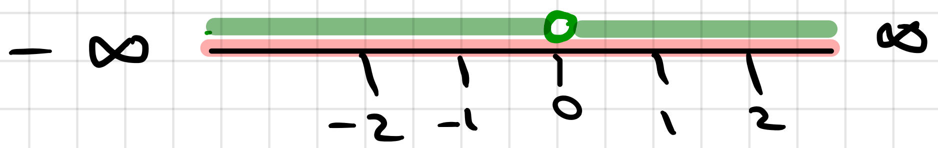
Theorem 2.4.3: [The Limit and Continuity of a Composite Function]
Let f and g be two functions and let a and L be two real numbers.

1. If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$.
2. If g is continuous at a and f is continuous at $g(a)$, then the composite function $(f\circ g)(x) = f(g(x))$ is continuous at a.

Example 1: Discuss the continuity of the following function

$F(x) = \sin\left(\frac{1}{x}\right)$ $\quad F = g(h(x))$

- $g(x) = \sin x$ is continuous on $\mathbb{R} = (-\infty, \infty)$
- $h(x) = \frac{1}{x}$ is continuous on $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$
- $\therefore F$ is continuous on $(-\infty, 0) \cup (0, \infty)$



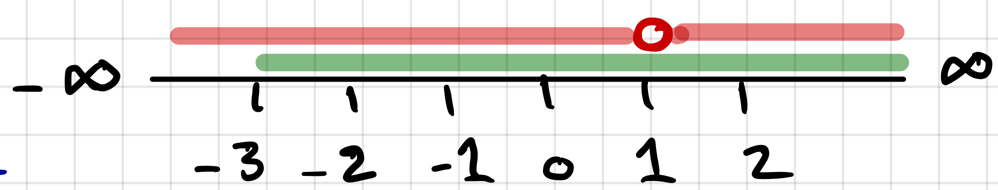
$$F(x) = \frac{\sqrt{x+3} - 2}{x-1}$$

$$F(x) = (\sqrt{x+3} - 2) \cdot \frac{1}{x-1}$$

$g(x) = \sqrt{x+3} - 2$ is continuous iff

$$x+3 \geq 0 \Rightarrow x \geq -3$$

$\therefore g(x)$ is continuous on $[-3, \infty)$



$h(x) = \frac{1}{x-1}$ is continuous iff

$$x-1 \neq 0 \Rightarrow x \neq 1$$

$\therefore h$ is continuous on $\mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$

$\therefore F$ is continuous on $[-3, 1) \cup (1, \infty)$

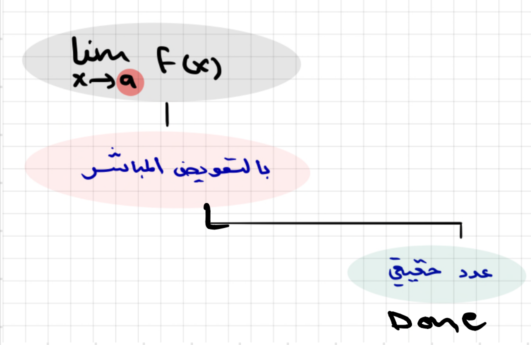
Limits of composite function

Example 1: Find the limits

$$\lim_{x\to 0^+} \cos\left(\frac{\pi}{3} e^{\sqrt{x}}\right) = \cos\left(\frac{\pi}{3} \lim_{x\to 0^+} e^{\sqrt{x}}\right)$$

$$= \cos\left(\frac{\pi}{3} e^{\sqrt{0}}\right)$$

$$= \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$



$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) = \sin^{-1}\left(\lim_{x\to 1} \frac{1-x}{1-x^2}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1-x}{(1-x)(1+x)}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1}{1+x}\right)$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\lim_{x\to\infty} \log_3 (x^2-1) = \log_3\left(\lim_{x\to\infty} x^2-1\right)$$

$$= \log_3(\infty^2-1)$$

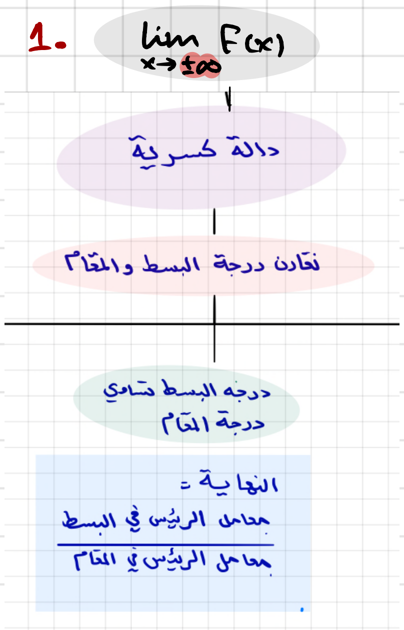
$$= \log_3(\infty) = \infty \qquad , \; 3^{\infty} = \infty$$

$$\lim_{x\to\infty} \ln\left(\frac{4+x}{x-1}\right) = \ln\left(\lim_{x\to\infty} \frac{4+x}{x-1}\right)$$

درجة البسط =
درجة المقام.

$$= \ln(1) = 0$$

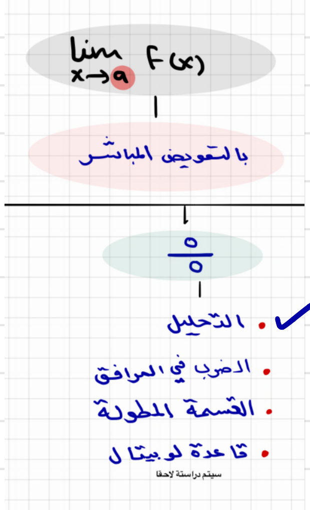
1.



$$\lim_{x\to\infty} \cos^{-1}\left(\frac{2+x}{2x+1}\right) = \cos^{-1}\left(\lim_{x\to\infty} \frac{2+x}{2x+1}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

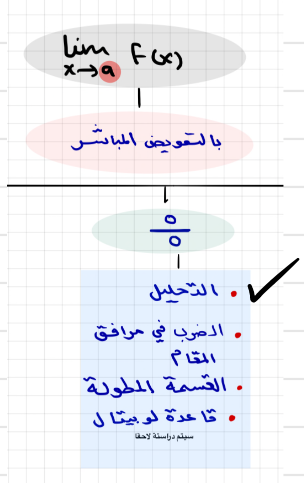
$$\lim_{x\to 2} \sin\left(\frac{\pi(x-2)}{x^2-4}\right) = \sin\left(\lim_{x\to 2} \frac{\pi(x-2)}{x^2-4}\right)$$



$$= \sin\left(\lim_{x\to 2} \frac{\pi\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}\right)$$

$$= \sin\left(\lim_{x\to 2} \frac{\pi}{x+2}\right)$$

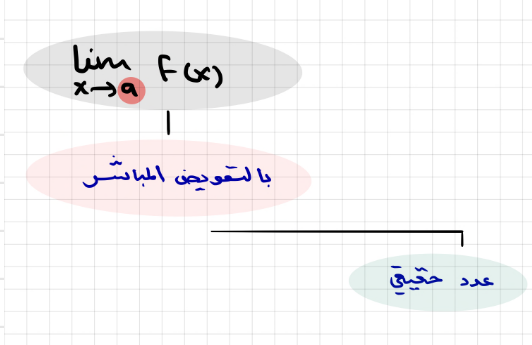
$$= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$



$$\lim_{x\to\infty} \cos\left(\frac{\pi(2x^2-2)}{x^2-4}\right) = \cos\left(\lim_{x\to\infty} \frac{\pi(2x^2-2)}{x^2-4}\right)$$

$$= \cos(2\pi) = 1$$

$$\lim_{x\to 1^-} \sin^{-1} x = \sin^{-1}(1) = \frac{\pi}{2}$$



$$\lim_{x\to -1^-} \sin^{-1}(x) = \sin^{-1}(-1) = -\frac{\pi}{2}.$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\sin 0}{0} = \frac{0}{0} \text{ case}$$

$$\to \lim_{x\to 0} \frac{\cos x}{1} = \cos(0) = 1$$

$\therefore$ **Rule:** $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

$$\lim_{x\to -4^-} \tan^{-1} \sqrt[5]{\frac{x-3}{x+4}}$$

$$= \tan^{-1}\left(\lim_{x\to -4^-} \sqrt[5]{\frac{x-3}{x+4}}\right)$$

$$= \tan^{-1}\left(\sqrt[5]{\lim_{x\to -4^-} \frac{x-3}{x+4}}\right)$$

$$= \tan^{-1}\left(\sqrt[5]{\lim_{x\to -4^-} x-3 \cdot \lim_{x\to -4^-} \frac{1}{x+4}}\right)$$

$$= \tan^{-1}\left(\sqrt[5]{-7 \cdot -\infty}\right)$$

$$= \tan^{-1}\left(\sqrt[5]{\infty}\right)$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

why $-\infty$??

$$\lim_{x\to 0} \cos\left(\frac{\pi}{\sqrt{17-\sec x}}\right)$$

$$= \cos\left(\lim_{x\to 0} \frac{\pi}{\sqrt{17-\sec x}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{17-\lim_{x\to 0} \sec x}}\right)$$

$$= \cos\left(\frac{\pi}{\sqrt{17-1}}\right)$$

$$= \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Remember:

$\sec x = \frac{1}{\cos x}$

$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$

Example 2: **Discuss the continuity of the following function at** $x = 0$

$$g(x) = \begin{cases} \frac{\sin x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

$g(0) = 1$

$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{\sin x}{x}$ $\left[\frac{0}{0} \text{ case}\right]$

$= \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = 1$

Rule

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\because g(0) = 1 = \lim_{x \to 0} g(x)$

$\therefore$ g is continuous.

$$F(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

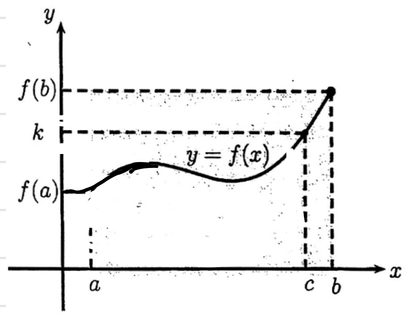
$F(0) = 0$

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = \text{DNE}$ $\quad \underset{\sin\left(\frac{1}{0}\right)}{\text{undefined}}$

f is discontinuity at $x = 0$.

The Intermediate Value Theorem I.V.T

Theorem 2.4.4: [Intermediate Value Theorem]
If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c)=k$.



اذا كانت f داله متصلة في الفترة [a,b] وكان k عدد حقيقي محصور بين f(a) و f(b) فإنه يوجد على الأقل عدد واحد c في الفترة [a,b] بحيث f(c)=k

Example 1: Show that the function $f(x) = x^3 + 2x^2 - 1$ has a zero in the interval $[0,1]$.

$f(c) = 0$

$f(0) = 0^3 + 2(0)^2 - 1 = -1$

$f(1) = 1^3 + 2(1) - 1 = 2$

$f(0) = -1 < 0 < 2 = f(1).$ ✅

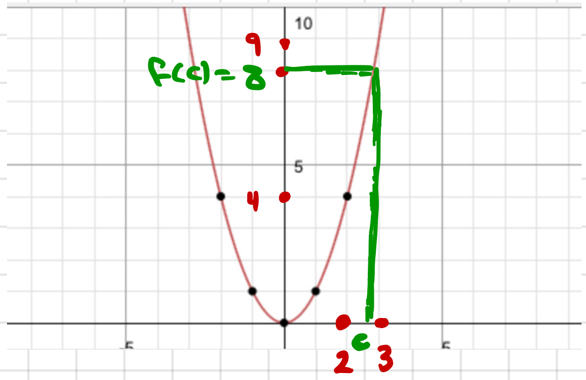
Example: Show that the function $f(x) = x^2$ has value 8 in $[2,3]$

$f(c) = 8$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$

$4 < 8 < 9$ ✅



ماهي قيمة c التي اذا عوضنا بها في المعادلة f(c)=8

$f(c) = c^2$

$8 = c^2$

$\sqrt{8} = c$

$2\sqrt{2} = c \in [2,3]$

$f(2\sqrt{2}) = (2\sqrt{2})^2 = 4 \cdot 2 = 8$

💡 What about $g(x) = x^2$ has value 3 in $[4,9]$

$f(c) = 3$

$g(2) = 4$

$g(3) = 9$

but

3 is not in the interval $[4,9]$. ❎

ملاحظه نظرية القيمة المتوسطه تؤكد فقط وجود حل للمعادلة f(c)=k دون الحاجه الى تعيين قيمة c

Example: Show that the function $f(x) = x^3 + x$ has value of 9 in the interval

$f(1) = 1^3 + 1$ -

$f(2) = 2^3 + 2 = 10$

$2 < 9 < 10$ ✅

f(1) f(c) f(b)

Differentiation

Definition of the Derivative

Basic of Differentiation Rule

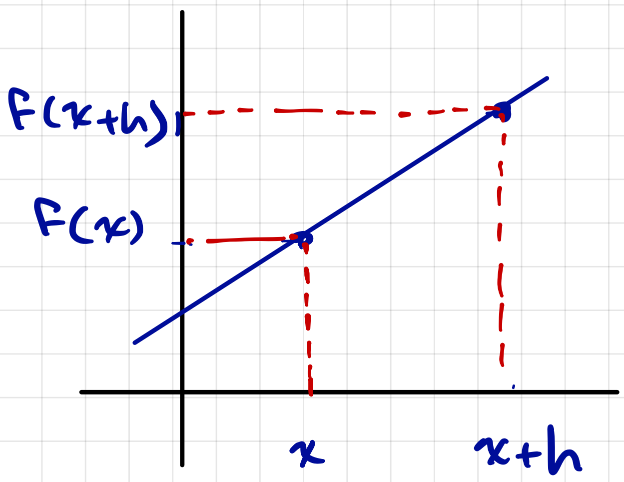
Derivative of Exponential and Logarithmic Functions

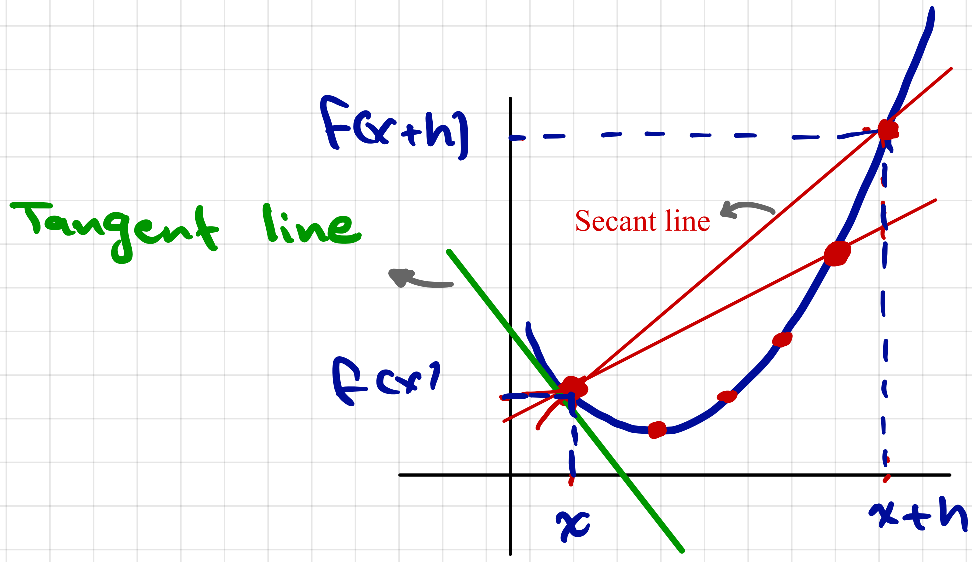
Derivative of Trigonometric Functions

The Chain Rule

Implicit Differentiation and Higher Derivative

Definition of the Derivative





Slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{\cancel{x} + h - \cancel{x}}$$

$$= \frac{f(x+h) - f(x)}{h}$$

$$m = \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{\text{تعريف التفاضل}}$$

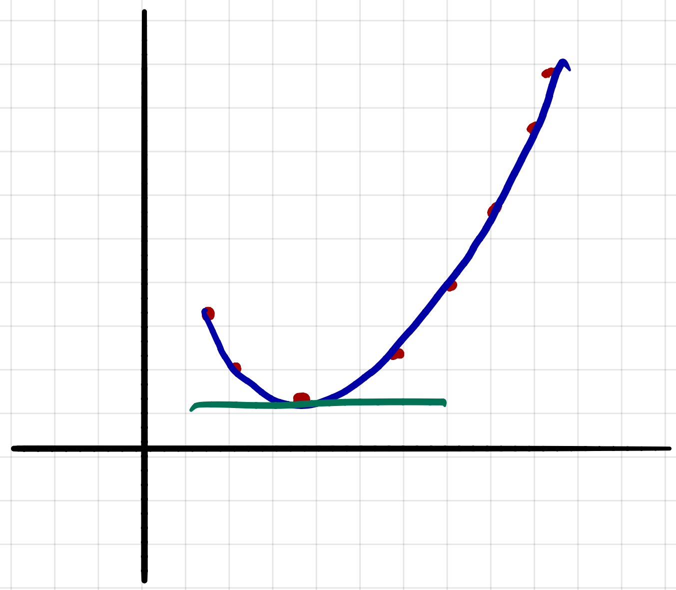
التفاضل = ميل المماس لمنحنى الداله عند نقطه

$$\therefore m = f'(a)$$

What happen if the tangent line is horizontal?

If the tangent line is horizontal then

$m = f'(a) = 0$



Example 1: Let $f(x) = 2x^2$ and $a = 2$. Find $f'(a)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = 2(x+h)^2 = 2(x^2 + 2xh + h^2)$$

$$f(x) = 2x^2 \qquad = 2x^2 + 4xh + 2h^2$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\cancel{2x^2} + 4xh + 2h^2 - \cancel{2x^2}}{h}$$

$$= \frac{4xh + 2h^2}{h} = \frac{2\cancel{h}(2x+h)}{\cancel{h}} = 4x + h$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 4x + h = 4x.$$

$$f'(a) = f'(2) = 4 \cdot 2 = 8.$$

Example 2: let $f(x) = \frac{1}{x}$ and $a = 1$, find $f'(a)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = \frac{1}{x+h} \quad \text{and} \quad f(x) = \frac{1}{x}$$

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} = \frac{\cancel{x} - \cancel{x} - h}{x(x+h)} = \frac{-h}{x(x+h)}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{-h}{x(x+h)}}{h} = \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \frac{-1}{x(x+h)}$$

$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

$$f'(a) = \frac{-1}{a^2} = \frac{-1}{1} = -1$$

Example 3: Let $f(x) = \sqrt{x}$. Find the equation of the tangent to the graph of $f(x)$ at $x = 4$.

$$y - y_1 = m(x - x_1)$$

m = slope

(x_1, y_1) points.

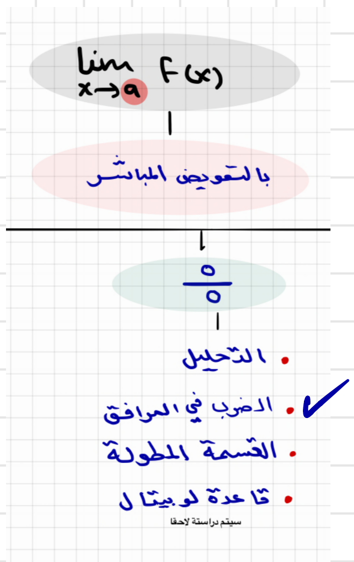
$m = f'(x)$ at $x = 4$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \left[\frac{0}{0}\right]$$

الضرب في المرافق

Conjugate Method



$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

• $f'(4) = \frac{1}{\sqrt{4} + \sqrt{4}} = \frac{1}{2+2} = \frac{1}{4} \rightarrow m$

$$\therefore \; y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}(x - 4) + 2$$

$$= \frac{1}{4}x - \frac{1}{4} \cdot 4 + 2$$

$$= \frac{1}{4}x - 1 + 2$$

$$y = \frac{1}{4}x + 1$$

Example 4: Find the derivative of $y = 2x^2 + 3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = 2(x+h)^2 + 3 = 2(x^2 + 2xh + h^2) + 3$$

$$= 2x^2 + 4xh + 2h^2 + 3$$

$$f(x) = 2x^2 + 3$$

$$\therefore \frac{f(x+h) - f(x)}{h} = \frac{2x^2 + 4xh + 2h^2 + 3 - (2x^2+3)}{h}$$

$$= \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{+3} \cancel{-2x^2} \cancel{-3}}{h}$$

$$= \frac{4xh + 2h^2}{h} = \frac{2\cancel{h}(2x+h)}{\cancel{h}} = 2(2x+h).$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2(2x+h) = 4x.$$

Example5: Let $f(x) = \sqrt{x-1}$. Sowh that $f'(x) = \dfrac{1}{2\sqrt{x-1}}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \qquad \left[\tfrac{0}{0} \text{ conjugate method}\right]$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \cdot \frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x+1}}$$

$$= \lim_{h \to 0} \frac{x+h-1 - (x-1)}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h \cancel{-1} \cancel{-x} \cancel{+1}}{h\left(\sqrt{x+h-1} + \sqrt{x-1}\right)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-1} + \sqrt{x+1}} = \frac{1}{\sqrt{x-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

Basic Differentiation Rules

القوانين الأساسيه للإشتقاق

For $y = f(x)$, all of the following are used to represent the derivative: $f'(x)$, y', $\frac{dy}{dx}$, $D_x y$, $\frac{d}{dx}[f(x)]$.

The constant Rule:

$F(x) = c$ then $F'(x) = 0$

Ex: $F(x) = e$, $F'(x) = 0$

Power Rule:

$F(x) = x^n$ then $F'(x) = n\, x^{n-1}$

Ex: $F(x) = x^4$, $F'(x) = 4x^3$

Radical Power Rule

$F(x) = x^{\frac{1}{n}}$ then $F'(x) = \frac{1}{n} x^{\frac{1}{n}-1}$

Ex: Let $F(x) = \sqrt{x}$. Find $F'(x)$

$$F(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$F'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$$

Note:

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

It is useful to know :

$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$ and $\frac{d}{dx}\left[\sqrt{\text{ما بداخل الجذر}}\right] = \dfrac{\text{تفاضل مابداخل الجذر}}{\text{الجذر نفسه* 2}}$

For example:

- $F(x) = \sqrt{x}$

  $F'(x) = \frac{1}{2\sqrt{x}}$

- $F(x) = \sqrt{x^2+1}$ then

  $F'(x) = \frac{2x}{2\sqrt{x^2+1}}$

Theorem 3.2.5: [The Constant Multiple, The Sum and Difference Rules]
Let c be a constant. If $f(x)$ and $g(x)$ are differentiable, then $cf(x)$ and $f(x) \pm g(x)$ are also differentiable, and

i) $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}(f(x))$

ii) $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

iii) $\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$.

Example :- Let $f(x) = 2x^3 - \sqrt{x}$. Find $f'(x)$.

$f'(x) = 6x^2 - \frac{1}{2\sqrt{x}}$

Example : Let $f(x) = x^2 + 4^3$. Find $f'(x)$.

عدد ثابت

$f'(x) = 2x + 0 = 2x$

Example : let $y = 3x^4$. Find y'

$y' = 3.4\,x^3 = 12x^3$.

Product Rule :

$(f(x)\,g(x))' = f(x).\,g'(x) + g(x).\,f'(x)$

For example: $f(x) = (3x - 2x^2)(5+4x)$

$f'(x) = (3x-2x^2)(5+4x)' + (5+4x).(3x-2x^2)'$

$= (3x - 2x^2)(4) + (5+4x)(3-4x)$

$= (12x - 8x^2) + (15 - 20x + 12x - 16x^2)$

$= (12x - 8x^2) + (15 - 8x - 16x^2)$

$= -24x^2 + 4x + 15$

Quotient Rule: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)\,g(x) - g'(x)\,f(x)}{g^2(x)}$

For example:

$$f(x) = \frac{3x - 2x^2}{5 + 4x}$$

$$f'(x) = \frac{(3x - 2x^2)'(5+4x) - (5+4x)'(3x - 2x^2)}{(5+4x)^2}$$

$$= \frac{(3 - 4x)(5+4x) - (4)(3x - 2x^2)}{(5+4x)^2}$$

$$= \frac{15 + \cancel{12x} - 20x - 16x^2 - \cancel{12x} + 8x^2}{(5+4x)^2}$$

$$= \frac{15 - 20x - 8x^2}{(5+4x)^2}$$

Derivative of Exponential Function

The Exponential functions have the form

$$f(x) = a^x$$

Variable — متغير → Any real number

Base — اساس → $a > 0,\ a \neq 1$

The Exponential function with base e is given by :

$$f(x) = e^x$$

Note:

لماذا الأساس $a>0$ بمعنى الأساس دائماً موجب وليس سالب؟

فرضاً: $f(x) = -4^x$

x: عبارة عن اي عدد حقيقي

$f(2) = (-4)^2 = 16 \in R$

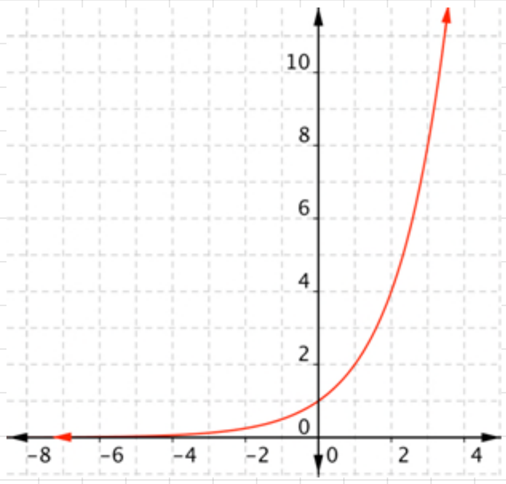
$f(\frac{1}{2}) = (-4)^{1/2}$

$= \sqrt{-4} \notin R$

$\because$ ندرس مجالات الدالة في R

ونظراً لاحتمال ظهور أعداد مركبة عندما يكون الأساس سالب لذلك تم استبعاده .

Properties:



1. $\lim_{x \to \infty} e^x = \infty$
2. $\lim_{x \to -\infty} e^x = 0$
3. $e^x \cdot e^y = e^{x+y}$
4. $\frac{e^x}{e^y} = e^{x-y}$
5. $e^{-x} = \frac{1}{e^x}$
6. $e^0 = 1$

Derivative of Exponential Function

For any constant a,

1. $\frac{d}{dx}[e^x] = e^x$
2. $\frac{d}{dx}[a^x] = a^x \ln a$

Example: Find y' if $y = 5^x$

$$y' = 5^x \ln 5$$

Example: Find y' if $y = x^2 e^x$

$$y = x^2 e^x$$

$$y' = (x^2)'(e^x) + (x^2)(e^x)'$$

$$= 2x\, e^x + x^2 e^x$$

$$= e^x (2x + x^2)$$

Note:
الداله عبارة عن حاصل
ضرب دالة أسيه بدالة
كثيرة حدود لذلك نطبق
قانون تفاضل حاصل
ضرب دالتين

Example: Find y' if $y = \dfrac{3^x}{x + e^x}$

$$y' = \frac{(3^x)' \cdot (x + e^x) - (x + e^x)' \cdot 3^x}{(x + e^x)^2}$$

$$= \frac{3^x \ln 3 \,(x + e^x) - (1 + e^x) \cdot 3^x}{(x + e^x)^2}$$

$$= \frac{3^x \ln 3 x + 3^x \ln 3 e^x - 3^x - 3^x e^x}{(x + e^x)^2}$$

$$= \frac{3^x (\ln 3 x - 1) + 3^x e^x (\ln 3 - 1)}{(x + e^x)^2}$$

$$= \frac{3^x (\ln 3x - 1) + (3e)^x (\ln 3 - 1)}{(x + e^x)^2}$$

Example: Find the points (x, y) on the curve $y = x^2 e^x + 1$ at wich the tangent line is horizontal $y' = 0$

$$y = x^2 e^x + 1$$
$$y' = (x^2)' e^x + x^2 (e^x)' + 0$$
$$= 2x\, e^x + x^2 e^x$$
$$= (x^2 + 2x)\, e^x$$

Horizontal tangents : $y' = 0$

$$(x^2 + 2x)\, e^x = 0$$
$$x^2 + 2x = 0$$
$$x\,(x+2) = 0$$
$$x = 0 \quad \text{or} \quad x + 2 = 0 \Rightarrow x = -2$$

$\therefore$ Points: $(0, f(0))$ and $(-2, f(-2))$

$$f(0) = 0^2 \cdot e^0 + 1 = 1$$
$$f(-2) = (-2)^2 e^{-2} + 1$$
$$= 4e^{-2} + 1$$

$\therefore$ The curve has a horizontal line at $(0, 1)$ and $(-2, 4e^{-2} + 1)$

Example: For what value of x does the curve $f(x) = 2x - e^x$ have any horizontal tangents? Also for what value of x does the tangent line to the curve parallel to $y = -3x$

Horizontal tangents : $f'(x) = 0$

$$2 - e^x = 0$$
$$2 = e^x$$
$$x = \ln 2$$

3 How to solve Exp. and Log. Function Equations

1. Exponintial Function

نفصل الدالة الاسيه 1. Isolate the exponintial expression

يكون لدينا حالتين 2. we will have two possible cases.

Case 1 نفس الأساس
Same base
or
can be written to
have the same base
How to solve
1. Apply Exponintial ruls.
2. Solve for x

Case 2 أساس مختلف
Not the same base
How to solve
ناخذ اللوغاريتم للطرفين 1. Take log of both sides
نطبق خصائص اللوغاريتم 2. Apply logs properties
3. Solve for x

Parallel tangent : $f'(x)$ = slop of the given line $y = mx+b$

$\because y = \overset{m}{-3}x \quad \therefore f'(x) = -3$

$2 - e^x = -3$

$2+3 = e^x$

$5 = e^x$

$\ln 5 = \ln(e^x)$

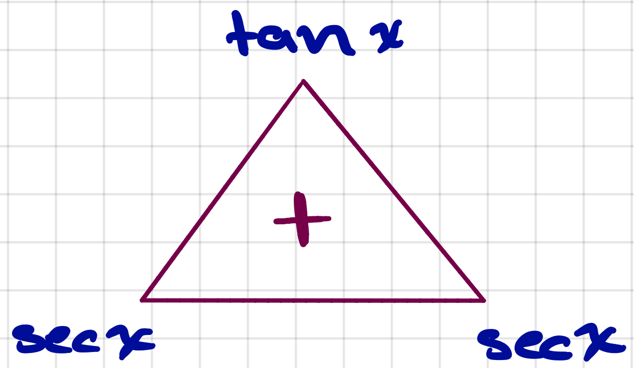
$\ln 5 = x$

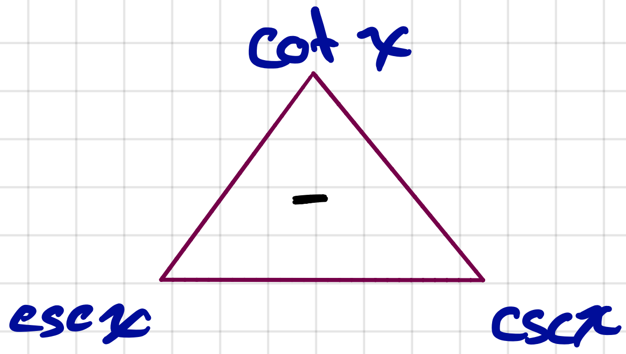
Derivatives of Trigonometric Function

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$





Example: Find y' if

$y = (\sin x + \cos x)\sec x$

Remember:

$\tan x = \dfrac{\sin x}{\cos x}$

$\sec x = \dfrac{1}{\cos x}$

$y = \sin x \sec x + \cos x \sec x$

$= \sin x \cdot \dfrac{1}{\cos x} + \cos x \cdot \dfrac{1}{\cos x}$

$= \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\cos x}$

$= \tan x + 1$

$\because y = \tan x + 1 \Rightarrow y' = \sec^2 x$

$y = \tan x + \sqrt{x}$

$y' = \sec^2 x + \frac{1}{2\sqrt{x}}$

$y = x^2 \cos x - 2x \sin x$

$y' = 2x \cos x + x^2(-\sin x) - 2[1 \cdot \sin x + x \cdot \cos x]$

$= \cancel{2x \cos x} - x^2 \sin x - 2\sin x - \cancel{2x \cos x}$

$= \sin x(-x^2 - 2)$

$y = \frac{\cot x}{1 + \cot x}$

$y' = \frac{(-\csc^2 x) \cdot (1 + \cot x) - (-\csc^2 x)(\cot x)}{(1 + \cot x)^2}$

$= \frac{-\csc^2 x - \cancel{\csc^2 x \cot x} + \cancel{\csc^2 x \cot x}}{(1 + \cot x)^2}$

$= \frac{-\csc^2 x}{(1 + \cot x)^2}$

$y = \sin x \cos x$

$y' = (\sin x)'(\cos x) + (\sin x)(\cos x)'$

$= \cos x(\cos x) + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x$

$y = \tan x + x^2 \cot x$

$y' = \sec^2 x + (2x \cot x + x^2(-\csc^2 x))$

$= \sec^2 x + 2x \cot x - x^2 \csc^2 x$

$$y = \frac{\sin x}{x}$$

$$y' = \frac{\cos x \cdot x - 1 \cdot \sin x}{x^2}$$

$$= \frac{x\cos x - \sin x}{x^2}$$

$$y = \sec x \tan x$$

$$y' = (\sec x)' \cdot (\tan x) + (\sec x) \cdot (\tan x)'$$

$$= (\sec x \tan x)(\tan x) + \sec x \cdot \sec^2 x$$

$$= \sec x \tan^2 x + \sec x \cdot \sec^2 x$$

$$= \sec x \,(\tan^2 x + \sec^2 x).$$

$$y = \cos x \csc x$$

$$y = \cos x \cdot \frac{1}{\sin x}$$

$$y = \frac{\cos x}{\sin x} = \cot x$$

$$\therefore\ y' = -\csc^2 x$$

Remember

$$\csc x = \frac{1}{\sin x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$y = \sin x \csc x$$

$$y = \sin x \cdot \frac{1}{\sin x} = \frac{\sin x}{\sin x} = 1$$

$\therefore\ y = 1$ and $y' = 0$

$y = \dfrac{\tan x}{\sec x}$

$y = \tan x \cdot \dfrac{1}{\sec x}$

$= \dfrac{\sin x}{\cos \cancel{x}} \cdot \cos \cancel{x} = \sin x$

$\therefore y' = \cos x.$

$y = \cos x \sec x$

$y = \cancel{\cos} x \cdot \dfrac{1}{\cancel{\cos} x} = 1$

$\therefore y' = 0$

Example: Find all points on the curve

$$y = 3\tan x \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

where the tangent line is parallel to the line $y = 6x$.

$f'(x) = 6$

$3\sec^2 x = 6$

$3 \dfrac{1}{\cos^2 x} = 6$

$\Rightarrow \dfrac{3}{\cos^2 x} = 6$

$\Rightarrow 6\cos^2 x = 3$

$\Rightarrow \cos^2 x = \dfrac{3}{6} = \dfrac{1}{2}$

$\Rightarrow \cos x = \pm \dfrac{1}{\sqrt{2}}$

$$\Rightarrow \cos x = \frac{1}{\sqrt{2}} \qquad [\cos x \geq 0 \text{ on } -\frac{\pi}{2} < x < \frac{\pi}{2}]$$

$$\Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \Rightarrow x = \pm \frac{\pi}{4}$$

$$\therefore F\left(\frac{\pi}{4}\right) = 3\tan\left(\frac{\pi}{4}\right) = 3.1 = 3$$

$$F\left(-\frac{\pi}{4}\right) = 3\tan\left(-\frac{\pi}{4}\right) = 3.(-1) = -3$$

- The equation of the tangent line at $\left(\frac{\pi}{4}, 3\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = 6\left(x - \frac{\pi}{4}\right)$$

$$\Rightarrow y = 6\left(x - \frac{\pi}{4}\right) + 3$$

- The equation of the tangent line at $\left(-\frac{\pi}{4}, -3\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 3 = 6\left(x + \frac{\pi}{4}\right)$$

$$\Rightarrow y = 6\left(x + \frac{\pi}{4}\right) - 3$$

The Chain Rule

$f(x) = \cancel{\cos(2x+1)}$

$\sin$ $x+1)$.

2

$y = f(g(x))$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

$$\boxed{y' = f'(g(x)) \cdot g'(x)}$$

$f(y) = h(g(y))$

$f'(y) = h'(g(y)) \cdot g'(y)$

$\cos$

Example: If $y = (x^2+1)^3$. Find y'

Method 1

let $u = x^2+1$, $y = u^3$

$\frac{du}{dx} = 2x$, $\frac{dy}{du} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 3u^2 \cdot 2x$$

$$= 3(x^2+1) \cdot 2x$$

$$= 6x(x^2+1)$$

Method 2

$$y' = 3(x^2+1) \cdot 2x$$

$$= 6x(x^2+1)$$

The General power Rule

Example: Find y' if

1. $y = \sqrt[3]{1+x^2}$

$$y = (1+x^2)^{\frac{1}{3}}$$

$$\therefore y' = \frac{1}{3}(1+x^2)^{\frac{1}{3}-1} \cdot 2x$$

$$= \frac{1}{3}(1+x^2)^{-\frac{2}{3}} \cdot 2x$$

$$= \frac{2x}{3\sqrt[3]{(1+x^2)^2}}$$

2. $y = \frac{1}{x^2-1}$

$$y = (x^2-1)^{-1}$$

$$y' = -(x^2-1)^{-1-1} \cdot 2x$$

$$= -(x^2-1)^{-2} \cdot 2x$$

$$= \frac{-2x}{(x^2-1)^2}$$

Example: Let $g(x) = (3x+1)^6 \sqrt[3]{(2x-3)^5}$. Find $g'(x)$.

$$g(x) = (3x+1)^6 (2x-3)^{5/3}$$

$$g'(x) = [(3x+1)^6]'(2x-3)^{5/3} + (3x+1)^6 \cdot [(2x-3)^{5/3}]'$$

$$= [6(3x+1)^5 \cdot 3](2x-3)^{5/3} + (3x+1)^6 \cdot \left[\frac{5}{3}(2x-3)^{\frac{5}{3}-1} \cdot 2\right]$$

$$= 18(3x+1)^5 (2x-3)^{5/3} + (3x+1)^6 \cdot \frac{10}{3}(2x-3)^{2/3}$$

$$= 18(3x+1)^5 \sqrt[3]{(2x-3)^5} + (3x+1)^6 \cdot \frac{10}{3}\sqrt[3]{(2x-3)^2}$$

Example: Find all the points on the graph of

$$g(x) = \sqrt[3]{(x^2-4)^2}$$

for which $g'(x) = 0$ and those for which $g'(x)$ DNE.

$$g(x) = (x^2-4)^{\frac{2}{3}}$$

$$g'(x) = \frac{2}{3}(x^2-4)^{\frac{2}{3}-1} \cdot 2x$$

$$= \frac{4x}{3}(x^2-4)^{-\frac{1}{3}}$$

$$= \frac{4x}{3\sqrt[3]{x^2-4}}$$

$$g'(x) = 0 \iff \frac{4x}{3\sqrt[3]{x^2-4}} = 0 \qquad \frac{a}{b} = 0 \Rightarrow a = 0$$

$$\iff 4x = 0$$

$$\iff x = 0$$

$$g'(x) \text{ DNE} \iff 3\sqrt[3]{x^2-4} = 0 \qquad \frac{a}{b} \text{ DNE} \Rightarrow b = 0$$

3 مجرد عدد ثابت لايؤثر

$$\iff x^2-4 = 0$$

$$\iff x = \pm 2$$

Example : Find y' if $y = (3x - x^2 + \sqrt{x})^6$

$$y' = 6\,(3x - x^2 + \sqrt{x})^5 \cdot \left(3 - 2x + \frac{1}{2\sqrt{x}}\right)$$

Example : If $y = t^2$ and $x = \dfrac{t-1}{t+1}$ find $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$y = t^2$ and $\dfrac{dy}{dt} = 2t$

$x = \dfrac{t-1}{t+1}$ and

$$\frac{dx}{dt} = \frac{1\cdot(t+1) - 1(t-1)}{(t+1)^2} = \frac{\cancel{t} + 1 - \cancel{t} + 1}{(t+1)^2} = \frac{2}{(t+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \cancel{2}t \cdot \frac{(t+1)^2}{\cancel{2}} = t\,(t+1)^2$$

قلبنا لإنا بج

Trigonometric Function and the Chain Rule

Example: Find the derivative of the following functions:

$f(x) = \cos(3x)$

$f'(x) = -\sin(3x) \cdot 3 = -3\sin(3x).$

$f(x) = x^2 + \sin(x^3)$

$f'(x) = 2x + \cos(x^3) \cdot 3x^2$

$= 2x + 3x^2 \cos(x^3).$

$f(x) = \sec^2(\sqrt{x})$

$f(x) = (\sec(\sqrt{x}))^2$

$f'(x) = 2(\sec(\sqrt{x}))^1 (\sec(\sqrt{x}) \tan(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}})$

$= \cancel{2} \cdot \sec^2(\sqrt{x}) \tan(\sqrt{x}) \cdot \frac{1}{\cancel{2}\sqrt{x}}$

$= \dfrac{\sec^2(\sqrt{x}) \tan(\sqrt{x})}{\sqrt{x}}$

$g(x) = \tan(x + \sqrt{x})$

$g'(x) = \sec^2(x + \sqrt{x}) \cdot (1 + \frac{1}{2\sqrt{x}})$

$f(x) = \csc(\cos x)$

$f'(x) = -\csc(\cos x) \cot(\cos x) \cdot (-\sin x)$

$= \sin x \csc(\cos x) \cot(\cos x)$

$g(x) = \tan(\sin(\cos x))$

$g'(x) = \sec^2(\sin(\cos x)) \cdot \cos(\cos x) \cdot (-\sin x)$

$= -\sec^2(\sin(\cos x)) \cdot \cos(\cos x)\sin(x)$

Example:

Let $g(t) = 3t^2 - \cos(t)$ and $f(x) = \sec(x)$.

① Set $y = f(g(t))$. ② Find $\frac{dy}{dt}$

① $y = f(3t^2 - \cos t) = \sec(3t^2 - \cos t)$

② $\frac{dy}{dt} = \underbrace{\sec(3t^2 - \cos t) \cdot \tan(3t^2 - \cos t)}_{\text{sec تفاضل ال}} \cdot \underbrace{(6t + \sin t)}_{\text{sec تفاضل مابداخل ال}}$

الحل بطريقه أخرى

$y = \sec(3t^2 - \cos t)$

let $u = 3t^2 - \cos t$ and $y = \sec(u)$

✓ $\frac{du}{dt} = 6t + \sin t$ and ✓ $\frac{dy}{du} = \sec(u) \cdot \tan(u)$

$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$

$= \sec(u) \cdot \tan(u) \cdot (6t + \sin t)$

$= \sec(3t^2 - \cos t) \cdot \tan(3t^2 - \cos t) \cdot (6t + \sin t).$

Exp.Function and the Chain Rule

If $y = a^{f(x)}$ then $y' = a^{f(x)} \cdot f'(x) \cdot \ln a$

$y = e^{g(x)}$ then $y' = e^{g(x)} \cdot g'(x)$

Example: Find y' if

$$y = 5^{\sqrt{x}}$$

$$y' = 5^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot \ln 5$$

$$= \frac{\ln(5) \cdot 5^{\sqrt{x}}}{2\sqrt{x}}$$

Example: Find the derivative of the following:

$$y = e^{\sec(4x)}$$

$$y' = e^{\sec(4x)} \cdot \sec(4x) \cdot \tan(4x) \cdot 4$$

$$= 4\sec(4x)\tan(4x)\, e^{\sec(4x)}$$

$$f(x) = 2^{x + \csc x}$$

$$f'(x) = 2^{x+\csc x} \cdot (1 - \csc x \cot x) \cdot \ln 2$$

$$= \ln 2\,(1 - \csc x \cot x)\, 2^{x+\csc x}$$

Log.Function and the Chain Rule

$$\frac{d}{dx}\left[\log_a (g(x))\right] = \frac{g'(x)}{g(x)\ln a}$$

$$\frac{d}{dx}\left[\ln (g(x))\right] = \frac{g'(x)}{g(x)}$$

Example: Find y' if

$y = \ln(\sin x)$

$$y' = \frac{\cos x}{\sin x} = \cot x.$$

أين يكون خط المماس أفقي للمنحنى المعطى ؟
آي ماقيمه x التي يكون عندها مماس أفقي للمنحنى؟

Example: Find where the tangent line to the graph $y = \ln(x^3 - x^2 + 4)$ is horizontal

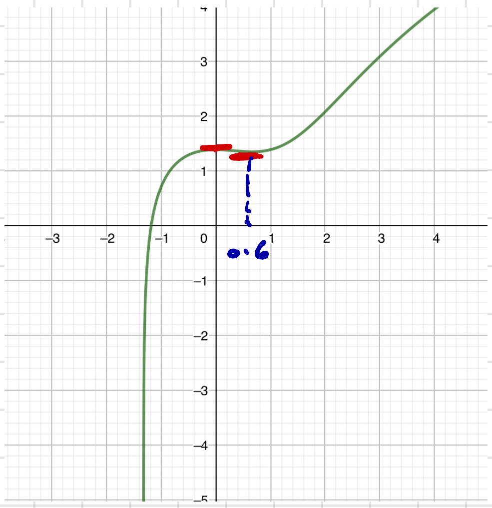
$f'(x) = 0$ ← tangent line horizontal

$$f'(x) = \frac{3x^2 - 2x}{x^3 - x^2 + 4} = \frac{x(3x - 2)}{x^3 - x^2 + 4} = 0$$

$\Rightarrow x(3x - 2) = 0$

$\Rightarrow x = 0$ or $3x - 2 = 0$

$\Rightarrow 3x = 2 \Rightarrow x = \frac{2}{3}$



The graph has two horizontal tangent lines at $x = 0$ and $x = \frac{2}{3}$

Implicit Differentiation and Higher Derivatives

$$\boxed{\frac{d}{dx}[y^n] = ny^{n-1}\,y'}$$

Example: If $x^2+y^2=5$, find the following

1) $$2x + 2yy' = 0$$

$$\Rightarrow 2yy' = -2x$$

$$\Rightarrow y' = \frac{-2x}{2y} = \frac{-x}{y}.$$

2) Equation of the tangent line to $x^2+y^2=5$ at the point $(3/\sqrt{5}, 4/\sqrt{5})$.

$$y - y_1 = m(x - x_1)$$

Remember $m = f'(a)$ or $m = y'\big|_{x=a}$

$$\therefore m = y'(3/\sqrt{5}, 4/\sqrt{5}) = \frac{-3/\sqrt{5}}{4/\sqrt{5}}$$

$$= \frac{-3}{\cancel{\sqrt{5}}} \cdot \frac{\cancel{\sqrt{5}}}{4} = \frac{-3}{4}$$

$$\therefore y - \frac{4}{\sqrt{5}} = \frac{-3}{4}\left(x - \frac{3}{\sqrt{5}}\right)$$

$$\Rightarrow y = \frac{-3}{4}\left(x - \frac{3}{\sqrt{5}}\right) + \frac{4}{\sqrt{5}}$$

$$= \frac{-3}{4}x + \frac{9}{4\sqrt{5}} + \frac{4}{\sqrt{5}}$$

$$= -\frac{3}{4}x + \frac{9\sqrt{5} + 16\sqrt{5}}{4\cdot 5}$$

$$= \frac{-3}{4} x + \frac{\cancel{25}\sqrt{5}}{\cancel{20}}$$

$$= \frac{-3}{4} x + \frac{5\sqrt{5}}{4}$$

Example: If $y^3 + y^2 - 5y - x^2 = -4$. Find the following:

1)- y'

$$3y^2y' + 2yy' - 5y' - 2x = 0$$

$$y'(3y^2 + 2y - 5) = 2x$$

$$y' = \frac{2x}{3y^2 + 2y - 5}$$

2)- Equation of the tangent line to $y^3 + y^2 - 5y - x^2 = -4$ at the point $(3,-1)$

$$y - y_1 = m(x - x_1)$$

$$m = y'(3,-1) = \frac{2 \cdot 3}{3(-1)^2 + 2(-1) - 5} = \frac{6}{3-2-5} = \frac{6}{-4} = -\frac{3}{2}$$

$$\therefore \quad y - (-1) = -\frac{3}{2}(x - 3)$$

$$y = -\frac{3}{2}(x - 3) - 1$$

$$= -\frac{3}{2}x + \frac{9}{2} - 1$$

$$= -\frac{3}{2}x + \frac{7}{2}$$

Example: Compute the slope of the tangent line to the curve $\sin(xy) = x$ at the point $\left(\frac{1}{2}, \frac{\pi}{3}\right)$.

$$\text{slope} = y'(x, y).$$

$$\frac{d}{dx}\left[\sin(xy)\right] = \cos(xy) \cdot (x \cdot y' + 1 \cdot y) = 1$$

$$\cos(xy) \cdot (xy' + y) = 1$$

$$xy' + y = \frac{1}{\cos(xy)}$$

$$\Rightarrow xy' = \frac{1}{\cos(xy)} - y$$

$$y' = \frac{1}{x}\left(\frac{1}{\cos(xy)} - y\right)$$

$$\therefore \text{slope} = y'\left(\frac{1}{2}, \frac{\pi}{3}\right) = \frac{1}{(1/2)}\left[\frac{1}{\cos\left(\frac{1}{2} \cdot \frac{\pi}{3}\right)} - \frac{\pi}{3}\right]$$

$$= 2\left[\frac{1}{\cos(\pi/6)} - \frac{\pi}{3}\right]$$

$1 \cdot \frac{2}{\sqrt{3}}$

$$= 2\left[\frac{1}{\sqrt{3}/2} - \frac{\pi}{3}\right]$$

$$= 2\left[\frac{2}{\sqrt{3}} - \frac{\pi}{3}\right]$$

$$= \frac{4}{\sqrt{3}} - \frac{2\pi}{3}$$

Example : Find the equation of the tangent line to the graph of $y = 2x^2y - 3y = x$ at the point $(1,-1)$.

$$y - y_1 = m(x - x_1)$$

$$m = y'(1,-1)$$

$$\frac{d}{dx}\left[2x^2y - 3y = x\right] = 4xy + 2x^2y' - 3y' = 1$$

$$\Rightarrow 2x^2y' - 3y' = 1 - 4xy$$

$$\Rightarrow y'(2x^2 - 3) = 1 - 4xy$$

$$\Longrightarrow y' = \frac{1 - 4xy}{x^2 - 3}$$

2

$$\therefore m = y'(1,-1) = \frac{1 - 4(1)(-1)}{2(1)^2 - 3} = \frac{5}{-1} = -5$$

$$\therefore y - (-1) = -5(x - 1)$$

$$y + 1 = -5x + 5$$

$$y = -5x + 5 - 1$$

$$y = -5x + 4$$

Example: Find y' if $y = \cos(x-y) = x e^{x}$

تفاضل ال cos   تفاضل مابداخل ال cos   تفاضل ضرب دالتين

$$-\sin(x-y)(1-y') = 1\cdot e^{x} + x\cdot e^{x}$$

$$-\sin(x-y) + y'\sin(x-y) = e^{x}(1+x)$$

$$y'\sin(x-y) = e^{x}(1+x) + \sin(x-y)$$

$$y' = \frac{e^{x}(1+x)}{\sin(x-y)} + \frac{\sin(x-y)}{\sin(x-y)}$$

$$= \frac{e^{x}(1+x)}{\sin(x-y)} + 1$$

$$= e^{x}(1+x)\cdot\frac{1}{\sin(x-y)} + 1$$

$$= e^{x}(1+x)\cdot\csc(x-y) + 1$$

Remember $\frac{1}{\sin x} = \csc x$

Logarithmic Differentiation

- Taking natural logarithms for both sides.
- Applying the properties of logarithms.
- Differentiating with respect to x.
- Solving for y'.
- Replacing y by $f(x)$.

نأخذ اللوغاريتم للطرفين

نطبق خصائص اللوغاريتم

نفاضل بالنسبة لx

نحل بالنسبة لy^

نبدل الy بقيمة f(x) المعطاه في السؤال

If $y = [g(x)]^{f(x)}$ (دالة , دالة) we will use Log. D to find y'

Example: Find y' if $y = x^{x+2}$

$$y = x^{x+2}$$

$$\ln y = \ln x^{x+2}$$

$$\ln y = (x+2)\ln(x)$$

$$\frac{y'}{y} = (1)\cdot\ln(x) + (x+2)\cdot\frac{1}{x}$$

$$\frac{y'}{y} = \ln(x) + \frac{x+2}{x}$$

$$y' = y\left[\ln(x) + \frac{x+2}{x}\right]$$

$$y' = x^{x+2}\left[\ln(x) + \frac{x+2}{x}\right].$$

Example: Find y' if $y = \left(\frac{3^x}{x+e^x}\right)$

$$\ln y = \ln\left[\frac{3^x}{x+e^x}\right]$$

$$\ln y = \ln(3^x) - \ln(x+e^x)$$

$$\frac{y'}{y} = \frac{\cancel{3^x}\ln 3}{\cancel{3^x}} - \frac{1+e^x}{x+e^x}$$

$$y' = y\left[\ln(3) - \frac{1+e^x}{x+e^x}\right]$$

لإيجاد تفاضل هذه الداله نحتاج الى تطبيق
قانون تفاضل الداله الكسرية لكي نوجد y.
ماذا لو طبقنا Log. Differentiation ؟
هل سنحصل على نفس النتيجة.
Let’s try ! and see Ex3 in the lecture of Exp.function

$$y' = \frac{3^x}{x+e^x}\left[\frac{(x+e^x)\ln(3) - (1+e^x)}{x+e^x}\right]$$

$$= \frac{3^x(x+e^x)\ln(3) - 3^x(1+e^x)}{(x+e^x)^2}$$

$$= \frac{3^x \ln 3x + 3^x e^x \ln(3) - 3^x - 3^x e^x}{(x+e^x)^2}$$

$$= \frac{3^x(\ln 3x - 1) + 3^x e^x(\ln 3 - 1)}{(x+e^x)^2}$$

Same result

اذا من الممكن إيجاد تفاضل
الدالة الكسرية بطريقة
Log.Differentiation

H.W: Find y' if $y = \dfrac{(2x-1)^2(x^2+1)^3}{\sqrt{x^4+1}}$

Higher Order Derivatives

The following notations for higher derivatives, with y= f(x) are usually used

$f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x), \cdots f^{(n)}(x)$
$y', y'', y''', y^{(4)}, \cdots y^{(n)}$
$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \cdots, \frac{d^ny}{dx^n}$
$D_xy, D_x^2y, D_x^3y, D_x^4y, \ldots, D_x^ny$

Example : Find the <u>third</u> derivative of $f(x) = x^{\frac{1}{2}} + x^3$

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} + 3x^2$$

$$f''(x) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1} + 3\cdot 2x$$

$$= -\frac{1}{4} x^{-\frac{3}{2}} + 6x$$

$$f'''(x) = \left(-\frac{3}{2}\right)\left(-\frac{1}{4}\right) x^{-\frac{3}{2}-1} + 6$$

$$= \frac{3}{8} x^{-\frac{5}{2}} + 6$$

$$= \frac{3}{8}\cdot\frac{1}{\sqrt[2]{x^5}} + 6$$

$$= \frac{3}{8\sqrt{x^5}} + 6$$

Remember

$$x^{\frac{m}{n}} = \sqrt[n]{x^m}$$

Note:

إذا كانت $n=2$

هذا يعني الجذر

التربيعي وبالتالي

نعمل عدم كتابته

Example : Find y'' if $xy^3 = 2$

$$1 \cdot y^3 + x \cdot 3y^2 y' = 0$$

$$3xy^2 y' = -y^3$$

$$y' = \frac{-\cancel{y^3}}{3x\cancel{y^2}} = \frac{-y}{3x} = -\frac{1}{3}\left(\frac{y}{x}\right)$$

$$y'' = -\frac{1}{3}\left[\frac{y' \cdot x - 1 \cdot y}{x^2}\right]$$

$$= -\frac{1}{3}\left[\frac{\left(\frac{-y}{3\cancel{x}}\right) \cdot \cancel{x} - y}{x^2}\right]$$

$$= \frac{-1}{3}\left[\frac{\frac{-y}{3} - y}{x^2}\right]$$

$$\frac{-y}{3} - \frac{y}{1} = \frac{-y-3y}{3} = \frac{-4y}{3}$$

$$= -\frac{1}{3}\left[\frac{\frac{-4y}{3}}{x^2}\right]$$

$$= \frac{\frac{4y}{3}}{3x^2} = \frac{4y}{3} \cdot \frac{1}{3x^2} = \frac{4y}{9x^2}$$

Example : Find $D_x^{25}(\sin x)$

$$D_x^{1}(\sin x) = \cos x$$

$$D_x^{2}(\sin x) = -\sin x$$

$$D_x^{3}(\sin x) = -\cos x$$

$$D_x^{4}(\sin x) = \sin x$$

$$25 \div 4 = 6, \quad 25 - 24 = 1$$

$$\therefore D_x^{25}(\sin x) = D_x^{1}(\sin x) = \cos x.$$

Example : Find the n^{th} derivatives of the function

$$f(x) = x^4 - x^3 + x^2 - \pi x + 4$$

$$f'(x) = 4x^3 - 3x^2 + 2x - \pi$$

$$f''(x) = 12x^2 - 6x + 2$$

$$f'''(x) = 24x - 6$$

$$f^{(4)}(x) = 24$$

$$f^{(5)}(x) = 0$$

Applications of Differentiation

L'Hopital's Rule

Maximum and Minimum values

Rollie's Theorem and the Main Value Theorem

Monotonicity and the First Derivative Test

Concavity and Second Derivative Test

L'Hopital's Rule

كميات غير محدده

Indeterminate Forms : I.F

The following expression are called I.F

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, ∞^0 , 0^0 , 1^∞ and $\infty - \infty$

L'Hopital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

where a can be real number.

Example:-

$$\lim_{x \to \infty} \frac{x^3+1}{x^2-1} = \frac{\infty^3+1}{\infty^2-1} = \frac{\infty}{\infty} \quad (I.F)$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{3x^2}{2x} \qquad \left(\frac{\infty}{\infty} \text{ I.F again}\right)$$

$$\overset{H}{=} \lim_{x \to \infty} \frac{6x}{2}$$

$$= \lim_{x \to \infty} 3x = \infty$$

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \frac{\ln(\infty)}{\sqrt[3]{\infty}} = \frac{\infty}{\infty} \quad (I.F)$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3x^{2/3}}} \xrightarrow{\text{How}}$$

$$\sqrt[3]{x} = x^{1/3} \quad \text{and}$$

$$\frac{d}{dx}[x^{1/3}] = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x} \cdot \frac{3x^{\frac{2}{3}}}{1} \right)$$

$$= \lim_{x\to\infty} \frac{3\cancel{x}^{\cancel{\frac{2}{3}}}}{\cancel{x}}$$

$$x^{\frac{2}{3}-1} = x^{\frac{2-3}{3}} = x^{-\frac{1}{3}}$$

$$= \lim_{x\to\infty} \frac{3}{x^{1/3}} = \frac{3}{(\infty)^{1/3}} = 0$$

Example:

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{1-1}{1+1} = \frac{0}{2} = 0 \quad (\text{not } I.F)$$

no need to use L.R

لا داعي لإستخدام قاعدة لوبيتال

What happen if we use L.R

ماذا يحدث لو استخدمنا قاعدة لوبيتال

اجابة خاطئة

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \lim_{x\to 1} \frac{2x}{1} = 2 \quad (\text{wrong answer})$$

$$\lim_{x\to 0}\frac{\sin x}{x^2}=\frac{\sin(0)}{0^2}=\frac{0}{0}\quad (I.F)$$

$$\overset{H}{=}\lim_{x\to 0}\frac{\cos x}{2x}=\frac{\cos(0)}{2\cdot 0}=\frac{1}{0}$$

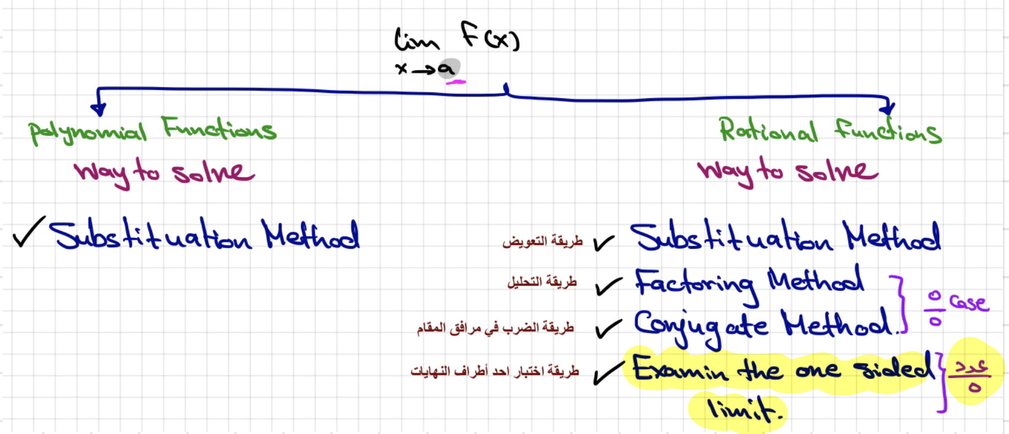
نختبر النهايات عن يمين الحد ويسارة

$$\lim_{x\to 0^+}\frac{\cos x}{2x}=+\infty$$

$$\lim_{x\to 0^-}\frac{\cos x}{2x}=-\infty$$

$$\lim_{x\to 0}\frac{\cos x}{2x}\quad \text{D.N.E}$$

Remember:



لمزيد من المعلومات راجعي درس Limits في ملف
رياضيات ١ على الرابط التالي
https://t.me/MadaAltiary

Example:

$$\lim_{x\to(\frac{\pi}{2})^+}\frac{2\tan x}{1+\sec x}=\frac{2\tan(\frac{\pi}{2})}{1+\sec(\frac{\pi}{2})}=\frac{\infty}{\infty}\quad (I.F)$$

$$\overset{H}{=}\lim_{x\to(\pi/2)^+}\frac{2\cancel{\sec}^{2}x}{\cancel{\sec}x\tan x}$$

$$=\lim_{x\to(\pi/2)^+}\frac{2\sec x}{\tan x}$$

$$=\lim_{x\to(\pi/2)^+}\frac{2\cdot\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

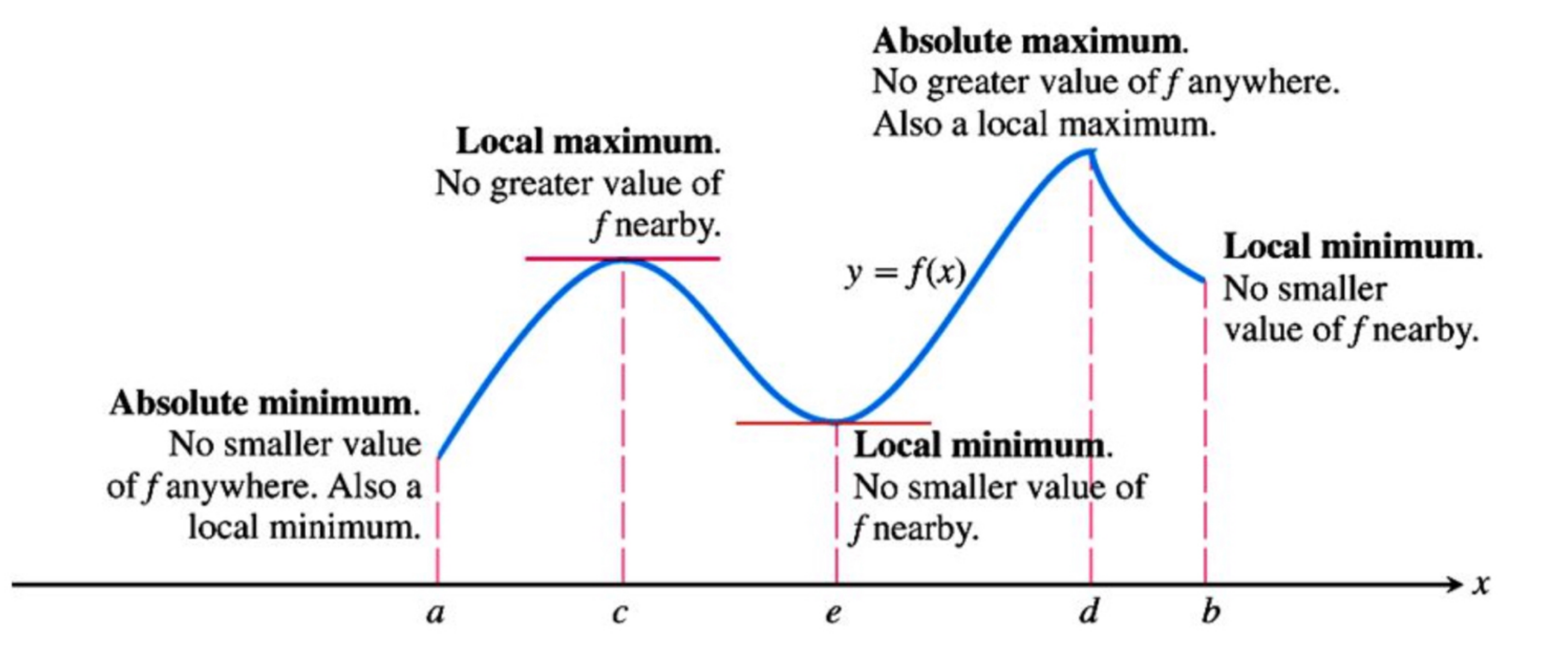
$$=\lim_{x\to(\pi/2)^+}\frac{2}{\cancel{\cos x}}\cdot\frac{\cancel{\cos x}}{\sin x}$$

$$=\lim_{x\to(\pi/2)^+}\frac{2}{\sin x}=\frac{2}{\sin(\frac{\pi}{2})}=\frac{2}{1}=2$$

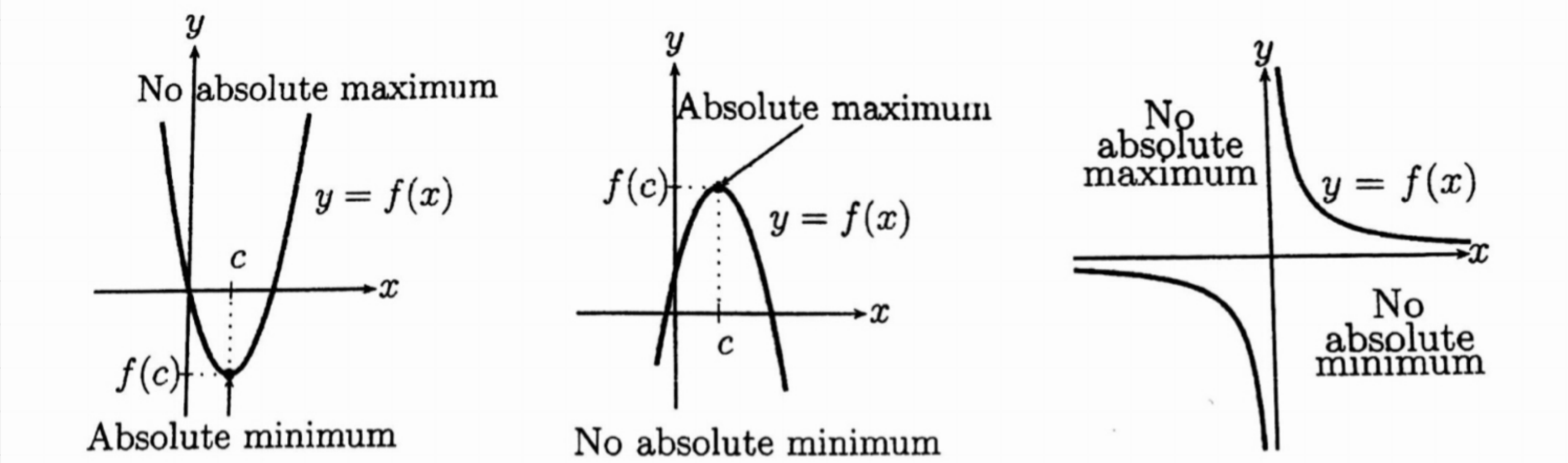
Maximum and Minimum Values

Extreme values:

max and min

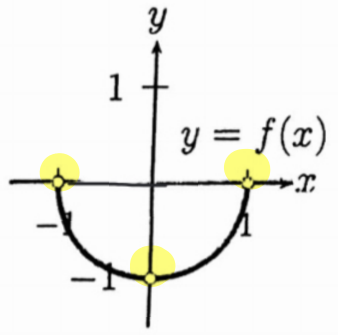


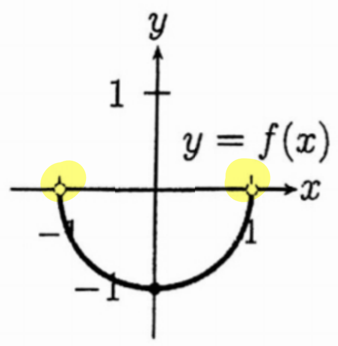
Absolute Extreme Values	Local Extreme Values
$f(c)$ is an :	$f(c)$ is an
• Absolute min of f if: $f(c) \leq f(x)\ \forall x \in D(f)$	Local min of f: $f(c) \leq f(x)\ \forall x$ in some open interval containing a.
• Absolute max of f if: $f(c) \geq f(x)\ \forall x \in D(f)$	Local max of f: $f(c) \geq f(x)\ \forall x$ in some open interval containing a.

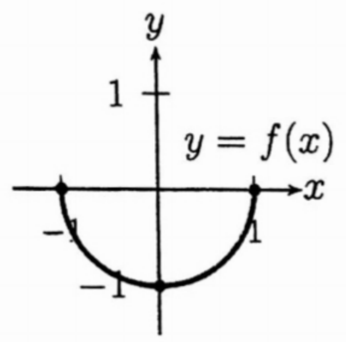


Remark: Every absolute extremum is a local extremum but the converse is not true always.

Example 1: Determine the absolute extreme for the given graphs.





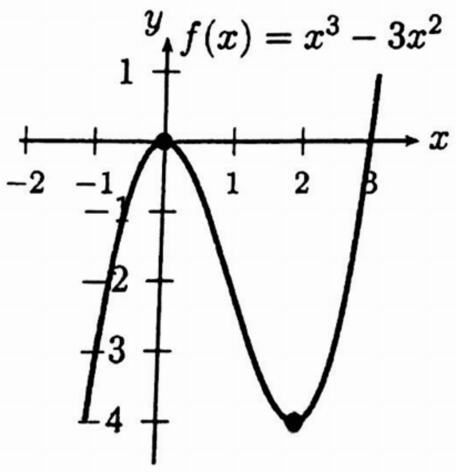


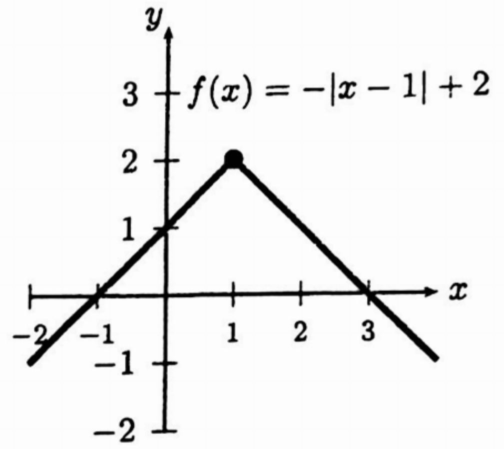
f has no absolute max nor absolute min	f has absolute min at $x=0$ with the value $f(0) = -1$	f has absolute max at $x=\pm 1$ with value $f(\pm 1) = 0$ f has absolute min at $x=0$ with value $f(0) = -1$

Critical numbers :

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ DNE

Example: Find the value of the derivative at each of the local extremum shown in the following figures.





$f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x$

$f'(0) = 3(0)^2 - 6(0) = 0$

$f'(2) = 3(2)^2 - 6(2) = 0$

$\Rightarrow x = 0, 2$ are critical numbers of f

$f(x) = -|x-1| + 2$

$$f(x) = \begin{cases} -(x-1) + 2, & x \geq 1 \\ (x-1) + 2, & x \leq 1 \end{cases}$$

$$f'(x) = \begin{cases} -1, & x \geq 1 \\ 1, & x \leq 1 \end{cases}$$

$\Rightarrow x = 1$ is the critical number of f since $f'(1) = \pm 1$

$\Rightarrow f'(1)$ DNE

$f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x$

$f'(0) = 3(0)^2 - 6(0) = 0$

$f'(2) = 3(2)^2 - 6(2) = 0$

$\Rightarrow x = 0, 2$ are critical numbers of f.

Example: Find the critical numbers of $f(x) = x^3 - \frac{3}{2}x^2 + 1$

$$f'(x) = 3x^2 - \frac{3}{2} \cdot 2x$$
$$= 3x^2 - 3x$$
$$= 3x(x-1)$$

$f'(x) = 0 \Rightarrow 3x(x-1) = 0$

$\Rightarrow 3x = 0 \Rightarrow x = 0$ or $x - 1 = 0 \Rightarrow x = 1$

$\because D(f) = \mathbb{R} \Rightarrow x = 0, 1$ are the critical numbers

Example: Find the critical numbers of $f(x) = 3x^{\frac{1}{3}} + \frac{3}{2}x^{\frac{4}{3}}$

$$f'(x) = 3 \cdot \frac{1}{3} x^{-2/3} + \frac{\cancel{3}}{2} \cdot \frac{4}{\cancel{3}} x^{\frac{1}{3}}$$
$$= x^{-2/3} + 2x^{\frac{1}{3}}$$
$$= x^{-2/3}(1 + 2x)$$
$$= \frac{1+2x}{x^{2/3}}$$

$$x^{-\frac{2}{3}} \cdot x$$
$$= x^{-2/3+1}$$
$$= x^{\frac{1}{3}}$$

$f'(x) = 0 \Rightarrow 1 + 2x = 0$

$\Rightarrow 2x = -1 \Rightarrow x = \frac{-1}{2}$

$f'(x)$ undefined $\Rightarrow x^{2/3} = 0 \Rightarrow x = 0$

$\because D(f) = \mathbb{R} \Rightarrow x = 0, -\frac{1}{2}$ are the critical numbers.

Example: Find the critical numbers of $f(x) = \frac{x^2-1}{x^3}$.

$$f'(x) = \frac{2x \cdot x^3 - 3x^2(x^2-1)}{(x^3)^2}$$

$$= \frac{2x^4 - 3x^4 + 3x^2}{x^6}$$

$$= \frac{3x^2 - x^4}{x^6}$$

$$= \frac{x^2(3-x^2)}{x^6}$$

$$= \frac{3-x^2}{x^4}$$

$f'(x) = 0 \Rightarrow 3 - x^2 = 0 \Rightarrow 3 = x^2$

$\Rightarrow x = \pm\sqrt{3}$

$f'(x)$ undefined $\Rightarrow x^4 = 0 \Rightarrow x = 0$

$\because D(f) = \mathbb{R} - \{0\} \Rightarrow x = \pm\sqrt{3}$ are the only critical number of f

Fermat's theorem:-

If f has local extremum at c, then c is a critical number of f.

Finding Extreme on a closed interval

كيفية إيجاد القيم العظمى والصغرى في فترة مغلقة :-

١- نوجد النقاط الحرجة للدالة

٢- نعوض بالنقاط الحرجة في الدالة

٣- نعوض بأطراف الفترة في الدالة.

٤- أصغر قيمة من القيم الناتجة تكون القيمة الصغرى المطلقة
وأكبر قيمة منهم تكون القيمة العظمى المطلقة.

Example:- Find the absolute maximum and minimum of $f(x) = x^2 - 4x$ on $[0,3]$

$$f'(x) = 2x - 4$$

$$f'(x) = 0 \Rightarrow 2x - 4 = 0$$

$$\Rightarrow 2x = 4 \Rightarrow x = 2$$

$f(0) = 0 \rightarrow$ Absolute max.

$f(3) = 3^2 - 4\cdot3 = -3$ ✓

$f(2) = 2^2 - 4\cdot2 = -4 \rightarrow$ Absolute min

Example: Find the absolute maximum and minimum of $f(x) = 3x^{2/3} - 2x$ on $[-1, 8]$.

$$f'(x) = \cancel{3}\cdot\frac{2}{\cancel{3}}\, x^{-\frac{1}{3}} - 2$$

$$= 2x^{-\frac{1}{3}} - 2$$

$$= x^{-1/3}\left(2 - 2x^{\frac{1}{3}}\right)$$

$$= \frac{2 - 2x^{1/3}}{x^{1/3}}$$

$$f'(x) = 0 \Rightarrow 2 - 2x^{1/3} = 0$$

$$\Rightarrow 2x^{1/3} = 2$$

$\Rightarrow x^{1/3} = 1 \Rightarrow x = 1$

$f'(x)$ undefined $\Rightarrow \quad x^{1/3} = 0 \Rightarrow x = 0$

$\because \; 0, 1 \in [-1, 8] \Rightarrow 0, 1$ are the critical

numbers of f.

$f(-1) = 3(-1)^{2/3} - 2(-1) = 3 + 2 = 5 \rightarrow$ Absolute max

$f(1) = 3(1)^{2/3} - 2(1) = 3 - 2 = 1$

$f(0) = 3(0)^{2/3} - 2(0) = 0$

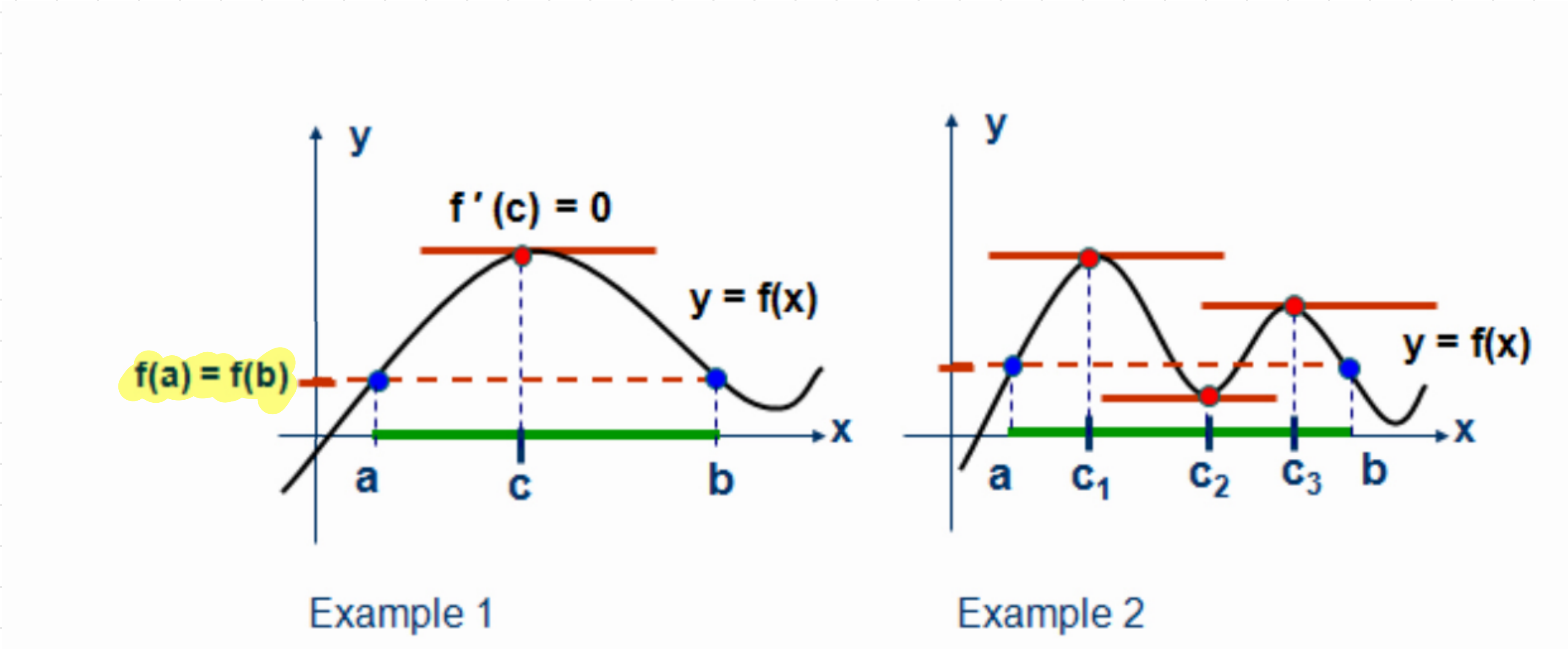
$f(8) = 3(8)^{2/3} - 2(8) = 12 - 16 = -4$ Absolute min

Rolle's Theorem and the Mean Value Theorem

Rolle's theorem: Let f be

1. continuous function on closed interval $[a,b]$
2. differentiable on open interval (a,b), and
3. $f(a) = f(b)$.

then there is a number $c \in (a,b)$ s.t $f'(c) = 0$



Example:- Let $f(x) = x^4 - 2x^2$. Find all value of c in the interval $[-2, 2]$ s.t $f'(c) = 0$.

1)- f is cont. on $[-2, 2]$

2)- f is diff on $(-2, 2)$

3)- $\left.\begin{array}{l} f(-2) = (-2)^4 - 2(-2)^2 = 16 - 8 = 8 \\ f(2) = 2^4 - 2(2)^2 = 16 - 8 = 8 \end{array}\right\}$ $f(-2) = f(2)$

$\exists\ c \in (-2, 2)$ s.t $f'(c) = 0$

$$f'(x) = 4x^3 - 4x$$

$$f'(c) = 0 \Rightarrow 4c^3 - 4c = 0$$

$$\Rightarrow 4c(c^2 - 1) = 0$$

$$\Rightarrow 4c = 0 \quad \text{or} \quad c^2 - 1 = 0$$

$$\Rightarrow c = 0 \quad \text{or} \quad c^2 = 1$$

$$\Rightarrow c = 0 \quad \text{or} \quad c = \pm 1$$

Example: Let $f(x) = (1-x)^{2/3} + 1$. Show that $f(0) = f(2)$ but there is no $c \in (0, 2)$ s.t $f'(c) = 0$

$$f(0) = (1-0)^{2/3} + 1 = 1 + 1 = 2$$

$$f(2) = (1-2)^{2/3} + 1 = (-1)^{2/3} + 1 = \sqrt[3]{(-1)^2} + 1 = \sqrt[3]{1} + 1 = 2$$

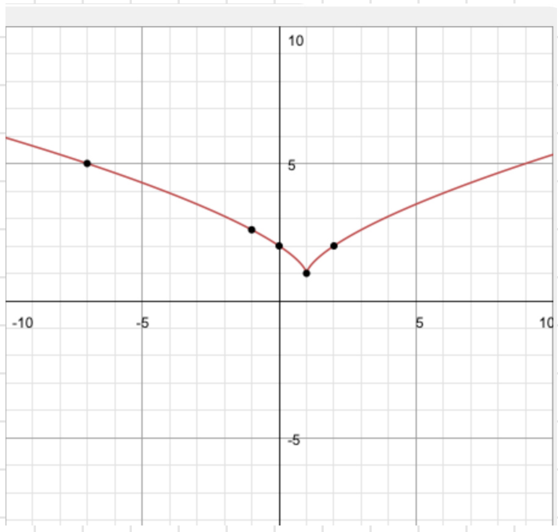
$$\therefore f(0) = f(2)$$

$$f'(x) = \frac{2}{3} \cdot (1-x)^{2/3 - 1} \cdot (-1)$$

$$= -\frac{2}{3}(1-x)^{-\frac{1}{3}}$$

$$= \frac{-2}{3(1-x)^{\frac{1}{3}}}$$

$$= \frac{-2}{3\sqrt[3]{1-x}}$$



f is not differentiable at $x = 1$.

Since $f'(x)$ undefined at $x = 1$

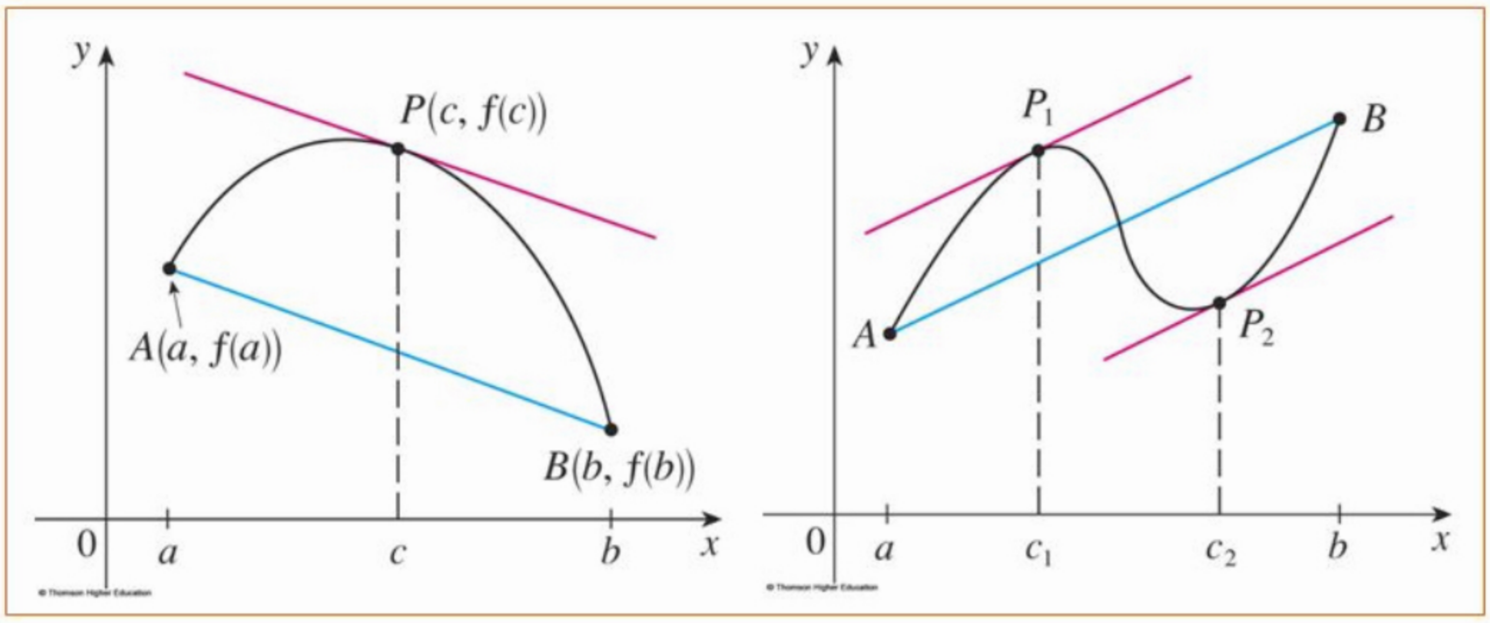
The Mean Value Theorem MVT

MVT Let f be

1) continuous function on closed interval $[a,b]$

2) differentiable function on open interval (a,b)

then there is a number $c \in (a,b)$ s.t $f'(c) = \dfrac{f(b)-f(a)}{b-a}$



Example: Let $f(x) = 2 - \dfrac{3}{x}$ Find all values of c in the interval $(1, 3)$ s.t $f'(c) = \dfrac{f(3) - f(1)}{3-1}$

$\because$ $D(f) = \mathbb{R} - \{0\} \Rightarrow f$ is continuous on $[1,3]$.

f is differentiable on $(1, 3)$.

$\therefore \exists\, c \in (1,3)$ s.t $f'(c) = \dfrac{f(3)-f(1)}{3-1}$

$f(3) = 2 - \dfrac{3}{3} = 2 - 1 = 1$

$f(1) = 2 - \dfrac{3}{1} = 2 - 3 = -1$

$$\therefore f'(c) = \frac{f(3) - f(1)}{3-1} = \frac{1-(-1)}{2} = \frac{2}{2} = 1$$

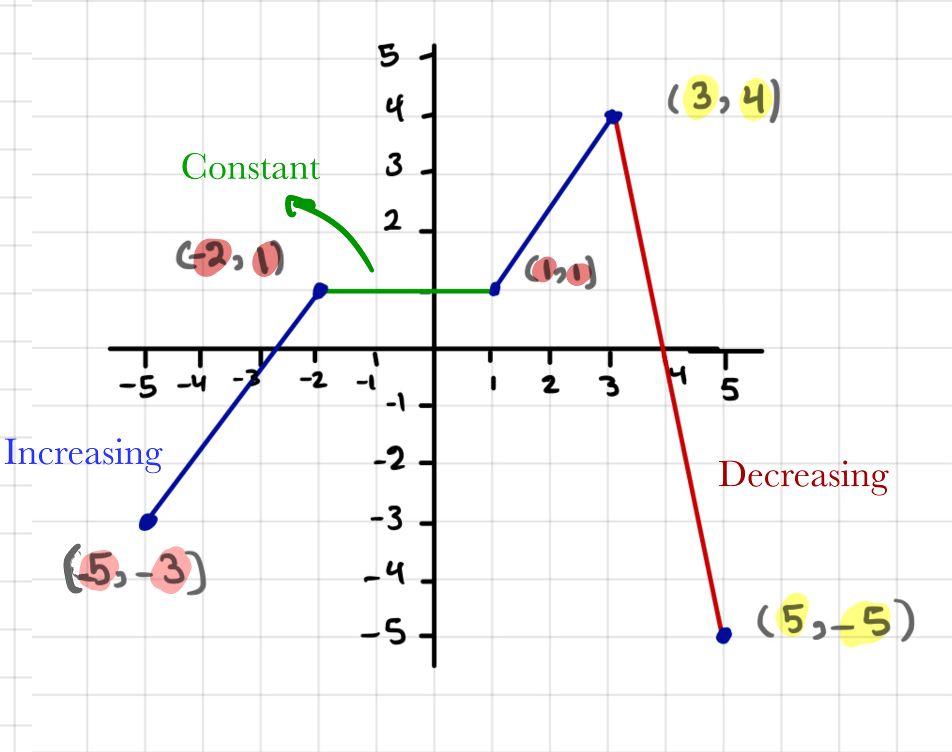
$$f'(x) = \frac{3}{x^2} \Rightarrow f'(c) = \frac{3}{c^2}$$

$$\Rightarrow \frac{3}{c^2} = 1 \Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}$$

$$\because c = -\sqrt{3} \notin (1, 3) \Rightarrow c = \sqrt{3}.$$

Monotonocity and The First Derivative Test

Definition for Monotonic Function:



Increasing: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Decreasing: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Constant: $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$

Test for Monotonic Function:

Increasing: $f'(x) > 0$

Decreasing: $f'(x) < 0$

constant: $f'(x) = 0$

Example 1:

Find the intervals on which $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ is increasing or decreasing

$f'(x) = 0$

1. $12x^3 - 12x^2 - 24x = 0$

$12x(x^2 - x - 2) = 0$

$12x(x-2)(x+1) = 0$

$\Rightarrow x = 0 \quad , x = 2 , \quad x = -1$

$\because D(f) = \mathbb{R} \Rightarrow -1, 0$ and 2 are critical numbers.

2)

	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, \infty)$
Test value	-2	-0.5	1	3
f'(Test value)	-96	7.5	-24	144
Sing of f'	$-$	$+$	$-$	$+$
	Dec	Inc	Dec	Inc

3) f increasing on the intervals $(-1, 0) \cup (2, \infty)$

decreasing on the intervals $(-\infty, -1) \cup (0, 2)$

Example 2:

Find the intervals on which $f(x) = (x^2 - 1)^{2/3}$ is increasing or decreasing

1)_ $f'(x) = \frac{2}{3}(x^2-1)^{-1/3}(2x)$

$= \dfrac{4x}{3(x^2-1)^{1/3}}$

$= \dfrac{4x}{3\sqrt[3]{(x+1)(x-1)}}$

$f'(x) = 0$

$4x = 0$

$x = 0$

$f'(x)$ undefined

$\sqrt[3]{(x+1)(x-1)} = 0$

$\Rightarrow x = \pm 1$

$\because D(f) = \mathbb{R}$

$\therefore -1, 0, 1$ are critical numbers of f.

2)_

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Test value	-2	-0.5	1	2
f'(Test value)	-1.8	0.7	-0.7	1.8
Sing of f'	$-$	$+$	$-$	$+$
	Dec	Inc	Dec	Inc

3)_ f is increasing on $(-1, 0) \cup (1, \infty)$

Decreasing on $(-\infty, -1) \cup (0, 1)$

First Derivative test and local Extremum:

1) If $f'(x)$ changes from $+$ to $-$ at c $\Rightarrow$ f has a local max at c

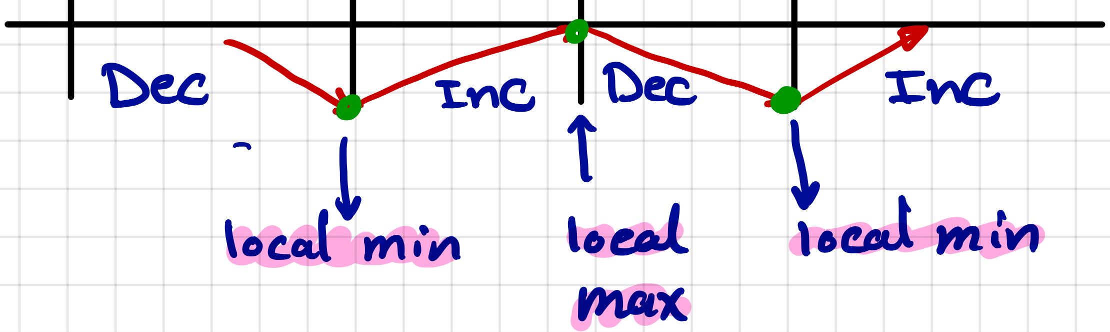
2) If $f'(x)$ changes from $-$ to $+$ at c $\Rightarrow$ f has a local min at c

3) If $f'(x)$ does n't change at c $\Rightarrow$ f has no local max or min at c.

Example 1 : Find the local extreme for $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

From Example 1 we have:

	$-\infty$ to -1	-1 to 0	0 to 2	2 to ∞
Test value	-2	-0.5	1	3
f'(Test value)	-96	7.5	-24	144
sing of f'	$-$	$+$	$-$	$+$



$\therefore$ f has local max at $x = 0$ with value $f(0) = 1$

local min at $x = -1$ with value $f(-1) = -4$

local min at $x = 2$ with value $f(2) = -31$

Example 2: Find the local extreme for $f(x) = \frac{x}{2} + \sin x$ in the interval $(0, 2\pi)$

$$f'(x) = \frac{1}{2} + \cos x$$

$$f'(x) = 0 \Rightarrow \frac{1}{2} + \cos x = 0$$

$$\Rightarrow \cos x = -\frac{1}{2}$$

$$\Rightarrow x = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow x = \frac{2\pi}{3}, \quad x = \frac{4\pi}{3}$$

$\because D(f) = (0, 2\pi)$

$\therefore \frac{2\pi}{3}, \frac{4\pi}{3}$ are the critical numbers.

$0 \qquad \frac{2\pi}{3} \qquad \frac{4\pi}{3} \qquad 2\pi$

Test Value	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
f'(Test Value)	$f'(\frac{\pi}{2}) = \frac{1}{2}$	$f'(\pi) = \frac{-1}{2}$	$f'(\frac{3\pi}{2}) = \frac{1}{2}$
sign of $f'(x)$	+	−	+
Concl	Inc.	Dec.	Inc.

local max

$$F\left(\frac{2\pi}{3}\right) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\pi}{3} + \sin\left(\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

local min

$$F\left(\frac{4\pi}{3}\right) = \frac{1}{2} \cdot \frac{4\pi}{3} + \sin\left(\frac{4\pi}{3}\right)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

Example 3: Find the local extreme for $F(x) = \frac{x}{x^2+1}$ and Find the intervals on which f is increasing and decreasing.

$$F(x) = x(x^2+1)^{-1}$$

$$F'(x) = (1)(x^2+1)^{-1} - 2x(x^2+1)^{-2} \cdot (x)$$

$$= (x^2+1)^{-1} - 2x^2(x^2+1)^{-2}$$

$$= (x^2+1)\frac{(x^2+1)^{-1}}{(x^2+1)} - 2x^2(x^2+1)^{-2}$$

$$= (x^2+1)(x^2+1)^{-2} - 2x^2(x^2+1)^{-2}$$

$$= (x^2+1)^{-2}\left[x^2+1 - 2x^2\right]$$

$$= (x^2+1)^{-2}\left[-x^2+1\right]$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$= \frac{(1-x)(1+x)}{(x^2+1)^2}$$

never zero

$F'(x)=0 \Rightarrow (1-x)(1+x)=0$

$\Rightarrow x = \pm 1$

$\because D(f) = \mathbb{R} \Rightarrow \pm 1$ are critical numbers for f.

	$-\infty$		∞
Test value	-2	0	2
f'(Test value)	$-3/25$	0	$-1/2$
Sing of f'	$-$	$+$	$-$
	Dec	Inc	Dec

local min $f(-1) = -\frac{1}{2}$

local max $f(1) = \frac{1}{2}$

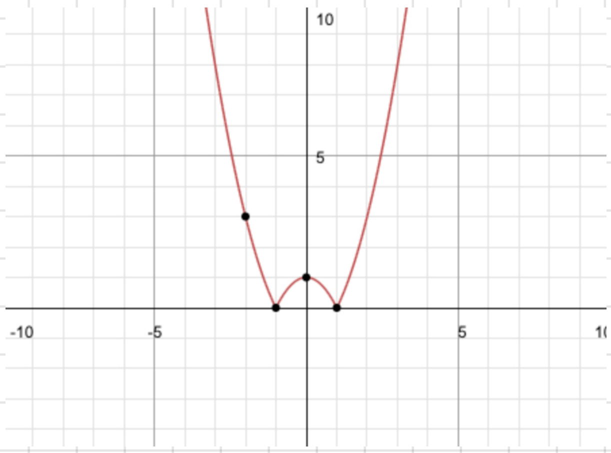
f is increasing on $(-1, 1)$

is decreasing on $(-\infty, -1) \cup (1, \infty)$

Example 4: Find the local extreme for $f(x) = |x^2 - 1|$ and find the intervals on which f is increasing and decreasing.

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \text{ or } x \leq -1 \\ -(x^2 - 1) & \text{if } -1 \leq x \leq 1 \end{cases}$$

$$f'(x) = \begin{cases} 2x & \text{if } x > 1 \text{ or } x < -1 \\ -2x & \text{if } -1 < x < 1 \end{cases}$$



$f'(x) = 0$

$-2x = 0$

$x = 0$

$f'(x)$ undefined

$x = \pm 1$

$\because D(f) = \mathbb{R} \Rightarrow -1, 0, 1$ are the critical numbers.

	$-\infty$	$f'(x) = 2x$	-1	$f'(x) = -2x$	0	$f'(x) = -2x$	1	$f'(x) = 2x$	∞
Test Value		-2		$\frac{-1}{2}$		$\frac{1}{2}$		2	
f'(Test Value)		$f'(-2) = -4$		$f'(\frac{-1}{2}) = 1$		$f'(\frac{1}{2}) = -1$		$f'(2) = 4$	
sign of $f'(x)$		$-$		$+$		$-$		$+$	
Concl		Dec.		Inc.		Dec.		Inc.	
			local min $f(-1) = 0$		local max $f(0) = 1$		local min $f(1) = 0$		

f is increasing on $(-1, 0) \cup (1, \infty)$

Decreasing on $(-\infty, -1) \cup (0, 1)$

Example 5: Find the local extreme for $f(x) = \ln(9 - x^2)$ and find the intervals on which f is increasing and decreasing.

$$f'(x) = \frac{-2x}{9 - x^2}$$

$f'(x) = 0$

$-2x = 0$

$x = 0$

$f'(x)$ undefined

$9 - x^2 = 0$

$9 = x^2$

$\pm 3 = x$

$\because D(f) = (-3, 3)$ because

$9 - x^2 > 0$

$9 > x$

$\pm 3 > x$

$\therefore x = 0$ is the only critical number.

	-3 to 0	0 to 3
Test value	-1	1
f'(Test value)	$f'(-1) = \frac{1}{4}$	$f'(1) = -\frac{1}{4}$
Sing of f'	$+$	$-$
	Inc	Dec

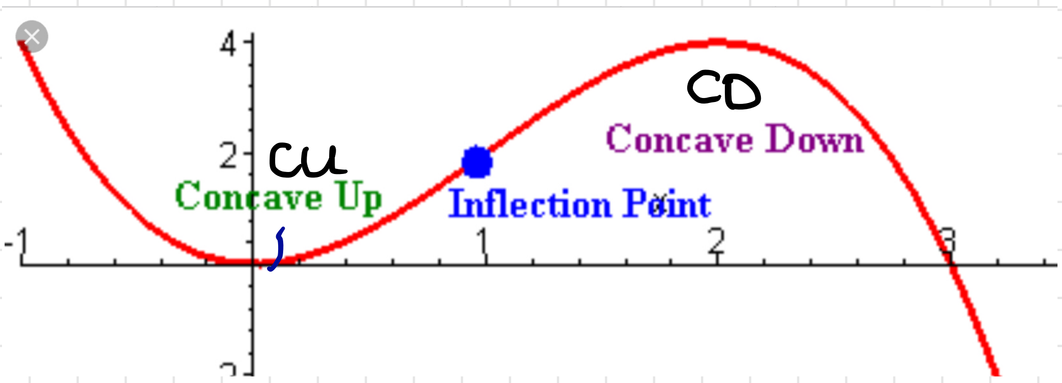
local max

$$f(0) = \ln 9 = \ln 3^2 = 2\ln 3$$

f is increasing on $(-3, 0)$

decreasing on $(0, 3)$

Concavity and the Second Derivative Test



Definition and Test for Concavity

I: open interval
f: differentiable on I

By Defintion | By Test

f' is increasin on I $\rightarrow$ CU
f' is decreasing on I $\rightarrow$ CD

$f''(x) > 0 \rightarrow$ CU
local min

$f''(x) < 0 \rightarrow$ CD
local max

Points of Inflection:

A point at which the graph of a function f changes concavity.

هي النقطة التي يتغير عندها تقعر منحنى الداله f

Example 1: Find where the graph of $f(x) = \frac{1}{8}x^4 - \frac{1}{2}x^3 + \frac{1}{8}$ is concave up and concave down and points of inflection

$$f'(x) = \frac{1}{8} \cdot 4x^3 - 3 \cdot \frac{1}{2}x^2$$

$$= \frac{1}{2}x^3 - \frac{3}{2}x^2$$

$$f''(x) = \frac{3}{2}x^2 - \frac{6}{2}x$$

$$= \frac{3}{2}x^2 - \frac{3}{2}x \cdot 2$$

$$= \frac{3}{2}x(x-2)$$

$f''(x)=0 \Rightarrow \frac{3}{2}x\,(x-2)=0$

$\Rightarrow \frac{3}{2}x=0$ or $x-2=0$

$\Rightarrow x=0$ or $x=2$

ملاحظه : نقط الإنقلاب هي
نفس النقاط التي يكون
التفاضل الثاني الداله يساوي
الصفر أو غير موجود
وتنتمي الى مجال الداله

	$-\infty$ … 0	0 … 2	2 … ∞
Test Value	-1	1	3
f''(Test Value)	$f''(-1)=\frac{9}{2}$	$f''(1)=\frac{-3}{2}$	$f''(3)=\frac{9}{2}$
sign of $f''(x)$	$+$	$-$	$+$
Concl	Up	Down	Up
	local min	local max	local min

$\therefore$ f is CU on $(-\infty,0)\cup(2,\infty)$
is CD on $(0,2)$

The inflection points are 0 and 2.

Example 2:

Find where the graph of $f(x)=\dfrac{x^2+1}{x^2-1}$ is concave up and concave down.

$$f'(x)=\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$=\frac{\cancel{2x^3}-2x-\cancel{2x^3}-2x}{(x^2-1)^2}$$

$$=\frac{-4x}{(x^2-1)^2}$$

$$f''(x)=\frac{(x^2-1)^2(-4)-(-4x)(2)(x^2-1)(2x)}{(x^2-1)^4}$$

$$=\frac{4(3x^2+1)}{(x^2-1)^3}$$

$f''(x) = 0$

$3x^2 + 1 = 0$

but there is no such x

$f''(x)$ undefined

$(x^2-1)^3 = 0$

$\sqrt[3]{(x+1)(x-1)} = 0$

$x = \pm 1$

$\therefore D(f) = \mathbb{R} - \{\pm 1\}$

$\therefore$ f has no inflection points.

	$-\infty$ to -1	-1 to 1	1 to ∞
Test Value	-2	0	2
f''(Test Value)	$f''(-2) = \frac{52}{27}$	$f''(0) = -4$	$f''(2) = \frac{52}{27}$
sign of $f''(x)$	$+$	$-$	$+$
Concl	Up	Down	Up
	local min	local max	local min

$\therefore$ f is CU on $(-\infty, -1) \cup (1, \infty)$

is CD on $(-1, 1)$

Example: Find where the graph of $f(x) = \dfrac{x}{x^2-1}$ is concave up and concave down and points of inflection.

$$f'(x) = \frac{(x^2-1)(1) - x(2x)}{(x^2-1)^2}$$

$$= \frac{x^2-1-2x^2}{(x^2-1)^2}$$

$$= \frac{-x^2-1}{(x^2-1)^2}$$

$$= \frac{-(x^2+1)}{(x^2-1)^2}$$

$$f''(x) = \frac{(x^2-1)^2(-2x) + 2(x^2+1)\cdot 2(x)(x^2-1)}{(x^2-1)^4}$$

$$= \frac{2x(x^2+3)}{(x^2-1)^3}$$

$f''(x) = 0$

$2x\,(x^2+3) = 0$ → never zero

$2x = 0$

$x = 0$

$f''(x)$ undefined

$(x^2-1)^3 = 0$

$\Rightarrow x = \pm 1$

$\because D(f) = \mathbb{R} - \{\pm 1\}$

$\therefore$ f has point of inflection at $x = 0$

	$-\infty$ to -1	-1 to 0	0 to 1	1 to ∞
Test Value	-2	$\frac{-1}{2}$	$\frac{1}{2}$	2
f''(Test Value)	$f''(-2) = \frac{-28}{27}$	$f''(\frac{-1}{2}) = \frac{208}{27}$	$f''(\frac{1}{2}) = \frac{-208}{27}$	$f''(2) = \frac{28}{27}$
sign of $f''(x)$	$-$	$+$	$-$	$+$
Concl	Down	Up	Down	Up

$\therefore$ f is CD on $(-\infty, -1) \cup (0, 1)$

is CU on $(-1, 0) \cup (1, \infty)$

Theorem 4.8.2: ***[The Second Derivative Test]***

Suppose that f'' is continuous on the open interval containing c such that $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a local minimum.
2. If $f''(c) < 0$, then $f(c)$ is a local maximum.

Example: Find the local exterme for $f(x) = 2\sin x + \cos 2x$,

$0 \leq x \leq 2\pi$.

$f'(x) = 2\cos x - 2\sin 2x$

$= 2\cos x - 4\sin x \cos x$

$= 2\cos x\,(1 - 2\sin x)$

$f'(x) = 0 \Rightarrow 2\cos x\,(1 - 2\sin x) = 0$

$\Rightarrow 2\cos x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow \cos x = 0$ or $\sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad x = \frac{\pi}{6}, \frac{5\pi}{6}$

$f''(x) = -2\sin x - 4\cos(2x)$

$f''(\pi/6) = -2\sin(\pi/6) - 4\cos(2\cdot\pi/6)$

$= -2\cdot\frac{1}{2} - 4\cdot\frac{1}{2}$

$= -1 - 2 = -3 < 0$

$F(\pi/6) = 2\sin(\pi/6) + \cos(2\cdot\pi/6)$

$= 2\cdot\frac{1}{2} + \frac{1}{2}$

$= 1 + \frac{1}{2} = \frac{3}{2}$ is a local max

$$F''(5\pi/6) = -2\sin(5\pi/6) - 4\cos(2\cdot 5\pi/6)$$

$$= -2\cdot\frac{1}{2} - 4\cdot\frac{1}{2}$$

$$= -1 - 2 = -3 < 0$$

$$F(5\pi/6) = 2\sin(5\pi/6) + \cos(2\cdot 5\pi/6)$$

$$= 2\cdot\frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{1}{2} = \frac{3}{2} \text{ is a local max}$$

$$F''(\pi/2) = -2\sin(\pi/2) - 4\cos(2\cdot\pi/2)$$

$$= -2(1) - 4(-1)$$

$$= -2 + 4 = 2 > 0$$

$$F(\pi/2) = 2\sin(\pi/2) + \cos(2\cdot\pi/2)$$

$$= 2\cdot(1) + (-1)$$

$$= 2 - 1 = 1 \text{ is a local min}$$

$$f''(3\pi/2) = -2\sin(3\pi/2) - 4\cos(2 \cdot 3\pi/2)$$

$$= -2(-1) - 4(-1)$$

$$= 2 + 4 = 6 > 0$$

$$f(3\pi/2) = 2\sin(3\pi/2) + \cos\left(2 \cdot \frac{3\pi}{2}\right)$$

$$= 2 \cdot (-1) + (-1)$$

$$= -2 - 1 = -3$$ is a local min

Integrals

Definition of integral
Integral of powerFunction
Integral of Exponential Function
Integral of Logarithmic Function
Integral of Trigonometric Function

Antiderivative(integrals)

عكس التفاضل　　　　التكامل

Fundamental theorem of Calculus

For $F'(x) = f(x)$ then

Indefinite integral

$$\int f(x)\,dx = F(x) + c$$

Definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F'(x) = \underbrace{2x}_{f(x)}$

What is $F(x)$

$F(x) = x^2$

$$\int 2x\,dx = x^2 + c$$

باستخدام قوانين التكامل سنحصل
على الدالة الاصليه F(x)

Note.

سبب وجود ال c في التكامل

$x^2 + 1 \xrightarrow{\text{مشتقه}} 2x$

$x^2 \longrightarrow 2x$

∴ تكامل $2x$ قد تكون الدالة
الأصلية له x^2 أو x^2+1
ولذلك نضع c

Properties of Integral

لوكان لدينا داخل التكامل
دالتين بينهم عملية جمع او
طرح فان التكامل يتوزع

(1)- $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(2)- $\int c\,f(x)\,dx = c \int f(x)\,dx$

العدد الثابت يكون خارج
التكامل

(3)- $\int_{-a}^{a} f(x)\,dx = 0$ if f is odd

Ex: $\int_{-2}^{2} x^3\,dx = 0$　odd　check!!

اذا كان التكامل من سالب العدد الى موجب العدد
وكانت الدالة فردية فان التكامل يساوي صفر

(4)- $\int_{-a}^{a} f(x)\,dx = 2\left(\int_0^a f(x)\,dx\right)$ if f is even

Ex: $\int_{-2}^{2} x^2\,dx = 2\int_0^2 x^2\,dx = 2 \cdot \frac{x^3}{3}\Big|_0^2 = 2 \cdot \left(\frac{8}{3}\right) = \frac{16}{3}$　even

اذا كان التكامل من سالب العدد الى
موجب العدد وكانت الدالة زوجيه فإن
الناتج عبارة عن التكامل من صفر الى
موجب العدد مضروب في ٢

Integral of Power Function

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$$

$$\int (f(x))^n \cdot f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1} + C$$

Example 1: Evaluate $\int x^5 \, dx$

$$\int x^5 \, dx = \frac{x^6}{6} + C$$

Example 2: Evaluate $\int x^2 + 7 \, dx$

$$\int x^2 + 7 \, dx = \int x^2 \, dx + \int 7 \, dx$$ تطبيق الخاصيه ١ (توزيع التكامل)

$$= \int x^2 \, dx + 7 \int 1 \, dx$$

$$= \frac{x^3}{3} + 7x + C$$

Note:

$\int 1 \, dx = x$

السبب: $x^0 = 1$ و

$\int x^0 \, dx$

$= \frac{x^{0+1}}{0+1} = x^1$

Example 3: Evaluate $\int_0^1 x^2 \, dx$

$$\int_0^1 x^2 \, dx = \frac{x^3}{3} \Big|_0^1 =$$

$$= \frac{(1)^3}{3} - \frac{(0)^3}{3} = \frac{1}{3}$$

داله مرفوعة لأس

مشتقتها

Example 4: Evaluate $\int \sin^2 x \cos x \, dx$

$$\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3} + C$$

نأخذ الداله المرفوعة للأس ونضيف عليه واحد ونقسم على الاس

Example 5: Evaluate $\int_0^1 (3x-1)^3 \, dx$

دالة مرفوعة لأس

$$\int_0^1 (3x-1)\,dx = \frac{1}{3}\int_0^1 (3x-1)^3 \cdot 3 \, dx$$

في هذا المثال لاتوجد مشتقه الداله
ولكي اطبق القانون لابد ان اضرب
في مشتقة الداله وأقسم عليه

$$= \frac{1}{3} \left. \frac{(3x-1)^4}{4} \right|_0^1$$

هنا طبقنا القانون لان التكامل
اصبح على الصوره الداله في
مشتقتها

$$= \frac{1}{3}\left[\frac{(3(1)-1)^4}{4} - \frac{(3(0)-1)^4}{4}\right]$$

$$= \frac{1}{3}\left[\frac{(2)^4}{4} - \frac{(-1)^4}{4}\right]$$

$$= \frac{1}{3}\left[\frac{16}{4} - \frac{1}{4}\right]$$

$$= \frac{1}{3}\left(\frac{15}{4}\right) = \frac{15}{12}$$

$$= \frac{5}{4}$$

Example 6: Evaluate $\int_{-2}^{2} x^3 \, dx$

$$\int_{-2}^{2} x^3 \, dx = \left. \frac{x^4}{4} \right|_{-2}^{2} = (2)^4 - (-2)^4 = 16 - 16 = 0$$

Example : Evaluate $\int_{-2}^{2} x^2 \, dx$

$$\int_{-2}^{2} x^2 \, dx = \left. \frac{x^3}{3} \right|_{-2}^{2} = \frac{1}{3}\left[2^3 - (-2)^3\right] = \frac{1}{3}(8+8) = \frac{16}{3}$$ تطبيق قانون التكامل

or

$$\int_{-2}^{2} x^{\overset{\text{even}}{2}} \, dx = 2\int_0^2 x^2 \, dx = 2 \cdot \left. \frac{x^3}{3} \right|_0^2 = 2 \cdot \left(\frac{8}{3}\right) = \frac{16}{3}$$ تطبيق الخاصيه ٤

Integral of Logarithmic Function

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int \frac{f'(x)}{\sqrt{x}}\,dx = 2\sqrt{x} + c$$

Example 1: Compute $\int \frac{5}{x+1}\,dx$

$$\int \frac{5}{x+1}\,dx = 5\int \frac{1}{x+1}\,dx$$

تفاضل الداله

داله

نطبق القانون ٢ مباشره

$$= 5\ln|x+1| + c$$

Example 2: Compute $\int \tan x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

مشتقة الداله

داله

في هذا المثال نطبق القانون 2

لان مشتقة ال cosx هي ‎-sinx وإشارة السالب غير موجودة لذلك نضرب في السالب ونقسم عليه

$$= -\int \frac{-\sin x}{\cos x}\,dx$$

$$= -\ln|\cos x| + c$$

تم تطبيق القانون

$$= \ln|\cos x|^{-1} + c$$

تطبيق خواص الداله اللوغاريتميه

$$= \ln\left|\frac{1}{\cos x}\right| + c$$

$$= \ln|\sec x| + c$$

Example 3: $\int_1^{e^2} \frac{3}{x}\,dx$

$$\int_1^{e^2} \frac{3}{x}\,dx = 3\int_1^{e^2} \frac{1}{x}\,dx$$

نطبق القانون ١ مباشره

$$= 3\ \ln(x)\Big|_1^{e^2}$$

$$= 3\left[\ln(e^2) - \ln(1)\right]$$

$$= 3\left[2\ln(e) - \ln(1)\right]$$

$$= 3\left[2-0\right] = 6$$

Example 4: Compute $\int_0^2 \frac{e^x}{e^x+1}\,dx$

مشتقة الدالة

داله

$$\int_0^2 \frac{e^x}{e^x+1}\,dx = \ln\left|e^x+1\right|\Big|_0^2$$

نطبق القانون ٢ مباشره

$$= \ln\left|(e^2+1) - (e^0+1)\right|$$

$$= \ln\left|(e^2+1) - 2\right|$$

$$= \ln(e^2+1) - \ln(2)$$

Example 5: Evaluate $\int \frac{1}{\cos^2 x\sqrt{\tan x}}\,dx$

تطبيق القانون ٣

$$\frac{1}{\cos^2 x\sqrt{\tan x}}\,dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$$

$$= 2\sqrt{\tan x} + c$$

Example 6 : Evaluate $\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx$

لاتوجد مشتقه

داله تحت الجذر

$$\int_0^4 \frac{1}{\sqrt{2x+1}}\,dx = = \frac{1}{2}\int_0^4 \frac{2}{\sqrt{2x+1}}\,dx$$

مشتقة الداله

داله تحت الجذر

نضرب التكامل في ٢
وهو مشتقة الداله
ونقسم عليه لكي
نستطيع تطبيق قانون
التكامل

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{2x+1}\,\Big|_0^4$$

طبقنا القانون ٣ بعد تعديل
التكامل

$$= \sqrt{2(4)+1} - \sqrt{2(0)+1}$$

$$= \sqrt{9} - \sqrt{1}$$

$$= 3-1 = 2$$

Integral of Exponential function

$$\boxed{\begin{aligned} \int e^{kx}\,dx &= \frac{1}{k}e^{kx} + c \\ \int a^x\,dx &= \frac{a^x}{\ln a} + c \end{aligned}}$$

Example 1 :- Compute $\int e^{2x}\,dx$

$$\int e^{2x}\,dx = \frac{e^{2x}}{2} + c$$

Example 2: Compute $\int_0^{\ln 5} 5e^x \, dx$

$$\int_0^{\ln 5} 5 e^x \, dx = 5\int_0^{\ln 5} e^x \, dx$$

$$= 5 e^x \Big|_0^{\ln 5}$$

$$= 5\left[e^{\ln 5} - e^0\right]$$

$$= 5\left[5 - 1\right]$$

$$= 5(4) = 20$$

Example 3: Compute $\int 2^x \, dx$

$$\int 2^x \, dx = \frac{2^x}{\ln 2} + C$$

Example 4: Evaluate $\int_0^1 2^x \, dx$

$$\int_0^1 2^x \, dx = \frac{2^x}{\ln 2}\Big|_0^1$$

$$= \frac{2^1}{\ln 2} - \frac{2^0}{\ln 2}$$

$$= \frac{2}{\ln 2} - \frac{1}{\ln 2}$$

$$= \frac{1}{\ln 2}.$$

Integral of Trigonometric Function

1) $\int \sin(kx)\,dx = \frac{-1}{k}\cos(kx) + c$

2) $\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + c$

3) $\int \sec(kx)\tan(kx)\,dx = \frac{1}{k}\sec(kx) + c$

4) $\int \cos(kx)\,dx = -\frac{1}{k}\sin(kx) + c$

5) $\int \csc^2(kx)\,dx = -\frac{1}{k}\cot(kx) + c$

6) $\int \csc(kx)\cot(kx)\,dx = -\frac{1}{k}\csc(kx) + c$

Exampl 1: Evaluate $\int \sin(2x)\,dx$

$$\int \sin(2x)\,dx = \frac{-1}{2}\cos(2x) + C$$

Example 2: Evaluate $\int_0^{2\pi} \sin x\,dx$

$$\int_0^{2\pi} \sin x\,dx = -\cos x \Big|_0^{2\pi}$$

$$= -\cos(2\pi) + \cos(0)$$

$$= -1 + 1 = 0$$

Example 3: Evaluate $\int \cos(3x)\,dx$

$$\int \cos(3x)\,dx = \frac{1}{3}\sin x + C$$

Example 4: Evaluate $\int_0^{\pi/3} \cos(3x)\,dx$

$$\int_0^{\pi/3} \cos(3x)\,dx = \frac{1}{3}\sin(3x)\Big|_0^{\pi/3}$$

$$= \frac{1}{3}\left[\sin(3\cdot\pi/3) - \sin(3\cdot 0)\right]$$

$$= \frac{1}{3}\left[\sin(\pi) - \sin(0)\right]$$

$$= \frac{1}{3}\left[0 - 0\right] = 0$$

Example 5: Evaluate $\int \sec^2(3x)\,dx$

$$\int \sec^2(5x)\,dx = \frac{1}{5}\tan(5x) + C$$

Example 6: Evaluate $\int_0^{\pi} \sec^2 \quad dx$

$$\int_0^{\pi} \sec^2\left(\frac{x}{4}\right)dx = \int_0^{\pi} \sec^2\left(\frac{1}{4}x\right)\,dx$$

$$= \frac{1}{\frac{1}{4}}\tan\left(\frac{1}{4}x\right)\Big|_0^{\pi}$$

$$= 4\tan\left(\frac{1}{4}x\right)\Big|_0^{\pi}$$

$$= 4\left[\tan\left(\frac{1}{4}\cdot\pi\right) - \tan\left(\frac{1}{4}\cdot 0\right)\right]$$

$$= 4\left[\tan\left(\frac{\pi}{4}\right) - \tan(0)\right]$$

$$= 4\left[1 - 0\right] = 4.$$

Example 7: Evaluate $\int \csc^2(3x)\,dx$

$$\int \csc^2(3x)\,dx = -\frac{1}{3}\cot(3x) + C$$

Example 8: Evaluate $\int \sec(4x)\tan(4x)\,dx$

$$\int \sec(4x)\tan(4x)\,dx = \frac{1}{4}\sec(4x) + C$$

Example 9: Evaluate $\int_{-\pi/4}^{\pi/4} \sec(4x)\tan(4x)\,dx$

$$\int \sec(4x)\tan(4x)\,dx = \frac{1}{4}\sec(4x)\Big|_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{4}\left[\sec\left(4\cdot\frac{\pi}{4}\right) - \sec\left(4\cdot\frac{-\pi}{4}\right)\right]$$

$$= \frac{1}{4}\left[\sec(\pi) - \sec(-\pi)\right]$$

$$= \frac{1}{4}\left[-1 - (-1)\right]$$

$$= \frac{1}{4}\left[-1+1\right] = \frac{1}{4}(0) = 0.$$

Example 10: Evaluate $\int \csc(2x)\cot(2x)\,dx$

$$\int \csc(2x)\cot(2x)\,dx = -\frac{1}{2}\csc(2x) + C$$