3.1 - Derivatives of Polynomials and Exponential Functions

1) Derivative of a Constant function

$$\frac{d}{dx}[c] = 0 \quad \text{for all } c \in \mathbb{R}$$

Example:-

$\frac{d}{dx}[\pi^2] = 0$ $\frac{d}{dx}[5^c] = 0$ $\frac{d}{dy}[18.5] = 0$

$\frac{d}{dx}[\sqrt{36}] = 0$ $\frac{d}{dx}[\ln(9)] = 0$

$\frac{d}{dx}[\sin(\frac{\pi}{2})] = 0$ $\frac{d}{dx}[\cos^2(5)] = 0$

if $f(x) = \sqrt{4 + c^2}$ then $f'(x) = 0$ ...

2) if $\underline{\underline{f(x) = ax}}$ for all $a \in \mathbb{R}$ then $\underline{\underline{f'(x) = a}}$

Example:

$\frac{d}{dx}[10x] = 10$

if $f(x) = \frac{-3}{4}x$ then $f'(x) = \frac{-3}{4}$ ...

if $f(x) = -x$ then $f'(x) = -1$.

$\frac{d}{dt}[2t] = 2$

if $f(\theta) = 18.5\theta$ then $f'(\theta) = 18.5$.

3) if $f(x) = x^n$ then $f'(x) = n x^{n-1}$

Example:

$\frac{d}{dx}[x^2] = 2x \quad \frac{d}{dx}[x^3] = 3x^2 \quad \frac{d}{dx}[x^4] = 4x^3$

$$\frac{d}{dx}\left[\frac{1}{x^5}\right] = \frac{d}{dx}[x^{-5}]$$

$$= -5x^{-5-1}$$

$$= -5x^{-6}$$

$$= \frac{-5}{x^6}$$

$$\frac{d}{dx}\left[\sqrt[3]{x^2}\right] = \frac{d}{dx}\left[(x^2)^{1/3}\right]$$

$$= \frac{d}{dx}\left[x^{2/3}\right]$$

$$= \frac{2}{3}x^{2/3-1}$$

$$= \frac{2}{3}x^{-\frac{1}{3}}$$

$$= \frac{2}{3x^{1/3}} = \frac{2}{3\sqrt[3]{x}}$$

$$\frac{d}{dx}\left[x^2\sqrt{x}\right] = \frac{d}{dx}\left[x^2 . x^{1/2}\right]$$

$$= \frac{d}{dx}\left[x^{2+\frac{1}{2}}\right]$$

$$= \frac{d}{dx}\left[x^{5/2}\right]$$

$$= \frac{5}{2}x^{5/2-1}$$

$$= \frac{5}{2}x^{3/2}$$

$$= \frac{5}{2}\sqrt{x^3}$$

4] $\frac{d}{dx}[cf(x)] = c.\frac{d}{dx}[f(x)]$

$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Example:

a) $\frac{d}{dx}[x^8+12x^5-4x^4+10x^3-6x+\sqrt{\frac{2}{5}}]$

$8x^7+12(5)x^4-4(4)x^3+10(3)x^2-6+0$

$8x^7+60x^4-16x^3+30x^2-6$

b) If $f(x)=(3x-2)^2$ then $f'(x) = ----$

$f(x) = 9x^2-2(3x)(2)+4$

$=9x^2-12x+4$

$f'(x)=18x-12$

c) $\frac{d}{dx}[x^2(1-2x)] = \frac{d}{dx}[x^2-2x^3]$

$=2x-6x^2$

d) $\frac{d}{dt}[\sqrt{t}(t-1)] = \frac{d}{dt}[t^{1/2}(t-1)]$

$= \frac{d}{dt}[t^{1/2}.t^1-t^{1/2}] = \frac{d}{dt}[t^{3/2}-t^{1/2}]$

$= \frac{3}{2}t^{3/2-1} - \frac{1}{2}t^{1/2-1}$

$= \frac{3}{2}t^{1/2} - 1/2\, t^{-1/2} = \frac{3}{2}\sqrt{t} - \frac{1}{2t^{1/2}}$

$= \frac{3}{2}\sqrt{t} - \frac{1}{2\sqrt{t}} = \frac{3\sqrt{t}.\sqrt{t}-1}{2\sqrt{t}}$

$= \frac{3t-1}{2\sqrt{t}}$

e) $\frac{d}{dx}\left[(2x+3)(4x-5)\right]$

$\frac{d}{dx}\left[2x(4x-5)+3(4x-5)\right]$

$\frac{d}{dx}\left[8x^2-10x+12x-15\right]$

$\frac{d}{dx}\left[8x^2+2x-15\right] = 16x+2$

f) $\frac{d}{dx}\left[(x-2)^3\right] = \frac{d}{dx}\left[x^3-3(2)x^2+3(4)x-2^3\right]$

$= \frac{d}{dx}\left[x^3-6x^2+12x-8\right]$

$= 3x^2-2(6)x+12$

$= 3x^2-12x+12$

g) $\frac{d}{dx}\left[x(2x+3)^2\right] = \frac{d}{dx}\left[x(4x^2+12x+9)\right]$

$= \frac{d}{dx}\left[4x^3+12x^2+9x\right]$

$= 12x^2+24x+9$

h) $f(t) = (3x^2+2)(x^3-5)$

$f'(t) =$ H.W

if $G(x) = \frac{5x^2 + 4x + 3}{x^2}$ then $G'(x) =$ ——

$$G(x) = \frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{3}{x^2}$$

$$= 5 + \frac{4}{x} + \frac{3}{x^2}$$

$$= 5 + 4x^{-1} + 3x^{-2}$$

$$G'(x) = 0 + 4(-1)x^{-1-1} + 3(-2)x^{-2-1}$$

$$= -4x^{-2} - 6x^{-3}$$

$$= \frac{-4}{x^2} - \frac{6}{x^3}$$

$$= \frac{-4x}{x^2, x} - \frac{6}{x^3}$$

$$= \frac{-4x}{x^3} - \frac{6}{x^3}$$

$$= \frac{-4x - 6}{x^3}$$

if $y = \frac{\sqrt{x} + x}{x^2}$ then $y' =$ - - .

$$y = \frac{x^{1/2} + x^1}{x^2} = \frac{x^{1/2}}{x^2} + \frac{x^1}{x^2} = x^{1/2 - 2} + x^{1-2}$$

$$= x^{-3/2} + x^{-1}$$

$$y' = -\frac{3}{2} x^{-\frac{3}{2}-1} - x^{-1-1} = -\frac{3}{2} x^{-5/2} - x^{-2} = \frac{-3}{2x^{5/2}} - \frac{1}{x^2} = \frac{-3}{2\sqrt{x^5}} - \frac{1}{x^2}$$

5] $\frac{d}{dx}[a^x] = a^x . \text{Ln}a$

$\frac{d}{dx}[e^x] = e^x$

Example

$\frac{d}{dx}[\pi^x] = \pi^x . \text{Ln}\pi = \text{Ln}(\pi) \cdot (\pi)^x$

$$\frac{d}{dx}[\sqrt{2^x}] = \frac{d}{dx}[(\sqrt{2})^x]$$
$$= (\sqrt{2})^x \cdot \text{Ln}\sqrt{2}$$
$$= (\sqrt{2})^x . \text{Ln}\, 2^{1/2}$$
$$= (\sqrt{2})^x . \frac{1}{2}\text{Ln}\, 2$$
$$= \frac{1}{2}\text{Ln}\, 2 \cdot (\sqrt{2})^x$$

$$\frac{d}{dx}[3^x + x^3] = \frac{d}{dx}[3^x] + \frac{d}{dx}[x^3]$$
$$= 3^x . \text{Ln}(3) + 3x^2$$

$$\frac{d}{dx}[e^x - x^e] = \frac{d}{dx}[e^x] - \frac{d}{dx}[x^e]$$
$$= e^x - e x^{e-1}$$
$$= e(e^{x-1} - x^{e-1})$$

if $y = e^{x+1} + x^2$ then find ① y''' or $\frac{d^3y}{dx^3}$

② $y^{(100)}$

$$y' = e^{x+1} + 2x$$

$$y'' = e^{x+1} + 2$$

$$y''' = e^{x+1}$$

$$y^{(4)} = e^{x+1}$$

$$y^{(5)} = e^{x+1}$$

$$\vdots$$

$$y^{(100)} = e^{x+1}$$

4.1 _ Maximum and Minimum Values

Definition(1)

let $\underline{\underline{c}}$ be a number in the domain $\underline{\underline{D}}$ of a function $\underline{\underline{f}}$.

Then 1) $f(c)$ is absolute maximum Value of f on D if $f(c) \geqslant f(x)$ for all x in D

2) $f(c)$ is absolute minimum Value of f on D if $f(c) \leqslant f(x)$ for all x in D

Definition(2)

The number $f(c)$ is a

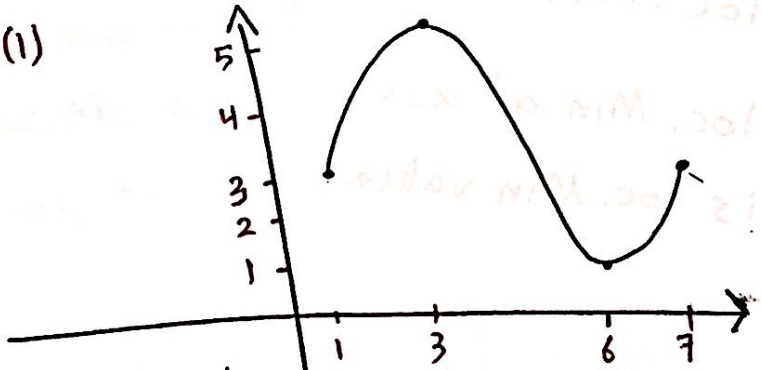
1) local maximum Value of $\underline{\underline{f}}$ if $f(c) \geqslant f(x)$ when $\underline{\underline{x}}$ is near $\underline{\underline{c}}$

2) local Minimum Value of $\underline{\underline{f}}$ if $f(c) \leqslant f(x)$ when x is near $\underline{\underline{c}}$

Note

Absolute maximum or minimum is Sometimes called global maximum or minimum

Example(1)



Find the Abs. Max and Abs. Min

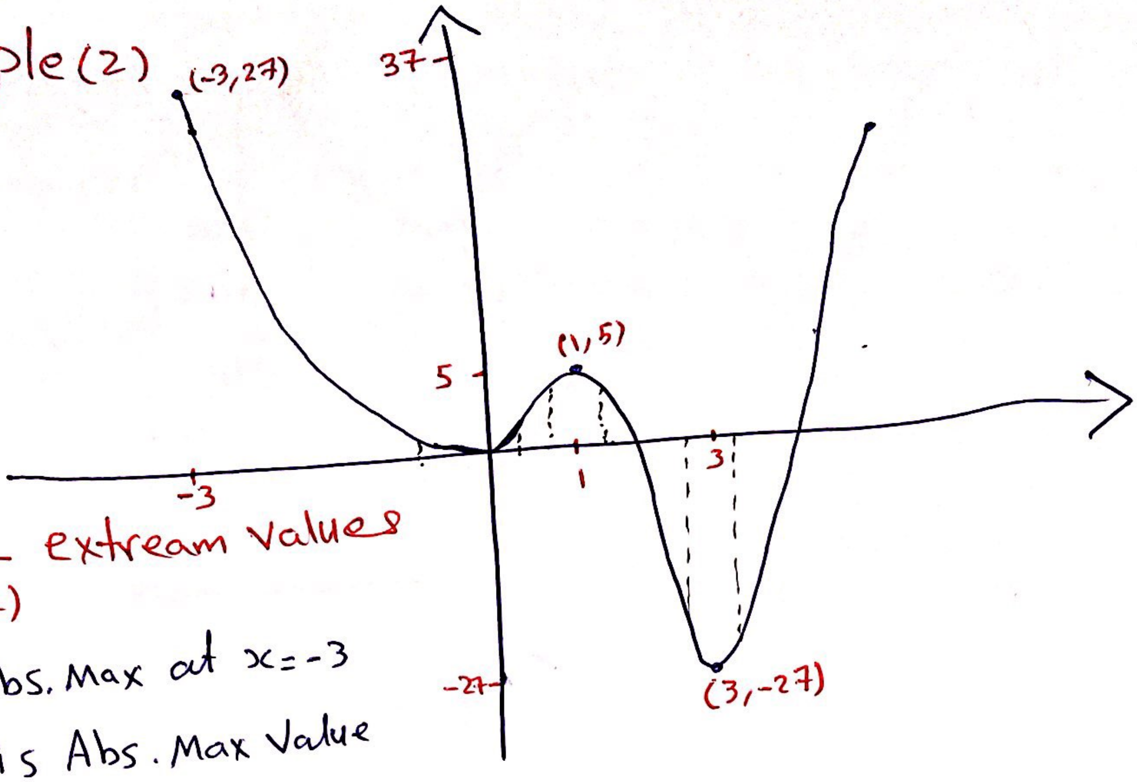
✓ $f(3) = 5$ is Abs. Max Value

or $f(x)$ has Abs max at $x = 3$

✓ $f(6) = 1$ is Abs. Min Value

or $f(x)$ has Abs. Min at $x = 6$

Example (2)



Find The extream values of f(x)

- ✓ f(x) has Abs. Max at $x = -3$
  $f(-3) = 37$ is Abs. Max Value
- ✓ f(x) has Abs. Min at $x = 3$
  $f(3) = -27$ is Abs. Min Value.
- ✓ f(x) has loc. Min at $x = 0$
  $f(0) = 0$ is loc. Min Value
- ✓ f(x) has loc. Max at $x = 1$
  $f(1) = 5$ is loc. Max Value
- ✓ f(x) has loc. Min at $x = 3$
  $f(3) = -27$ is loc. Min Value.

Example (4) : find the extream value of

① $f(x) = x^2$

② $f(x) = x^3$

③ $f(x) = \cos x$

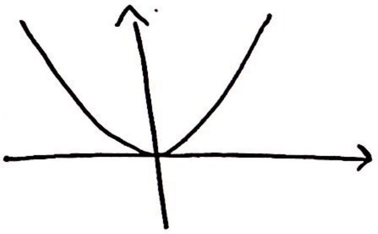
④ $f(x) = \sin x$

⑤ $f(x) = 2x$ [illegible] 1 on $[0, 3]$

⑥ $f(x) = x^2$ on $(-2, 2]$

Solution

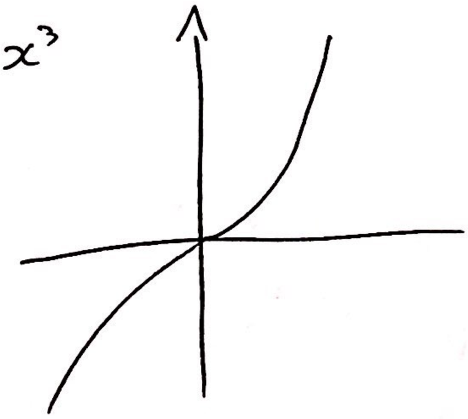
① $f(x) = x^2$



$f(0) = 0$ is Abs. Min and local. Min Value

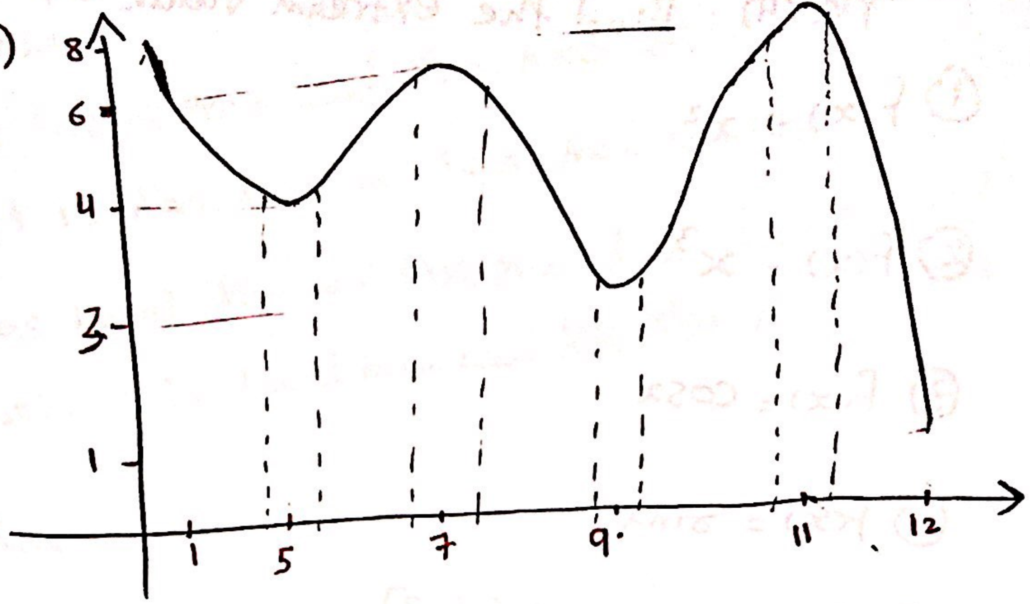
$F(x)$ has no Abs. Max and local. Max value.

② $f(x) = x^3$



$f(x)$ has no Max or Min

Example (3)



✓ f(x) has Abs. Max at $x=1$

$f(1)=8$ is Abs. Max value

✓ f(x) has Abs. Min at $x=12$

$f(12)=1$ is Abs. Min value

✓ f(x) has Loc. Min at $x=5$

$f(5)=4$ is loc. Min value

✓ f(x) has loc. Min at $x=9$

$f(9)=3$ is loc. Min value

✓ f(x) has loc. Max at $x=7$

$f(7)=6$ is loc. Max value

✓ f(x) has loc. Max at $x=11$

$f(11)=8$

✓ f(x) has Abs. Max at $x=11$

$f(11)=8$ is Abs. Max Value.

③ $f(x) = \cos x$

✓ $f(x)$ has local Max and Abs. Max at $x = 2n\pi$ $\forall n \in \mathbb{Z}$

$f(2n\pi) = 1$ is Abs. Max and local Max Value

✓ $f(x)$ has local Min and Abs. Min at $x = (2n+1)\pi$ $\forall n \in \mathbb{Z}$

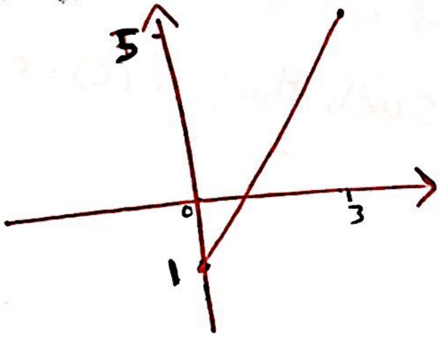
$f((2n+1)\pi) = -1$ is local Min and ~~Abs~~. Abs. Min value.

④ $f(x) = \sin x$

1 is Abs. Max and local Max Value

-1 is Abs. Min and local Min Value.

⑤ $f(x) = 2x-1$ on $[0,3]$



✓ $f(x)$ has Abs. Max at $x=3$

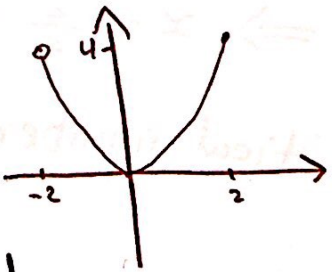
$f(3) = 5$ is Abs. Max Value

✓ $f(x)$ has Abs. Min at $x=0$

$f(0) = 1$ is Abs Min Value

✓ $f(x)$ has no local Min and Max.

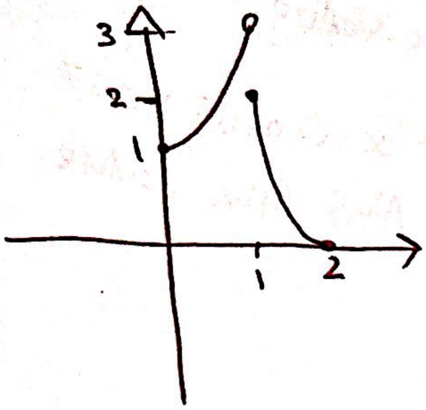
⑥ $f(x) = x^2$ on $(-2,2]$



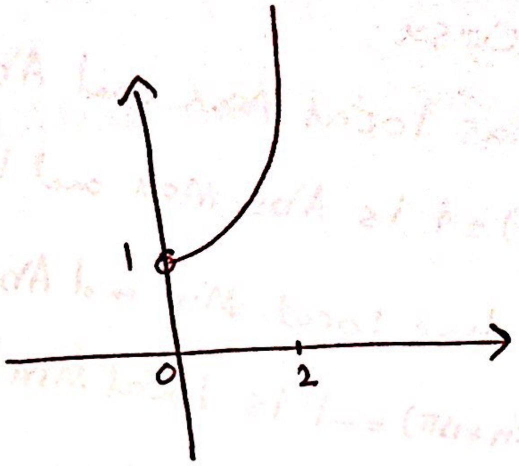
$f(x)$ has Abs. Min at $x=0$

$f(x)$ has Abs. Max at $x=2$

Example



This function has Min Value $f(2)=0$ but no Max. value



This function has no Maximum or Minimum

Definition

A critical number of a function f is a number c in the Domain of f such that $f'(c)=0$ or $f'(c)$ does not exist

Example

Find the Critical number of $f(x)=5x^2+4x$

① $D_f=\mathbb{R}$

② $f'(x)=10x+4$

③ $f'(x)=0 \Rightarrow 10x+4=0 \Rightarrow 10x=-4 \Rightarrow x=\frac{-4}{10}$

$\Rightarrow x=-\frac{2}{5} \in D_f$

$\Rightarrow$ the critical number is $x=-\frac{2}{5}$

Example

Find the critical numbers of $f(x) = x^{3/5}(4-x)$

① $D_{f(x)} = \mathbb{R}$

② $f'(x) = x^{3/5}(-1) + (4-x)\left(\frac{3}{5}x^{1-3/5}\right)$

$= -x^{3/5} + (4-x)\left(\frac{3}{5}x^{-2/5}\right)$

$= \frac{-x^{3/5}}{1} + \frac{3(4-x)}{5x^{2/5}}$

$= \frac{-5x^{3/5} . x^{2/5} + 3(4-x)}{5x^{2/5}}$

$= \frac{-5x^{\frac{3+2}{5}} + 3(4-x)}{5x^{2/5}}$

$= \frac{-5x + 12 - 3x}{5x^{2/5}}$

$= \frac{12 - 8x}{5x^{2/5}}$

③ $f'(x) = 0$ or $f'(x)$ D.N.E

$\frac{12-8x}{5x^{2/5}} = \frac{0}{1}$

$12 - 8x = 0$

$-8x = -12$

$x = \frac{12}{8}$

$x = 3/2 \in D_f$

$5x^{2/5} = 0$

$x^{2/5} = 0$

$(x^{2/5})^{5/2} = 0^{5/2}$

$x = 0 \in D_f$

$\Rightarrow$ the Critical numbers are $\frac{3}{2}$ and 0

Example : find the critical numbers of $f(x) = |3-6x|$

$$3-6x=0$$
$$-6x=-3$$
$$x=\frac{-3}{-6}$$
$$x=\frac{1}{2}$$

$f(x) = |3-6x|$ is not diff at $x = \frac{1}{2}$

i.e $f'(\frac{1}{2})$ D.N.E

$\Longrightarrow$ the Critical number is $\frac{1}{2}$

Note

If f has a local Maximum or Minimum at $\underline{c}$

then c is critical number of f

Example

Find the Absolute Maximum and Absolute Minimum values of

$$f(x) = x^3 - 3x^2 + 1 \quad \text{on } [-\tfrac{1}{2}, 4]$$

① $f(x)$ is cont on $\mathbb{R}$

$\Rightarrow f(x)$ is cont on $[-\frac{1}{2}, 4]$

② $f'(x) = 3x^2 - 6x$

③ $f'(x) = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$3x = 0 \Rightarrow x = 0 \in (-\frac{1}{2}, 4)$ $\boxed{c.n}$

or

$x - 2 = 0 \Rightarrow x = 2 \in (-\frac{1}{2}, 4)$ $\boxed{c.n}$

④ $f(-\frac{1}{2}) = (-\frac{1}{2})^3 - 3(-\frac{1}{2})^2 + 1 = -\frac{1}{8} - \frac{3}{4} + 1 = \frac{1}{8}$

$f(0) = (0)^3 - 3(0)^2 + 1 = 1$

$f(2) = (2)^3 - 3(2)^2 + 1 = 8 - 12 + 1 = -3$ is Abs. Min Value

$f(4) = (4)^3 - 3(4)^2 + 1 = 64 - 48 + 1 = 17$ is Abs. Max Valu.

3.2 - The Product and Quotient Rules

The Product Rule

$$\frac{d}{dx}\left[f(x).g(x)\right] = f(x).\frac{d}{dx}\left[g(x)\right] + g(x).\frac{d}{dx}\left[f(x)\right]$$

or $\left[f(x).g(x)\right]' = f(x).g'(x) + g(x).f'(x)$

Example a) If $f(x) = x.e^x$ then find $f'(x)$?

$$f'(x) = \frac{d}{dx}\left[xe^x\right]$$

$$= x.\frac{d}{dx}\left[e^x\right] + e^x\frac{d}{dx}\left[x\right]$$

$$= x.e^x + e^x(1)$$

$$= xe^x + e^x$$

$$= e^x(x+1)$$

b) find $f^{(n)}_{(x)}$

$$f''(x) = \frac{d}{dx}\left[e^x(x+1)\right]$$

$$= e^x.\frac{d}{dx}\left[x+1\right] + (x+1)\frac{d}{dx}\left[e^x\right]$$

$$= e^x(1) + (x+1)e^x$$

$$= e^x(1 + x + 1)$$

$$= e^x(x+2)$$

$$f'''(x) = \frac{d}{dx}\left[e^x(x+2)\right]$$

$$= e^x\frac{d}{dx}\left[(x+2)\right] + (x+2)\frac{d}{dx}\left[e^x\right]$$

$$= e^x(1) + (x+2)e^x = e^x(1+x+2) = e^x(x+3)$$

$$f^{(4)}(x) = e^x(x+4)$$

$$f^{(5)}(x) = e^x(x+5)$$

$$\vdots$$

$$f^{(n)}(x) = e^x(x+n)$$

Example

IF $f(x) = (1-e^x)(x+e^x)$ then find $f'(x)$

$$f'(x) = \frac{d}{dx}\left[(1-e^x)(x+e^x)\right]$$

$$= (1-e^x)\frac{d}{dx}\left[x+e^x\right] + (x+e^x)\frac{d}{dx}\left[1-e^x\right]$$

$$= (1-e^x)(1+e^x) + (x+e^x)(0-e^x)$$

$$= 1 - (e^x)^2 + (x+e^x)(e^x)$$

$$= 1 \underline{\underline{-1e^{2x}}} - xe^x \underline{\underline{-1e^{2x}}}$$

$$= 1 - 2e^{2x} - xe^x$$

Example

IF $f(x) = (x^2+2x)e^x$ then find $f'(x)$

$$f'(x) = \frac{d}{dx}\left[(x^2+2x)e^x\right] = \cancel{d}\,(x^2+2x)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2+2x$$

$$= (x^2+2x)e^x + e^x(2x+2)$$

$$= \left[x^2 + \underline{\underline{2x}} + \underline{\underline{2x}} + 2\right]e^x$$

$$= \left[x^2+4x+2\right]e^x$$

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).\frac{d}{dx}[f(x)] - f(x).\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

or $$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x).f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example

$$\frac{d}{dx}\left[\frac{x^2+x-2}{x^2+6}\right] = \frac{(x^2+6)\frac{d}{dx}[x^2+x-2] - (x^2+x-2)\frac{d}{dx}[x^2+6]}{(x^2+6)^2}$$

$$= \frac{(x^2+6)(2x+1) - (x^2+x-2)(2x)}{(x^2+6)^2}$$

$$= \frac{2x^3+12x+x^2+6-(2x^3+2x^2-4x)}{(x^2+6)^2}$$

$$= \frac{\underline{\cancel{2x^3}}\boxed{+12x}\underline{+x^2}+6\underline{-\cancel{2x^3}}\underline{-2x^2}\boxed{+4x}}{(x^2+6)^2}$$

$$= \frac{16x-x^2+6}{(x^2+6)^2}$$

Example

If $f(x) = \sqrt{x} . g(x)$ where $g(4) = 2$ and $g'(4) = 3$ then Find $f'(4)$

$$f(x) = \sqrt{x}\, g(x) = x^{1/2} . g(x)$$

$$f'(x) = \frac{d}{dx}\left[x^{1/2} . g(x)\right]$$

$$= x^{1/2}\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}\left[x^{1/2}\right]$$

$$= x^{1/2} g'(x) + g(x)\left[\frac{1}{2}x^{1/2 - 1}\right]$$

$$= x^{1/2} g'(x) + g(x)\left[\frac{1}{2}x^{-1/2}\right]$$

$$\boxed{f'(x) = x^{1/2} g'(x) + g(x)\left[\frac{1}{2\sqrt{x}}\right]}$$

$$f'(4) = \sqrt{4}\, g'(4) + g(4)\left[\frac{1}{2\sqrt{4}}\right]$$

$$= 2(3) + \cancel{2}\left[\frac{1}{\cancel{2}(2)}\right]$$

$$= 6 + \frac{1}{2}$$

$$= \frac{12+1}{2}$$

$$= \frac{13}{2}$$

Example

$$\frac{d}{dx}\left[\frac{e^x}{x^2}\right] = \frac{d}{dx}\left[e^x . x^{-2}\right] = e^x\frac{d}{dx}\left[x^{-2}\right] + x^{-2}\frac{d}{dx}\left[e^x\right]$$

$$= e^x\left[-2x^{-3}\right] + x^{-2}e^x$$

$$= e^x\left[-2x^{-3} + x^{-2}\right] = e^x\left[\frac{-2}{x^3} + \frac{1}{x^2}\right]$$

$$= e^x\left[\frac{-2}{x^3} + \frac{x}{x^3}\right] = e^x\left[\frac{-2+x}{x^3}\right] = \frac{(x-2)e^x}{x^3}$$

Example

Find an equation of tangent line to the curve $y = \frac{e^x}{1+x^2}$ at the point $(\underset{a}{1}, \underset{f(a)}{\tfrac{1}{2}e})$

$$y' = \frac{(1+x^2)\frac{d}{dx}[e^x] - e^x\frac{d}{dx}[1+x^2]}{(1+x^2)^2}$$

$$= \frac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2}$$

$$= \frac{(1+x^2-2x)e^x}{(1+x^2)^2} = \frac{(x^2-2x+1)e^x}{(1+x^2)^2}$$

$$= \frac{(x-1)(x-1)e^x}{(1+x^2)^2} = \frac{(x-1)^2e^x}{(1+x^2)^2}$$

$$\boxed{y' = \left(\frac{x-1}{1+x^2}\right)^2 \cdot e^x}$$

$$m = y'(a) = y'(1) = \left(\frac{1-1}{1+1^2}\right)^2 e^1 = \left(\frac{0}{2}\right)^2 e = 0 . e = 0$$

$\therefore \boxed{m=0} \Longrightarrow y = f(a)$

$\therefore y = \frac{1}{2}e$ is Horizontal Tangent

Note

If $m = 0$ then $y = f(a)$

If $m = \frac{1}{0}$ then $x = a$

Example

If $y = \frac{1-x}{x+2}$ then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$y' = \frac{(x+2)\frac{d}{dx}[1-x] - (1-x)\frac{d}{dx}[x+2]}{(x+2)^2}$$

$$= \frac{(x+2)(-1) - (1-x)(1)}{(x+2)^2}$$

$$= \frac{\cancel{-x} - 2 - 1 + \cancel{x}}{(x+2)^2}$$

$$y' = \frac{-3}{(x+2)^2} = \frac{-3}{x^2+4x+4}$$

$$y'' = \frac{(x^2+4x+4)\frac{d}{dx}[-3] - 3\frac{d}{dx}[x^2+4x+4]}{(x^2+4x+4)^2}$$

$$= \frac{(x^2+4x+4)(0) - 3(2x+4)}{(x^2+4x+4)^2}$$

$$= \frac{-3(2x+4)}{(x^2+4x+4)^2} = \frac{-6x-12}{((x+2)^2)^2} = \frac{-6x-12}{(x+2)^4}$$

Example

If $h(2) = 4$ and $h'(2) = -3$ then

find $\frac{d}{dx}\left(\frac{h(x)}{x}\right)\Big|_{x=2}$

$$\frac{d}{dx}\left[\frac{h(x)}{x}\right] = \frac{x\frac{d}{dx}[h(x)] - h(x)\frac{d}{dx}[x]}{x^2}$$

$$= \frac{xh'(x) - h(x)(1)}{x^2}$$

$$\frac{d}{dx}\left[\frac{h(x)}{x}\right]\Big|_{x=2} = \frac{2h'(2) - h(2)}{(2)^2} = \frac{2(-3) - 4}{4}$$

$$= -\frac{6-4}{4} = -\frac{10}{4} = -\frac{5}{2}$$

Example

If $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$ and $g'(4) = -3$ then

Find $h'(4)$

a) $h(x) = 3f(x) + 8g(x)$

$h'(x) = 3f'(x) + 8g'(x)$

$h'(4) = 3f'(4) + 8g'(4)$

$= 3(6) + 8(-3)$

$= 18 - 24$

$= -6$

b) $h(x) = f(x).g(x)$

$h'(x) = f(x).g'(x) + g(x)f'(x)$

$h'(4) = f(4).g'(4) + g(4).f'(4)$

$= 2(-3) + 5(6)$

$= -6 + 30$

$= 24$

c) $h(x) = \frac{f(x)}{g(x)}$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(4) = \frac{g(4)f'(4) - f(4)g'(4)}{[g(4)]^2}$$

$$= \frac{5(6) - 2(-3)}{[5]^2}$$

$$= \frac{30 + 6}{25}$$

$$= \frac{36}{25}$$

d) $h(x) = \frac{g(x)}{f(x)+g(x)}$

$$h'(x) = \frac{[f(x)+g(x)]g'(x) - g(x)[f'(x)+g'(x)]}{[f(x)+g(x)]^2}$$

$$= \frac{f(x)g'(x) + \cancel{g(x)g'(x)} - g(x)f'(x) - \cancel{g(x)g'(x)}}{(f(x)+g(x))^2}$$

$$= \frac{f(x)g'(x) - g(x)f'(x)}{(f(x)+g(x))^2}$$

$$h'(4) = \frac{f(4)g'(4) - g(4)f'(4)}{(f(4)+g(4))^2} = \frac{2(-3) - 5(6)}{(2+5)^2} = \frac{-6-30}{7^2}$$

$$= \frac{-36}{49}$$

Example

if $f(x) = 3^x$ then find

$f^{(n)}(x)$?

$$f(x) = 3^x$$

$$f'(x) = 3^x . \text{Ln}(3)$$

$$f''(x) = \text{Ln}(3) . \frac{d}{dx}\left[3^x\right]$$

$$= \text{Ln}(3) . 3^x . \text{Ln}(3)$$

$$= (\text{Ln}(3))^2 . 3^x$$

$$f'''(x) = (\text{Ln}(3))^2 . \text{Ln}(3) . 3^x$$

$$= (\text{Ln}(3))^3 . 3^x$$

$\vdots$

$$f^{(n)}(x) = (\text{Ln}(3))^n . 3^x$$

3.3 – Derivative of Trigonometric Function.

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x . \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x . \cot x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

Example

1) Differentiate $y = x^2 . \sin x$

$$y' = \frac{d}{dx}[x^2 \sin x]$$

$$= x^2 \frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x^2]$$

$$= x^2 \cos x + 2x \sin x$$

② $y = \csc\theta + e^\theta \cot\theta$

$$y' = \frac{d}{d\theta}[\csc\theta + e^\theta \cot\theta] = \frac{d}{d\theta}[\csc\theta] + \frac{d}{d\theta}[e^\theta \cot\theta]$$

$$= -\csc\theta . \cot\theta + e^\theta \frac{d}{d\theta}[\cot\theta] + \cot\theta \frac{d}{d\theta}[e^\theta]$$

$$= -\csc\theta . \cot\theta + e^\theta(-\csc^2\theta) + \cot\theta(e^\theta)$$

$$= -\csc\theta . \cot\theta - e^\theta \csc^2\theta + e^\theta \cot\theta$$

③ $$y = \frac{\sec\theta}{1+\sec\theta}$$

$$y' = \frac{(1+\sec\theta)\frac{d}{d\theta}[\sec\theta] - \sec\theta \frac{d}{dx}[1+\sec\theta]}{(1+\sec\theta)^2}$$

$$y' = \frac{(1+\sec\theta)\,\underline{\sec\theta.\tan\theta} - \sec\theta.\,\underline{\sec\theta.\tan\theta}}{(1+\sec\theta)^2}$$

$$= \frac{\sec\theta.\tan\theta[1+\cancel{\sec\theta}-\cancel{\sec\theta}]}{(1+\sec\theta)^2}$$

$$= \frac{\sec\theta.\tan\theta(1)}{(1+\sec\theta)^2}$$

$$= \frac{\sec\theta.\tan\theta}{(1+\sec\theta)^2}$$

④ $$y = \frac{\cos x}{1-\sin x}$$ find $y'(\frac{\pi}{6})$

$$y' = \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} = \frac{-\sin x + 1}{(1-\sin x)^2}$$

$$= \frac{\cancel{(1-\sin x)}}{\cancel{(1-\sin x)}(1-\sin x)} = \frac{1}{1-\sin x}$$

$$y'\left(\frac{\pi}{6}\right) = \frac{1}{1-\sin\left(\frac{\pi}{6}\right)}$$

$$= \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 2$$

⑤ $y = \dfrac{1-\sec x}{\tan x}$

$$y = \frac{1}{\tan x} - \frac{\sec x}{\tan x}$$

$$= \cot x - \frac{\left(\frac{1}{\cos x}\right)}{\left(\frac{\sin x}{\cos x}\right)}$$

$$= \cot x - \left(\frac{1}{\cancel{\cos x}}\right) \cdot \left(\frac{\cancel{\cos x}}{\sin x}\right)$$

$$= \cot x - \frac{1}{\sin x}$$

$$y = \cot x - \csc x$$

$$y' = -\csc^2 x - [-\csc x \cdot \cot x]$$

$$= -\csc^2 x + \csc x \cdot \cot x$$

$$= \csc x \cdot \cot x - \csc^2 x$$

$$y' = \csc x \left[\cot x - \csc x \right]$$

$$= \frac{1}{\sin x} \left[\frac{\cos x}{\sin x} - \frac{1}{\sin x} \right]$$

$$= \frac{1}{\sin x} \left[\frac{\cos x - 1}{\sin x} \right]$$

$$= \frac{\cos x - 1}{\sin^2 x}$$

$$= \frac{\cos x - 1}{1 - \cos^2 x}$$

$$= \frac{-\cancel{(1-\cos x)}}{\cancel{(1-\cos x)}(1+\cos x)}$$

$$= \frac{-1}{1+\cos x}$$

$$y = \frac{\tan x - 1}{\sec x}$$

$$y = \frac{\tan x}{\sec x} - \frac{1}{\sec x}$$

$$= \frac{\left(\frac{\sin x}{\cancel{\cos x}}\right)}{\left(\frac{1}{\cancel{\cos x}}\right)} - \cos x$$

$$= \frac{\sin x}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{1} - \cos x = \sin x - \cos x$$

$$y' = \cos x - (-\sin x) = \cos x + \sin x$$

⑥ $f(x) = xe^x \sec x$

$$f'(x) = \frac{d}{dx}[x]\, e^x \sec x + x \cdot \frac{d}{dx}[e^x] \sec x + x \cdot e^x \frac{d}{dx}[\sec x]$$

$$= e^x \sec x + xe^x \sec x + x \cdot e^x \sec x \cdot \tan x$$

$$= e^x \sec x \left[1 + x + x\tan x\right]$$

7) Find the equation of tangent line of $y = \sec x$ at $\left(\frac{\pi}{6}, \frac{2\sqrt{3}}{3}\right)$

$y = \sec x$

① $y' = \sec x \cdot \tan x$

② $m = y'\left(\frac{\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right) \cdot \tan\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{2}{3}$

③ $m = \frac{2}{3}$ $\left(\frac{\pi}{6}, \frac{2\sqrt{3}}{3}\right)$

the equation of tangent line :

$$y - y_1 = m(x - x_1)$$

$$y - \frac{2\sqrt{3}}{3} = \frac{2}{3}\left(x - \frac{\pi}{6}\right)$$

$$3y - 2\sqrt{3} = 2\left(x - \frac{\pi}{6}\right)$$

$$3y - 2\sqrt{3} = 2x - \frac{2\pi}{6}$$

$$3y - 2\sqrt{3} = 2x - \frac{\pi}{3}$$

$$3y = 2x - \frac{\pi}{3} + 2\sqrt{3}$$

$$y = \frac{2}{3}x - \frac{\pi}{9} + \frac{2\sqrt{3}}{3}$$

Example

$y = \sin x + \cos x \qquad (\underset{x_1}{0}, \underset{y_1}{1})$

ⓐ equation of tangent line

① $y' = \cos x - \sin x$

② $m = y'(a) = y'(0) = \cos(0) - \sin(0) = 1 - 0 = 1$

③ equation of tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 1(x - 0)$$

$$y - 1 = x$$

$\boxed{y = x + 1}$ or $\boxed{y - x = 1}$

or $\boxed{y - x - 1 = 0}$

Example

Find the 27th derivative of $f(x) = \cos x$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$\boxed{f^{(4)}(x) = \cos x}$

$$\begin{array}{r} 6 \\ 4\overline{)\,27} \\ 24 \\ \hline \boxed{3} \end{array}$$

$f^{(27)}(x) = f^{(3)}(x)$

$= \sin x$

$\frac{d^{17}}{dx^{17}}[\sin x]$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$\boxed{f^{(4)}(x) = \sin x}$

$$\begin{array}{r} 4 \\ 4\overline{)\,17} \\ 16 \\ \hline \boxed{1} \end{array}$$

$f^{(17)}(x) = f'(x) = \cos x$

$\frac{d^{99}}{dx^{99}}[\sin x]$

$$\begin{array}{r} 24 \\ 4\overline{)\,99} \\ 8 \\ \hline 19 \\ 16 \\ \hline \boxed{3} \end{array}$$

$f^{(99)}(x) = f^{(3)}(x) = -\cos x$

$$\frac{d^{24}}{dx^{24}}\left[\cos x\right]$$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

$$\begin{array}{r} 6 \\ 4\,\overline{)\,24} \\ \underline{24} \\ \boxed{00} \end{array}$$

$f^{(24)}(x) = f(x) = \cos x$

3.4_ The chain Rule

The chain Rule

① If $F(x) = (h \circ g)(x) = h(g(x))$ then

$$\boxed{F'(x) = h'(g(x)) \cdot g'(x)}$$

② If $y = f(u)$ and $u = g(x)$ then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Example:.

1) If $y = 3u^2$ and $u = \sin x$ then find $\frac{dy}{dx}$?

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$= 6u \cdot \cos x$

$= 6 \sin x \cdot \cos x$

$= 3\underline{(2) \sin x \cdot \cos x}$

$= 3 \underline{\sin(2x)}$

② $$\frac{d}{dx}\left[e^{x^2+3}\right] = e^{x^2+3} \cdot \frac{d}{dx}\left[x^2+3\right]$$

$$= e^{x^2+3} \cdot 2x$$

$$= 2x\, e^{x^2+3}$$

② $$\frac{d}{dx}\left[\left(\tfrac{1}{2}\right)^{\sin x}\right] = \left(\frac{1}{2}\right)^{\sin x} \cdot \ln\left(\tfrac{1}{2}\right) \cdot \frac{d}{dx}\left[\sin x\right]$$

$$= \left(\frac{1}{2}\right)^{\sin x} \cdot \ln\left(2^{-1}\right) \cdot \cos x$$

$$= -\left(\frac{1}{2}\right)^{\sin x} \cdot \ln 2 \cdot \cos x$$

③ $$\frac{d}{dx}\left[3^{x^3}\right] = 3^{x^3} \cdot \frac{d}{dx}\left[x^3\right] \cdot \ln 3$$

$$= \underline{3^{x^3} \cdot 3}\, x^2 \cdot \ln 3$$

$$= \underline{3}^{x^3+1} \cdot x^2 \cdot \ln 3$$

④ Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$

$$F(x) = (x^2+1)^{1/2}$$

$$F'(x) = \frac{1}{2}(x^2+1)^{1/2-1} \cdot \frac{d}{dx}\left[x^2+1\right]$$

$$= \frac{1}{\cancel{2}}(x^2+1)^{-1/2} \cdot \cancel{2}x = \frac{1}{(x^2+1)^{1/2}} \cdot \frac{x}{1} = \frac{x}{\sqrt{x^2+1}}$$

⑤ Differentiate ⓐ $y = \sin(x^2)$

$$y' = \cos(x^2) \cdot \frac{d}{dx}\left[x^2\right]$$

$$= \cos(x^2) \cdot 2x$$

$$= 2x\cos(x^2)$$

ⓑ $y = \sin^2 x$

$$y = \left[\sin x\right]^2$$

$$y' = 2\left[\sin x\right]^{2-1} \cdot \frac{d}{dx}\left[\sin x\right]$$

$$= 2\sin x \cdot \cos x$$

$$= \sin 2x$$

ⓒ $y = \tan(x^3 + 5x)$

$$y' = \sec^2(x^3 + 5x) \cdot \frac{d}{dx}\left[x^3 + 5x\right]$$

$$= \sec^2(x^3 + 5x) \cdot (3x^2 + 5)$$

$$= (3x^2 + 5) \cdot \sec^2(x^3 + 5x)$$

ⓓ $y = \tan^3(e^{3x})$

$$y = \left[\tan(e^{3x})\right]^3$$

$$y' = 3\left[\tan(e^{3x})\right]^{3-1} \cdot \frac{d}{dx}\left[\tan(e^{3x})\right]$$

$$= 3\left[\tan(e^{3x})\right]^2 \cdot \sec^2(e^{3x}) \cdot \frac{d}{dx}\left[e^{3x}\right]$$

$$= 3\tan^2(e^{3x}) \cdot \sec^2(e^{3x}) \cdot e^{3x} \cdot \frac{d}{dx}\left[3x\right]$$

$$= 3\tan^2(e^{3x}) \cdot \sec^2(e^{3x}) \cdot e^{3x} \cdot 3$$

$$= 3(3)e^{3x}\sec^2(e^{3x}) . \tan^2(e^{3x})$$

ⓔ $y = \sin(\cos(\tan(x^2)))$

$$y' = \cos(\cos(\tan(x^2))) \cdot \frac{d}{dx}\left[\cos(\tan(x^2))\right]$$

$$= \cos(\cos(\tan(x^2))) \cdot -\sin(\tan(x^2)) \cdot \frac{d}{dx}\left[\tan(x^2)\right]$$

$$= -\cos(\cos(\tan(x^2))) . \sin(\tan(x^2)) \cdot \sec^2(x^2)\frac{d}{dx}\left[x^2\right]$$

$$= -\cos(\cos(\tan(x^2))) . \sin(\tan(x^2)) . \sec^2(x^2) \cdot \underline{\underline{2x}}$$

$$= -2x\sec^2(x^2) . \sin(\tan(x^2)) . \cos(\cos(\tan(x^2)))$$

⑥ $y = (x^3-1)^{100}$

$$y' = 100(x^3-1)^{100-1} \cdot \frac{d}{dx}\left[x^3-1\right]$$

$$= 100(x^3-1)^{99} \cdot 3x^2$$

$$= 3(100)\,x^2(x^3-1)^{99}$$

$$= 300x^2(x^3-1)^{99}$$

⑦ $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^{9-1} \cdot \frac{d}{dt}\left[\frac{t-2}{2t+1}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^{8} \cdot \left[\frac{(2t+1)(1)-(t-2)(2)}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^{8} \cdot \left[\frac{\cancel{2t}\textcircled{+1}-\cancel{2t}\textcircled{+4}}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^{8} \cdot \left[\frac{5}{(2t+1)^2}\right]$$

$$= \frac{9(t-2)^8}{(2t+1)^8} \cdot \frac{5}{(2t+1)^2} = \frac{9(5)(t-2)^8}{(2t+1)^{8+2}}$$

$$= \frac{45(t-2)^8}{(2t+1)^{10}}$$

⑧ $y = (2x+1)^5 \cdot (x^3-2)^4$

$$y' = (2x+1)^5 \cdot \frac{d}{dx}\left[(x^3-2)^4\right] + (x^3-2)^4 \frac{d}{dx}\left[(2x+1)^5\right]$$

$$= (2x+1)^5 \cdot 4(x^3-2)^3 \cdot \frac{d}{dx}(x^3-2) + (x^3-2)^4 \cdot 5(2x+1)^4 \cdot \frac{d}{dx}(2x+1)$$

$$= \underline{(2x+1)^5} \cdot 4\underline{\underline{(x^3-2)^3}}(3x^2) + \underline{\underline{(x^3-2)^4}} \cdot 5\underline{(2x+1)^4} \cdot (2)$$

$$= (2x+1)^4 \cdot (x^3-2)^3 \left[(2x+1) \cdot 4(3x^2) + 5(x^3-2)(2)\right]$$

$$= (2x+1)^4(x^3-2)^3 \left[\underset{2\times 6}{12}x^2(2x+1) + \underset{2\times 5}{10}(x^3-2)\right]$$

$$= 2(2x+1)^4(x^3-2)^3 \left[6x^2(2x+1) + 5(x^3-2)\right]$$

$$= 2(2x+1)^4(x^3-2)^3 \left[\underline{12x^3} + 6x^2 \underline{+5x^3} - 10\right]$$

$$= 2(2x+1)^4(x^3-2)^3 (17x^3 + 6x^2 - 10)$$

$$= 2(2x+1)^4(x^3-2)^3$$

⑨ $y = \frac{r}{\sqrt{r^2+1}}$

$$y' = \frac{\sqrt{r^2+1} \cdot (1) - r \cdot \left[\frac{1}{\cancel{2}}(r^2+1)^{-1/2} \cdot \cancel{2}r\right]}{\left(\sqrt{r^2+1}\right)^2}$$

$$y' = \frac{\frac{\sqrt{r^2+1}}{1} - \frac{r^2}{\sqrt{r^2+1}}}{(r^2+1)} = \frac{\frac{\left(\sqrt{r^2+1}\right)^2 - r^2}{\sqrt{r^2+1}}}{\left(\frac{r^2+1}{1}\right)} \cancel{=}$$

$$y' = \frac{r^2+1-r^2}{\sqrt{r^2+1}} \div \left(\frac{r^2+1}{1}\right)$$

$$= \frac{1}{\sqrt{r^2+1}} \cdot \frac{1}{r^2+1}$$

$$= \frac{1}{(r^2+1)\sqrt{r^2+1}}$$

⑩ $y = \pi^{\sec(\sqrt{x})}$

$$y' = \pi^{\sec(\sqrt{x})} \cdot \frac{d}{dx}\left[\sec(\sqrt{x})\right] \cdot \ln\pi$$

$$= \pi^{\sec(\sqrt{x})} \cdot \sec(\sqrt{x}) \cdot \tan(\sqrt{x}) \cdot \frac{d}{dx}[\sqrt{x}] \cdot \ln\pi$$

$$= \pi^{\sec(\sqrt{x})} \cdot \sec(\sqrt{x}) \cdot \tan(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \cdot \ln\pi$$

$$= \frac{1}{2} \ln\pi \frac{1}{\sqrt{x}} \sec(\sqrt{x}) \cdot \tan(\sqrt{x}) \cdot \pi^{\sec(\sqrt{x})}$$

⑪ $y = \cot(e^t) + e^{\cot(t)}$

$$y' = -\csc^2(e^t) \cdot \frac{d}{dt}[e^t] + e^{\cot(t)} \cdot \frac{d}{dx}[\cot(t)]$$

$$= -\csc^2(e^t) \cdot e^t + e^{\cot(t)} \cdot (-\csc^2(t))$$

$$= -e^t . \csc^2(e^t) - \csc^2 t\, e^{\cot(t)}$$

⑫ $f(t) = \sqrt[3]{1 + \tan(t)} = (1 + \tan(t))^{1/3}$

$$f'(t) = \frac{1}{3}(1 + \tan(t))^{1/3 - 1} \cdot \frac{d}{dt}[1 + \tan(t)]$$

$$= \frac{1}{3}(1 + \tan(t)]^{-2/3} \cdot \sec^2(t)$$

$$= \frac{\sec^2(t)}{3[1 + \tan(t)]^{2/3}}$$

$$= \frac{\sec^2(t)}{3\sqrt[3]{(1 + \tan(t))^2}}$$

⑬ $y = \sqrt{1+2e^{3x}}$

$$y = (1+2e^{3x})^{1/2}$$

$$y' = \frac{1}{2}(1+2e^{3x})^{1/2-1} \cdot \frac{d}{dx}\left[1+2e^{3x}\right]$$

$$= \frac{1}{2}(1+2e^{3x})^{-1/2} \cdot 2e^{3x} \cdot \frac{d}{dx}[3x]$$

$$= \frac{1}{\cancel{2}}(1+2e^{3x})^{-1/2} \cdot \cancel{2}e^{3x} \cdot (3)$$

$$= \frac{3e^{3x}}{(1+2e^{3x})^{1/2}} = \frac{3e^{3x}}{\sqrt{1+2e^{3x}}}$$

⑭ $f(t) = e^{2t} \cdot \cos(4t)$

$$f'(t) = e^{2t}\frac{d}{dt}\left[\cos(4t)\right] + \cos(4t)\frac{d}{dt}\left[e^{2t}\right]$$

$$= -e^{2t}\sin(4t) \cdot \frac{d}{dt}[4t] + \cos(4t) \cdot e^{2t}\frac{d}{dt}[2t]$$

$$= -e^{2t}\sin(4t) \cdot (4) + \cos(4t) \cdot e^{2t}(2)$$

$$= 2e^{2t}\left[-2\sin(4t) + \cos(4t)\right]$$

⑮ $y = (2)^{3^{x^2}}$

$$y' = (2)^{3^{x^2}} \cdot \frac{d}{dx}\left[3^{x^2}\right] \cdot \ln(2)$$

$$= (2)^{3^{x^2}} \cdot 3^{x^2} \cdot \frac{d}{dx}\left[x^2\right] \cdot \ln(3) \cdot \ln(2)$$

$$= \underline{\underline{(2)}}^{3^{x^2}} \cdot 3^{x^2} \cdot \underline{\underline{\overset{1}{2}}}x \cdot \ln(3) \cdot \ln(2)$$

$$= (2)^{3^{x^2}+1} \cdot 3^{x^2} \cdot x \cdot \ln(3) \cdot \ln(2)$$

⑯ if $f(x) = \cos(x^2)$ then find $f''(x)$

$$F'(x) = -\sin(x^2) \cdot \frac{d}{dx}\left[x^2\right]$$

$$= -\sin(x^2) \cdot (2x)$$

$$= \underbrace{-2x}_{1} \cdot \underbrace{\sin(x^2)}_{2}$$

$$= -2x \frac{d}{dx}\left[\sin(x^2)\right] + \sin(x^2) \cdot \frac{d}{dx}\left[-2x\right]$$

$$= -2x\cos(x^2) \cdot \overset{\frac{d}{dx}[x^2]}{\uparrow} + \sin(x^2)(-2)$$

$$= -2x\cos(x^2) \cdot 2x - 2\sin(x^2)$$

$$= -4x^2\cos(x^2) - 2\sin(x^2)$$

⑰ find the equation of tangent line and normal line of $y = \sin(\sin x)$ at $(\pi/0)$

$$y' = \cos(\sin x) \cdot \frac{d}{dx}[\sin x]$$

$$= \cos(\sin x) \cdot \cos x$$

$$= \cos(x) \cdot \cos(\sin x)$$

$$m = y'(\pi) = \cos(\pi) \cdot \cos(\sin \pi)$$

$$= -1 \cdot \cos(0)$$

$$= -1 \cdot 1$$

$\boxed{m = -1}$; $m_\perp = -\frac{1}{m} = \frac{-1}{-1}$

$\boxed{m_\perp = 1}$

the equation of tanget line at $(\underset{x_1}{\pi} / \underset{y_1}{0})$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -1(x - \pi)$$

$\boxed{y = -x + \pi}$ → $y + x - \pi = 0$ or $y + x = \pi$

the equation on normal line at $(\pi, 0)$

$y - y_1 = m_\perp(x - x_1) \Rightarrow \boxed{y = x - \pi}$ → $y - x + \pi = 0$ or $y - x = -\pi$

⑱ $y = \cos^2 x$

$y = [\cos x]^2$

$y' = 2[\cos x]^{2-1} \cdot \frac{d}{dx}[\cos x]$

$= 2\cos x \cdot (-\sin x)$

$= -2\sin x \cdot \cos x$

$= -\sin(2x)$

$y'' = -\cos(2x) \cdot \frac{d}{dx}[2x]$

$= -\cos(2x) \cdot (2)$

$= -2\cos(2x)$

⑲ $y = \sin^3(x^2) = [\sin(x^2)]^3$

$y' = 3[\sin(x^2)]^{3-1} \cdot \frac{d}{dx}[\sin(x^2)]$

$y' = 3[\sin(x^2)]^2 \cdot \cos(x^2) \cdot \frac{d}{dx}[x^2]$

$= 3\sin^2(x^2) \cdot \cos(x^2) \cdot 2x$

$= 6x\sin^2(x^2) \cdot \cos(x^2)$

⑳ $y = 2^{e^x}$

$$y' = 2^{e^x} , \ln(2) . \frac{d}{dx}[e^x]$$

$$y' = 2^{e^x} . \ln(2) . e^x$$

$$= e^x . 2^{e^x} . \ln(2)$$

$$y'' = \frac{d}{dx}\left[e^x . 2^{e^x} , \underset{\text{عدد}}{\underline{\ln(2)}}\right]$$

$$= \ln(2) \frac{d}{dx}\left[\underset{1}{\underline{e^x}} . \underset{2}{\underline{2}}^{e^x}\right]$$

$$= \ln(2)\left[e^x \frac{d}{dx}[2^{e^x}] + 2^{e^x} \frac{d}{dx}[e^x]\right]$$

$$= \ln(2)\left[e^x . 2^{e^x} . \frac{d}{dx}(e^x)\ln 2 + 2^{e^x} . e^x\right]$$

$$= \ln(2)\left[\underline{e^x . 2}^{e^x} . e^x . \ln 2 + \underline{2^{e^x} . e^x}\right]$$

$$= \ln(2) . 2^{e^x} , e^x \left(e^x \ln 2 + 1\right)$$

㉑ if $F(x) = f(g(x))$ where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$
$g(5) = -2$, $g'(5) = 6$

Find $F'(5)$

$$F'(x) = f'(g'(x)) . g'(x) \Rightarrow F'(5) = f'(g(5)) . g'(5)$$

$$= f'(-2) . 6$$

$$= 4(6) = 24$$

(22) $h(x) = \sqrt{4+3f(x)}$ where $f(1) = 7$

$f'(1) = 4$

find $h'(1)$

$$h(x) = (4+3f(x))^{1/2}$$

$$h'(x) = \frac{1}{2}(4+3f(x))^{1/2-1} \cdot (3f'(x))$$

$$= \frac{1}{2}(4+3f(x))^{-1/2} \cdot (3f'(x))$$

$$= \frac{3f'(x)}{2\sqrt{4+3f(x)}}$$

$$h'(1) = \frac{3f'(1)}{2\sqrt{4+3f(1)}} = \frac{3(4)}{2\sqrt{4+3(7)}}$$

$$= \frac{12}{2\sqrt{4+21}}$$

$$= \frac{12}{2\sqrt{25}}$$

$$= \frac{6}{\sqrt{25}}$$

$$= \frac{6}{5}$$

3.5 - Implicit Differentiation

التفاضل الضمني

$y = f(x) \longrightarrow$ explicit function "دالة صريحة"

$f(x) + g(y) = h(x,y) \longrightarrow$ Implicit function "دالة ضمنية"

Example (1)

$y = \sin(x) + \sqrt{x} + e^{x} \longrightarrow$ Explicit function

$y^2 + \sin(x.y) - 2^{y} = e^{\sqrt{x}} + 5^{x+y} \longrightarrow$ Implicit function

Example (2)

If $x^2 + y^2 = 25$ then a) find y' or $\frac{dy}{dx}$

$$2x + 2y.y' = 0$$

$$2y.y' = -2x$$

$$y' = \frac{-2x}{2y}$$

$$\boxed{y' = -\frac{x}{y}} \quad \#$$

b) Find y'' or $\frac{d^2y}{dx^2}$

$$y' = -\frac{x}{y}$$

$$y'' = -\left[\frac{y(1) - x(1 \cdot y')}{y^2}\right]$$

$$y'' = -\left[\frac{y - xy'}{y^2}\right]$$

$$y'' = -\left[\frac{\frac{y}{1} - \frac{x}{1}\left(-\frac{x}{y}\right)}{y^2}\right]$$

$$y'' = -\left[\frac{\frac{y}{1} + \frac{x^2}{y}}{y^2}\right]$$

$$y'' = -\left[\frac{\left(\frac{y^2 + x^2}{y}\right)}{\left(\frac{y^2}{1}\right)}\right]$$

$$y'' = -\left(\frac{y^2+x^2}{y}\right) \div \frac{y^2}{1}$$

$$y'' = -\frac{25}{y} \cdot \frac{1}{y^2}$$

$$\boxed{y'' = -\frac{25}{y^3}} \quad \#$$

c) Find $y'(3,5)$

$$y' = -\frac{x}{y}$$

$$y'\Big|_{(3,5)} = -\frac{3}{5}$$

d) $\frac{d^2y}{dx^2}\Big|_{(2,3)} = \text{.......}$

$$\frac{d^2y}{dx^2}\Big|_{(2,3)} = \frac{-25}{y^3}\Big|_{(2,3)} = -\frac{25}{3^3} = -\frac{25}{27}$$

Example (3)

If $x^4 + y^4 = 16$ then find y''

$$4x^3 + 4y^3 \cdot y' = 0$$

$$4y^3 \cdot y' = -4x^3$$

$$y' = -\frac{4x^3}{4y^3}$$

$$\boxed{y' = -\frac{x^3}{y^3}}$$

$$y'' = -\left[\frac{y^3(3x^2) - x^3(3y^2 \cdot y')}{(y^3)^2}\right]$$

$$y'' = -\left[\frac{3x^2y^3 - 3x^3y^2y'}{y^6}\right]$$

$$y'' = -\left[\frac{3x^2y^3 - \frac{3x^3y^2}{1}\left(-\frac{x^3}{y^3}\right)}{y^6}\right]$$

$$y'' = -\left[\frac{\frac{3x^2y^3}{1} + \frac{3x^6}{y}}{y^6}\right]$$

$$y'' = -\left[\frac{\left(\frac{3x^2y^4 + 3x^6}{y}\right)}{\left(\frac{y^6}{1}\right)}\right]$$

$$y'' = -\left[\frac{3x^2(y^4 + x^4)}{y} \div \frac{y^6}{1}\right]$$

$$y'' = -\left[\frac{3x^2(16)}{y} \cdot \frac{1}{y^6}\right]$$

$$y'' = -\left[\frac{48x^2}{y^7}\right]$$

$$\boxed{y'' = \frac{-48x^2}{y^7}} \#$$

Scanned with CamScanner

Example (4)

If $x^3+y^3=6xy$ then find

ⓐ y'

$$x^3+y^3 = \underset{①}{6x}\,\underset{②}{y}$$

$$3x^2+3y^2\cdot y' = 6x\,\frac{d}{dx}[y] + y\cdot\frac{d}{dx}[6x]$$

$$3x^2+3y^2\cdot y' = 6x\,(1\cdot y') + y\cdot[6]$$

$$3x^2+3y^2y' = 6xy' + 6y$$

$$3y^2y' - 6xy' = 6y - 3x^2$$

$$3y'(y^2-2x) = 3(2y-x^2)$$

$$y' = \frac{3(2y-x^2)}{3(y^2-2x)}$$

$$\boxed{y' = \frac{2y-x^2}{y^2-2x}} \quad \#$$

ⓑ Find the Slope of tangent line at (3,2)

$$m = y'\Big|_{\underset{x\ y}{(3,2)}} = \frac{2y-x^2}{y^2-2x}\Big|_{(3,2)} = \frac{2(2)-3^2}{2^2-2(3)} = \frac{4-9}{4-6}$$

$$= \frac{-5}{-2} = \frac{5}{2}$$

Ⓒ Find the equation of tangent line and normal line at (3, 2)

tangent line

$m = \frac{5}{2}$; $(\underset{x_1}{3}, \underset{y_1}{2})$

the equation of tangat line :

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{5}{2}(x - 3)$

$2y - 4 = 5(x - 3)$

$2y - 4 = 5x - 15$

$2y - 5x - 4 + 15 = 0$

$\boxed{2y - 5x + 11 = 0}$

or

$2y - 5x = -15 + 4$

$\boxed{2y - 5x = -11}$

or

$2y = 5x - 15 + 4$

$2y = 5x - 11$

$\boxed{y = \frac{5}{2}x - \frac{11}{2}}$

Normal line

$m_{\perp} = -\frac{1}{m}$; $(\underset{x_1}{3}, \underset{y_1}{2})$

$= -\frac{2}{5}$

the equation of normal line :

$y - y_1 = m_{\perp}(x - x_1)$

$y - 2 = -\frac{2}{5}(x - 3)$

$5y - 10 = -2(x - 3)$

$5y - 10 = -2x + 6$

$5y + 2x - 10 - 6 = 0$

$\boxed{5y + 2x - 16 = 0}$

or

$5y + 2x = 6 + 10$

$\boxed{5y + 2x = 16}$

or

$5y = -2x + 6 + 10$

$5y = -2x + 16$

$\boxed{y = -\frac{2}{5}x + \frac{16}{5}}$

Example (5)

IF $\sin(x+y) = y^2\cos x$ then find y'

$$\cos(x+y)\left[1+y'\right] = y^2\frac{d}{dx}\left[\cos x\right] + (\cos x)\frac{d}{dx}\left[y^2\right]$$

$$\cos(x+y) + y'\cos(x+y) = -y^2\sin x + (\cos x)(2yy')$$

$$\cos(x+y) + \underline{y'\cos(x+y)} = -y^2\sin x + \underline{2yy'\cos x}$$

$$y'\cos(x+y) - 2yy'\cos x = -y^2\sin x - \cos(x+y)$$

$$y'\left[\cos(x+y) - 2y\cos x\right] = -y^2\sin x - \cos(x+y)$$

$$y' = \frac{-y^2\sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x}$$

$$y' = \frac{-\left[y^2\sin x + \cos(x+y)\right]}{\cos(x+y) - 2y\cos x}$$

$$y' = \frac{y^2\sin x + \cos(x+y)}{-\left[\cos(x+y) - 2y\cos x\right]}$$

$$y' = \frac{y^2\sin x + \cos(x+y)}{-\cos(x+y) + 2y\cos x}$$

$$y' = \frac{y^2\sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Example (6)

2 If $f(x) + x^2 [f(x)]^3 = 10$ and $f(1) = 2$

then find $f'(1)$

$$f(x) + \underbrace{x^2}_{①} \cdot \underbrace{[f(x)]^3}_{②} = 10$$

$$f'(x) + x^2 \cdot \frac{d}{dx}(f(x))^3 + (f(x))^3 \cdot \frac{d}{dx}[x^2] = 0$$

$$f'(x) + 3x^2 (f(x))^2 \cdot f'(x) + (f(x))^3 \cdot 2x = 0$$

$$f'(x)\left[1 + 3x^2 (f(x))^2\right] = -2x (f(x))^3$$

$$f'(x) = \frac{-2x(f(x))^3}{1+3x^2(f(x))^2}$$

$$f'(1) = \frac{-2(1)(f(1))^3}{1+3(1)^2(f(1))^2}$$

$$= \frac{-2(2)^3}{1+3(2)^2} = \frac{-2(8)}{1+3(4)}$$

$$= \frac{-16}{1+12} = -\frac{16}{13}$$

Example 7

If $1+x = \sin(xy^2)$ then find y'

$$1+x = \sin(xy^2)$$

$$1 = \cos(xy^2)\cdot(2xyy' + y^2)$$

$$1 = 2xyy'\cos(xy^2) + y^2\cos(xy^2)$$

$$1 - y^2\cos(xy^2) = 2xyy'\cos(xy^2)$$

$$\frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)} = y'$$

$$\frac{1}{2xy\cos(xy^2)} - \frac{y^2\cos(xy^2)}{2xy\cos(xy^2)} = y'$$

$$\frac{\sec(xy^2)}{2xy} - \frac{y^2}{2xy} = y'$$

$$\frac{\sec(xy^2) - y^2}{2xy} = y'$$

Example (8)

Find the equation of tangent line and normal line of $y\sin 2x = x\cos 2y$ at $(\frac{\pi}{2}, \frac{\pi}{4})$

✓ $y\sin(2x) = x\cos(2y)$

$$2y\cos(2x) + \sin(2x)\cdot y' = x\left(-2y'\sin(2y)\right) + \cos(2y)$$

$$2y\cos(2x) + \underline{y'\sin(2x)} = \underline{-2xy'\sin(2y)} + \cos(2y)$$

$$y'\sin(2x) + 2xy'\sin(2y) = \cos(2y) - 2y\cos(2x)$$

$$y'\left[\sin(2x) + 2x\sin(2y)\right] = \cos(2y) - 2y\cos(2x)$$

$$y' = \frac{\cos(2y) - 2y\cos(2x)}{\sin(2x) + 2x\sin(2y)}$$

✓ $$m = y'\Big|_{(\frac{\pi}{2}, \frac{\pi}{4})} = \frac{\cos(\frac{2\pi}{4}) - 2(\frac{\pi}{4})\cos(\frac{2\pi}{2})}{\sin(\frac{2\pi}{2}) + 2(\frac{\pi}{2})\sin(\frac{2\pi}{4})} = \frac{\cos(\frac{\pi}{2}) - \frac{\pi}{2}\cos(\pi)}{\sin(\pi) + \pi\sin(\frac{\pi}{2})}$$

$x \quad y$

$$= \frac{0 - \frac{\pi}{2}(-1)}{0 + \pi(1)} = \frac{(\frac{\pi}{2})}{(\frac{\pi}{1})} = \frac{\pi}{2} \cdot \frac{1}{\pi} = \frac{1}{2}$$

✓ $$m_{\perp} = -\frac{1}{m} = -2$$

*the equation of tangent line :

$m = \frac{1}{2}$; $(\frac{\pi}{2}, \frac{\pi}{4})$

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$

$2y - \frac{2\pi}{4} = (x - \frac{\pi}{2})$

$2y - \frac{\pi}{2} = x - \frac{\pi}{2}$

$2y = x - \frac{\pi}{2} + \frac{\pi}{2}$

$2y = x$ or $\boxed{2y - x = 0}$

or $\boxed{y = \frac{1}{2}x}$ #

*the equation of normal line :

$m_\perp = -2$; $(\frac{\pi}{2}, \frac{\pi}{4})$

$y - y_1 = m_\perp (x - x_1)$

$y - \frac{\pi}{4} = -2(x - \frac{\pi}{2})$

$y - \frac{\pi}{4} = -2x + 2(\frac{\pi}{2})$

$y - \frac{\pi}{4} = -2x + \pi \Rightarrow 4y - \pi = -8x + 4\pi$

$4y + 8x - \pi - 4\pi = 0$

$\boxed{4y + 8x - 5\pi = 0}$ #

3.6 - Derivatives of inverse Trigonometric Functions and Derivatives of logarithmic Functions

Derivatives of Inverse Trigonometric Functions

1) $\boxed{\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}}$; $\frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$

$\boxed{\frac{d}{dx}\left[\sin^{-1}(u)\right] = \frac{u'}{\sqrt{1-u^2}}}$; $\frac{d}{dx}\left[\cos^{-1}(u)\right] = -\frac{u'}{\sqrt{1-u^2}}$

2) $\boxed{\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}}$; $\frac{d}{dx}\left[\cot^{-1}(x)\right] = -\frac{1}{1+x^2}$

$\boxed{\frac{d}{dx}\left[\tan^{-1}(u)\right] = \frac{u'}{1+u^2}}$; $\frac{d}{dx}\left[\cot^{-1}(u)\right] = -\frac{u'}{1+u^2}$

3) $\boxed{\frac{d}{dx}\left[\sec^{-1}x\right] = \frac{1}{x\sqrt{x^2-1}}}$; $\frac{d}{dx}\left[\csc^{-1}x\right] = -\frac{1}{x\sqrt{x^2-1}}$

$\boxed{\frac{d}{dx}\left[\sec^{-1}(u)\right] = \frac{u'}{u\sqrt{u^2-1}}}$; $\frac{d}{dx}\left[\csc^{-1}(u)\right] = -\frac{u'}{u\sqrt{u^2-1}}$

Example (1)

Find the Derivative of the following function and Simplify where possible.

① $y = x \cdot \tan^{-1}(\sqrt{x})$

$$y' = x \cdot \frac{d}{dx}\left[\tan^{-1}(\sqrt{x})\right] + \tan^{-1}(\sqrt{x})\,\frac{d}{dx}[x]$$

$$= x \cdot \frac{1/2\sqrt{x}}{1+(\sqrt{x})^2} + \tan^{-1}(\sqrt{x}) \cdot (1)$$

$$= x \cdot \frac{\frac{1}{2\sqrt{x}}}{1+x} + \tan^{-1}(\sqrt{x})$$

$$= \frac{x}{1} \cdot \frac{1}{2\sqrt{x}(1+x)} + \tan^{-1}(\sqrt{x})$$

$$= \frac{x^1}{1} \cdot \frac{1}{2x^{1/2}(1+x)} + \tan^{-1}(\sqrt{x})$$

$$= \frac{x^{1-1/2}}{2(1+x)} + \tan^{-1}(\sqrt{x})$$

$$= \frac{x^{1/2}}{2(1+x)} + \tan^{-1}(\sqrt{x})$$

$$= \frac{\sqrt{x}}{2+2x} + \tan^{-1}(\sqrt{x})$$

② $y = \frac{1}{\sin^{-1}(x)}$

$$y = \frac{1}{[\sin^{-1}(x)]^{1}}$$

$$y = [\sin^{-1}(x)]^{-1}$$

$$y' = -1 \cdot [\sin^{-1}(x)]^{-1-1} \cdot \frac{d}{dx}[\sin^{-1}(x)]$$

$$y' = -1[\sin^{-1}(x)]^{-2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$y' = \frac{-1}{[\sin^{-1}(x)]^{2}} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$y' = \frac{-1}{[\sin^{-1}(x)]^{2} \cdot \sqrt{1-x^2}}$$

③ $y = \sqrt{\tan^{-1}x}$

$y = (\tan^{-1}(x))^{1/2}$

$y' = \frac{1}{2}[\tan^{-1}(x)]^{1/2 - 1} \cdot \frac{d}{dx}[\tan^{-1}(x)]$

$y' = \frac{1}{2}[\tan^{-1}(x)]^{-1/2} \cdot \frac{1}{1+x^2}$

$= \frac{1}{2[\tan^{-1}(x)]^{1/2}} \cdot \frac{1}{1+x^2}$

$= \frac{1}{2(1+x^2)\sqrt{\tan^{-1}(x)}}$

④ $y = \sin^{-1}(2x+1)$

$y' = \frac{\frac{d}{dx}(2x+1)}{\sqrt{1-(2x+1)^2}} = \frac{2}{\sqrt{1-(4x^2+4x+1)}}$

$= \frac{2}{\sqrt{\cancel{1}-4x^2-4x\cancel{-1}}} = \frac{2}{\sqrt{-4x^2-4x}}$

$= \frac{2}{\sqrt{4(-x^2-x)}} = \frac{2}{\sqrt{4}\cdot\sqrt{-x^2-x}} = \frac{2}{2\sqrt{-x^2-x}}$

$= \frac{1}{\sqrt{-x^2-x}}$

⑤ $y = \tan^{-1}\left(\frac{x}{5}\right)$

$$y = \tan^{-1}\left[\frac{1}{5}x\right]$$

$$y' = \frac{\frac{d}{dx}\left[\frac{1}{5}x\right]}{1 + \left(\frac{x}{5}\right)^2}$$

$$y' = \frac{\frac{1}{5}}{\frac{1}{1} + \frac{x^2}{25}} = \frac{\left(\frac{1}{5}\right)}{\left(\frac{25+x^2}{25}\right)}$$

$$y' = \frac{1}{5} \div \frac{25+x^2}{25}$$

$$y' = \frac{1}{5} \cdot \frac{25}{25+x^2}$$

$$y' = \frac{5}{25+x^2}$$

⑥ $y = \sin^{-1}\left(\frac{x}{3}\right)$

$$y = \sin^{-1}\left(\frac{1}{3}x\right)$$

$$y' = \frac{\frac{d}{dx}\left[\frac{1}{3}x\right]}{\sqrt{1 - \left(\frac{x}{3}\right)^2}}$$

$$= \frac{1/3}{\sqrt{\frac{1}{1} - \frac{x^2}{9}}} = \frac{\frac{1}{3}}{\sqrt{\frac{9-x^2}{9}}}$$

$$= \frac{1/3}{\frac{\sqrt{9-x^2}}{\sqrt{9}}} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{\sqrt{9-x^2}}{3}\right)}$$

$$= \frac{1}{3} \div \frac{\sqrt{9-x^2}}{3}$$

$$= \frac{1}{3} \cdot \frac{3}{\sqrt{9-x^2}}$$

$$= \frac{1}{\sqrt{9-x^2}}$$

⑦ $f(x) = \sec^{-1}(x^3)$

$$f'(x) = \frac{\frac{d}{dx}[x^3]}{x^3\sqrt{(x^3)^2 - 1}}$$

$$= \frac{3x^2}{x^3\sqrt{x^6 - 1}}$$

$$= \frac{3}{x\sqrt{x^6 - 1}}$$

⑧ $y = \underset{\textcircled{1}}{\underline{x}} \cdot \underset{\textcircled{2}}{\underline{\sin^{-1}(x)}} + \sqrt{1-x^2}$

$$y' = x \cdot \frac{d}{dx}\left[\sin^{-1}x\right] + \left(\sin^{-1}(x)\right) \cdot \frac{d}{dx}[x] + \frac{1}{\cancel{2}}(1-x^2)^{\frac{1}{2}-1} \cdot -\cancel{2}x$$

$$= \frac{x}{1} \cdot \frac{1}{\sqrt{1-x^2}} + \left(\sin^{-1}(x)\right) \cdot (1) + \frac{(1-x^2)^{-1/2}}{1} \cdot \frac{-x}{1}$$

$$= \cancel{\frac{x}{\sqrt{1-x^2}}} + \sin^{-1}(x) - \cancel{\frac{x}{\sqrt{1-x^2}}}$$

$$= \sin^{-1}(x)$$

⑨ $F(x) = \sqrt{1-x^2} \cdot \cos^{-1}(x)$

$$= \underbrace{(1-x^2)^{1/2}}_{①} \cdot \underbrace{\cos^{-1}(x)}_{②}$$

$$= (1-x^2)^{1/2} \cdot \frac{d}{dx}\left[\cos^{-1}(x)\right] + \left[\cos^{-1}(x)\right]\frac{d}{dx}(1-x^2)^{1/2}$$

$$= \sqrt{1-x^2} \cdot \frac{-1}{\sqrt{1-x^2}} + \left[\cos^{-1}x\right] \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot -2x$$

$$= -1 + \left[\cos^{-1}x\right] \cdot \frac{-x}{\sqrt{1-x^2}}$$

$$= \frac{-1}{1} - \frac{x\cos^{-1}x}{\sqrt{1-x^2}}$$

$$= \frac{-\sqrt{1-x^2} - x\cos^{-1}x}{\sqrt{1-x^2}}$$

Note: $\frac{d}{dt}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(t^{-1}\right)$

$= -1t^{-2}$

$= -\frac{1}{t^2}$

<u>Example</u>

$y = \cot^{-1}(t) + \cot^{-1}\left(\frac{1}{t}\right)$ find y'

$$y' = -\frac{1}{1+t^2} - \frac{-\frac{1}{t^2}}{1+\left(\frac{1}{t}\right)^2}$$

$$= -\frac{1}{1+t^2} + \frac{\left(\frac{1}{t^2}\right)}{\frac{1}{1}+\frac{1}{t^2}}$$

$$= -\frac{1}{1+t^2} + \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{t^2+1}{t^2}\right)}$$

$$= -\frac{1}{1+t^2} + \frac{1}{t^2} \cdot \frac{t^2}{t^2+1}$$

$$= -\frac{1}{1+t^2} + \frac{1}{t^2+1}$$

$$\boxed{y' = 0}$$

Derivative of logarithmic functions

① $$\frac{d}{dx}\left[Ln(x)\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[Ln(f(x))\right] = \frac{f'(x)}{f(x)}$$

② $$\frac{d}{dx}\left[\log_a(f(x))\right] = \frac{f'(x)}{Ln(a) \cdot f(x)}$$

$$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{X \; Ln(a)}$$

Example

① $$y = Ln(x^3+1)$$

$$y' = \frac{3x^2 \longrightarrow \text{مشتقة المقدار}}{x^3+1 \longrightarrow \text{المقدار}}$$

② $$y = \log_3(e^{2x}+5)$$

$$y' = \frac{2e^{2x}}{(e^{2x}+5)Ln3}$$

③ $$\frac{d}{dx} Ln(\sin 10x)$$

$$y' = \frac{10\cos(10x)}{\sin(10x)} = 10\cot(10x)$$

③ $$f(x) = \sqrt{Lnx}$$

$$= (Lnx)^{1/2}$$

$$f'(x) = \frac{1}{2}(lnx)^{1/2 - 1} \cdot \frac{1}{x}$$

$$= \frac{1}{2}(Lnx)^{-1/2} \cdot \frac{1}{x}$$

$$= \frac{1}{2\sqrt{Lnx} \cdot x}$$

⑤ $f(x) = \log_{10}(2 + \sin x)$

$$f'(x) = \frac{\cos x}{(2 + \sin x)\operatorname{Ln}(10)}$$

⑥ $f(x) = \operatorname{Ln}\left[\frac{x+1}{\sqrt{x-2}}\right]$

$$f^{*}(x) = \operatorname{Ln}(x+1) - \operatorname{Ln}\sqrt{x-2}$$

$$= \operatorname{Ln}(x+1) - \operatorname{Ln}(x-2)^{1/2}$$

$$\boxed{f(x) = \operatorname{Ln}(x+1) - \frac{1}{2}\operatorname{Ln}(x-2)}$$

$$f'(x) = \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

$$= \frac{1}{x+1} - \frac{1}{2(x-2)}$$

$$= \frac{2(x-2)}{2(x+1)(x-2)} - \frac{(x+1)}{2(x-2)(x+1)}$$

$$= \frac{2(x-2) - (x+1)}{2(x+1)(x-2)}$$

$$= \frac{2x - 4 - x - 1}{2(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

Note

if $f(x) = \ln|x|$ then $f'(x) = \frac{1}{x}$

Example

Find y' of the following function

① $$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

$$\ln y = \ln\left[\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right]$$

$$\ln y = \ln x^{3/4} + \ln(x^2+1)^{1/2} - \ln(3x+2)^5$$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$$\frac{y'}{y} = \frac{3}{4x} + \frac{\cancel{2}x}{\cancel{2}(x^2+1)} - \frac{5(3)}{3x+2}$$

$$\frac{y'}{y} = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$$

$$y' = y\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

$$y' = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

② $y = x^{\sqrt{x}}$

$$\ln y = \ln x^{\sqrt{x}}$$

$$\ln y = \sqrt{x} \ln x$$

$$\ln y = x^{1/2} \ln x$$

$$\frac{y'}{y} = x^{\frac{1}{2}} \cdot \frac{1}{x} + \frac{1}{2} x^{\frac{1}{2}-1} \ln x$$

$$\frac{y'}{y} = \frac{x^{1/2}}{x} + \frac{1}{2} x^{-1/2} \ln x$$

$$\frac{y'}{y} = x^{\frac{1}{2}-1} + \frac{\ln x}{2x^{1/2}}$$

$$\frac{y'}{y} = x^{-1/2} + \frac{\ln x}{2\sqrt{x}}$$

$$\frac{y'}{y} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$$

$$y' = y\left[\frac{2 + \ln x}{2\sqrt{x}}\right] = x^{\sqrt{x}}\left[\frac{2+\ln x}{2\sqrt{x}}\right]$$

③ $f(x) = (\sin x)^{x}$

$$\ln(f(x)) = \ln(\sin x)^{x} = \underset{1}{\underline{x}}\,\underset{2}{\underline{\ln(\sin x)}}$$

$$\frac{f'(x)}{f(x)} = 1\cdot\ln(\sin x) + x\cdot\frac{\cos x}{\sin x}$$

$$\frac{f'(x)}{f(x)} = \ln(\sin x) + x\cot x$$

$$f'(x) = f(x)\left[\ln(\sin x) + x\cot x\right]$$

$$f'(x) = (\sin x)^{x}\left[\ln(\sin x) + x\cot x\right]$$

④ $x^y = y^x$

$$\ln x^y = \ln y^x$$

$$y \ln x = x \ln y$$

$$y' \ln x + y\left(\frac{1}{x}\right) = \ln y + x\left(\frac{y'}{y}\right)$$

$$y' \ln x + \frac{y}{x} = \ln y + \frac{xy'}{y}$$

$$y' \ln x - \frac{xy'}{y} = \ln y - \frac{y}{x}$$

$$y'\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

$$y' = \frac{\left(\frac{x \ln y - y}{x}\right)}{\left(\frac{y \ln x - x}{y}\right)}$$

$$y' = \frac{x \ln y - y}{x} \cdot \frac{y}{y \ln x - x}$$

$$y' = \frac{xy \ln y - y^2}{xy \ln x - x^2}$$

Scanned with CamScanner

⑤ If $y = \ln(e^{-x} + x e^{-x})$ then find y'

$$y = \ln\left(\frac{1}{e^x} + \frac{x}{e^x}\right)$$

$$= \ln\left[\frac{1+x}{e^x}\right]$$

$$= \ln(1+x) - \ln(e^x)$$

$$\boxed{y = \ln(1+x) - x}$$

$$y' = \frac{1}{1+x} - 1$$

$$= \frac{1-(1+x)}{1+x}$$

$$= \frac{1-1-x}{1+x}$$

$$\boxed{y' = \frac{-x}{1+x}}$$ #

⑥ $y = 2^t . \log_2 t$

$$y' = 2^t \cdot \frac{1}{[Ln(2)] \cdot t} + \log_2 t \cdot 2^t \cdot Ln(2)$$

$$= 2^t \left[\frac{1}{[Ln(2)] \cdot t} + [Ln(2)] . \log_2 t \right]$$

7) $y = Ln\,(Lnx) \Rightarrow y' = \dfrac{\frac{d}{dx}[Lnx]}{Lnx}$

$$y' = \frac{\frac{1}{x}}{Lnx}$$

$$y' = \frac{1}{x} \div \frac{Lnx}{1}$$

$$= \frac{1}{x} \cdot \frac{1}{Lnx}$$

$$\boxed{y' = \frac{1}{xLnx}}$$

⑧ $f(x) = Ln(xe^{-2x})$

$$= Ln(x) + Ln(e^{-2x})$$

$$\boxed{f(x) = Ln(x) - 2x}$$

$$f'(x) = \frac{1}{x} - \frac{2}{1}$$

$$\boxed{f'(x) = \frac{1-2x}{x}}$$

⑨ $f(x) = \log_{10}(1 + \cos x)$

$$f'(x) = \frac{\frac{d}{dx}[1 + \cos x]}{1 + \cos x}$$

$$= \frac{-\sin x}{1 + \cos x}$$

⑩ $y = \sqrt[5]{Ln x}$

$$y = (Ln x)^{\frac{1}{5}}$$

$$y' = \frac{1}{5}(Ln x)^{\frac{1}{5}-1} \cdot \frac{d}{dx}[Ln x]$$

$$= \frac{1}{5}(Ln x)^{\frac{-4}{5}} \cdot \frac{1}{x} = \frac{1}{5x(Ln x)^{\frac{4}{5}}}$$

⑪ $y = Ln \sqrt[5]{x}$

$$= Ln x^{\frac{1}{5}}$$

$$= \frac{1}{5} Ln x$$

$$y' = \frac{1}{5} \cdot \frac{1}{x}$$

$$= \frac{1}{5x} \quad \#$$

⑫ $y = \sin(\ln x)$

$$y' = \cos(\ln x) \cdot \frac{d}{dx}[\ln x]$$

$$= \frac{\cos(\ln x)}{1} \cdot \frac{1}{x}$$

$$= \frac{\cos(\ln x)}{x}$$

⑬ $y = \ln(\sin x)$

$$= \frac{\frac{d}{dx}(\sin x)}{\sin x}$$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x$$

⑭ $g(x) = \ln(x \cdot \sqrt{x^2 - 1})$

$$= \ln x + \ln \sqrt{x^2 - 1}$$

$$= \ln x + \ln (x^2 - 1)^{1/2}$$

$$= \ln x + \frac{1}{2} \ln (x^2 - 1)$$

$$g'(x) = \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 - 1}$$

$$= \frac{1}{x} + \frac{x}{x^2 - 1}$$

$$= \frac{x^2 - 1 + x^2}{x(x^2 - 1)} = \frac{2x^2 - 1}{x^3 - x}$$

⑮ $y = (2x+1)^5 \cdot (x^4-3)^6$

$$\ln y = \ln\left[(2x+1)^5 \cdot (x^4-3)^6\right]$$

$$\ln y = \ln(2x+1)^5 + \ln(x^4-3)^6$$

$$\ln y = 5\ln(2x+1) + 6\ln(x^4-3)$$

$$\frac{1}{y}\cdot\frac{y'}{1} = \frac{5}{1}\cdot\frac{2}{2x+1} + \frac{6}{1}\cdot\frac{4x^3}{x^4-3}$$

$$\frac{y'}{y} = \frac{10}{2x+1} + \frac{24x^3}{x^4-3}$$

$$y' = y\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

$$y' = (2x+1)^5(x^4-3)^6\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

$$= \underset{2\times5}{10}\,\underline{(2x+1)^4}\,\underline{\underline{(x^4-3)^6}} + \underset{2\times12}{24x^3}\,\underline{(2x+1)^5}\,\underline{\underline{(x^4-3)^5}}$$

$$= 2(2x+1)^4(x^4-3)^5\left[5(x^4-3) + 12x^3(2x+1)\right]$$

$$= 2(2x+1)^4(x^4-3)^5\left[\underline{\underline{5x^4}} - 15 + \underline{\underline{24x^4}} + 12x^3\right]$$

$$= 2(2x+1)^4(x^4-3)^5\left[29x^4 + 12x^3 - 15\right]$$

⑯ $y = x^x$

$$\ln y = \ln(x^x)$$

$$\ln y = x \ln x$$

$$\frac{y'}{y} = x \cdot \frac{1}{x} + (\ln x)(1)$$

$$\frac{y'}{y} = 1 + \ln x$$

$$y' = y\left(1 + \ln x\right)$$

$$y' = x^x\left(1 + \ln x\right)$$

⑰ $y = x^{\sin x}$

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \cdot \ln x$$

$$\frac{y'}{y} = \frac{\sin x}{1} \cdot \frac{1}{x} + (\ln x)\cos x$$

$$y' = y\left[\frac{\sin x}{x} + (\ln x)\cos x\right]$$

$$y' = x^{\sin x}\left[\frac{\sin x}{x} + (\ln x)\cos x\right]$$

$$= \frac{x^{\sin x}}{1} \cdot \frac{\sin x}{x^{1}} + x^{\sin x} \cdot \ln x \cdot \cos x$$

$$= x^{\sin x - 1} \cdot \sin x + x^{\sin x} \cdot \ln x \cdot \cos x$$

⑱ $y = \sqrt{x}\, e^{x^2-x}(x+1)^{2/3}$

$$\text{Ln} y = \text{Ln}\left[\sqrt{x}\, . e^{x^2-x} . (x+1)^{2/3}\right]$$

$$= \text{Ln}\sqrt{x} + \text{Ln}(e^{x^2-x}) + \ln (x+1)^{2/3}$$

$$\ln y = \frac{1}{2}\ln x + x^2 - x + \frac{2}{3}\ln(x+1)$$

$$\frac{y'}{y} = \frac{1}{2} \cdot \frac{1}{x} + 2x - 1 + \frac{2}{3} \cdot \frac{1}{x+1}$$

$$\frac{y'}{y} = \frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}$$

$$y' = y\left[\frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\right]$$

$$y' = \sqrt{x}\, e^{x^2-x} . (x+1)^{2/3}\left[\frac{1}{2x} + 2x - 1 + \frac{2}{3(x+1)}\right]$$

(19) Find the equationt of tangent line to $y = \ln(x^2 - 3x + 1)$, (3,0)

$$y' = \frac{2x-3}{x^2-3x+1}$$

$$m = y'\Big|_{x=3} = \frac{2(3)-3}{3^2-3(3)+1} = \frac{6-3}{9-9+1} = \frac{3}{1} = 3$$

$m = 3$; $(\underset{x_1}{3}, \underset{y_1}{0})$

the equation of tangent line:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 3(x - 3)$$

$$\boxed{y = 3x - 9} \#$$

(20) If $f(x) = \cos(\ln x^2)$ find $f'(1)$

$$f(x) = \cos(2\ln x)$$

$$f'(x) = -\sin(2\ln x) \cdot 2 \cdot \frac{1}{x}$$

$$f'(1) = -\sin(2\ln(1)) \cdot 2\left(\frac{1}{1}\right)$$

$$= -\sin(0) \cdot 2$$

$$= (0)(2) = 0$$

㉑ if $f(x) = \ln x$ find $f^{(n)}_{(x)}$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = -1x^{-2} = -x^{-2}$$

$$f'''_{(x)} = (-1)(-2)x^{-3} = (1)(2)x^{-3} = 2x^{-3}$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4} = -(1)(2)(3)x^{-4} = -6x^{-4}$$

$$f^{(5)} = (-1)(-2)(-3)(-4)x^{-5} = (1)(2)(3)(4)x^{-5} = 29x^{-5}$$

$$\vdots$$

$$f^{(n)}_{(x)} = (-1)^{n-1}(n-1)!\,x^{-n}$$

Section 4.3

- increasing / Decreasing Test

a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

b) If $f'(x) < 0$ on interval, then f is decreasing on that interval.

The First Derivative Test

Suppose that c is a critical number of a continuous function f.

a) If f' change from +ve to -ve at c, then f has $\boxed{\text{a Local maximum}}$ at c

b) If f' change from -ve to +ve at c, then f has $\boxed{\text{a local Minimum}}$ at c

c) If f' is +ve to the left and right or -ve to the left and right at $\underline{\underline{c}}$ then f has no local maximum or minimum at $\underline{\underline{c}}$

Example

Find where the function $f(x)=3x^4-4x^3-12x^2+5$ is increasing and where it is decreasing and find a local Maximum and local minimum

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

① $f(x)$ is cont on $\mathbb{R}$

② $f'(x) = 12x^3 - 12x^2 - 24x$

③ $f'(x) = 0$

$$12x^3 - 12x^2 - 24x = 0$$

$$12x(x^2 - x - 2) = 0$$

$12x = 0$ or $x^2 - x - 2 = 0$

$\frac{12x}{12} = \frac{0}{12}$ $\qquad$ $(x-2)(x+1) = 0$

$x = 0 \in \mathbb{R}$

$x - 2 = 0$ or $x + 1 = 0$

$x = 2 \in \mathbb{R}$ $\qquad$ $x = -1 \in \mathbb{R}$

④ the critical numbers are 0 , 2 and -1

the critical points are $(0, f(0))$, $(2, f(2))$ and $(-1, f(-1))$

the Critical points are $(0, f(0)) = (0,5)$

$(2, f(2)) = (2,-27)$

$(-1, f(-1)) = (-1, 0)$

$\longleftarrow \quad -1 \quad\quad 0 \quad\quad 2 \quad \longrightarrow$

$12x$ اشارة	—	—	+	+
اشارة x^2-x-2	+	—	—	+
اشارة $f'(x)=12x(x^2-x-2)$	⊖ decreasing ↘	+ increasing ↗	⊖ decreasing ↘	+ increasing ↗

$f(x)$ is decreasing on the $(0,2) \cup (-\infty,-1)$

or $[0,2] \cup (-\infty,-1]$

$f(x)$ is increasing on the $(-1,0) \cup (2,\infty)$

or $[-1,0] \cup [2,\infty)$

$f(x)$ has local Maximum at $x=0$

or $f(0)=5$ is local Maximum Value

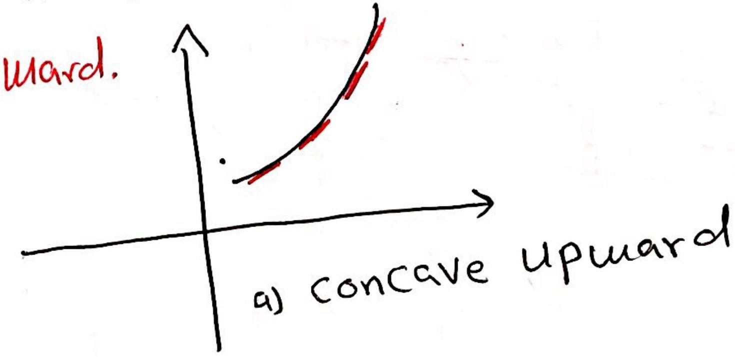
$f(x)$ ha~~s~~ve local Minimum at $x=-1$ and $x=2$

$f(-1) = 0$ is local Minimum Value

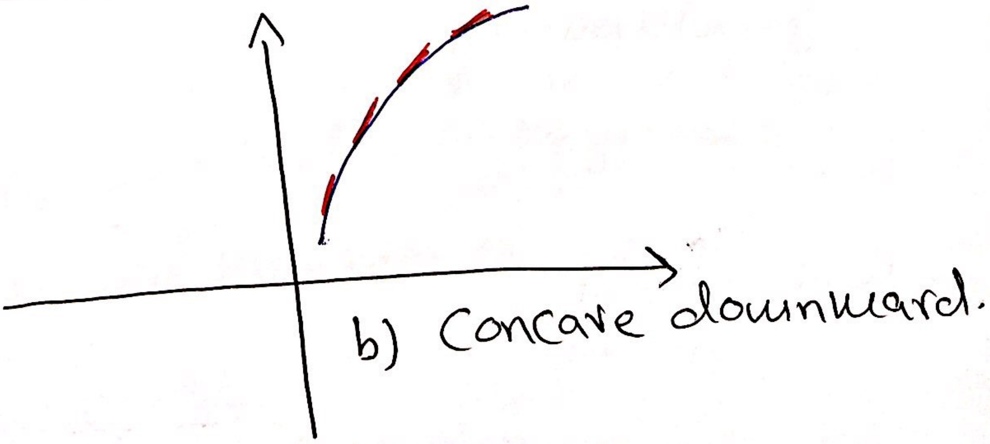
$f(2) = -27$ is local Minimum Value.

Definition

a) If the graph of $f(x)$ lies above all of its tangents on an interval I, then it is called Concave Upward.



b) If the graph of $f(x)$ lies below all of its tangents on an interval I, then it is called Concave downward.



Concavity test

a) If $\boxed{f''(x) > 0}$ for all x in I then the graph of f is $\boxed{\text{Concave upward}}$ on I

b) If $\boxed{f''(x) < 0}$ for all x in I then the graph of f is $\boxed{\text{Concave downward}}$ on I

Definition

A point P on a curve $y = f(x)$ is called an Inflection Point if f is continuous there and the curve Change from CU to CD or from CD to CU at P

The Second Derivative Test

Suppose f is continuous near c

a) If $\boxed{f'(c) = 0}$ and $\boxed{f''(c) > 0}$ then $f(x)$ has $\boxed{\text{local minmum at c}}$

b) If $\boxed{f'(c) = 0}$ and $\boxed{f''(c) < 0}$ then $f(x)$ has $\boxed{\text{local Maximum at c}}$

let $f(x) = x^4 - 4x^3$

a) find the intervals on which f is increasing or decreasing

b) Find the local Maximum and Minimum value of f Use the Second derivative Test

c) find the intervals of concavity and Inflection Points.

① $f(x) = x^4 - 4x^3$ is cont on $\mathbb{R}$

② $f'(x) = 4x^3 - 12x^2$

③ $f'(x) = 0 \Rightarrow 4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x-3) = 0$

$4x^2 = 0$ or $x - 3 = 0$

$\boxed{x = 3 \in \mathbb{R}}$

$\frac{4x^2}{4} = \frac{0}{4}$

$x^2 = 0$

$\sqrt{x^2} = \sqrt{0}$

$\boxed{x = 0 \in \mathbb{R}}$

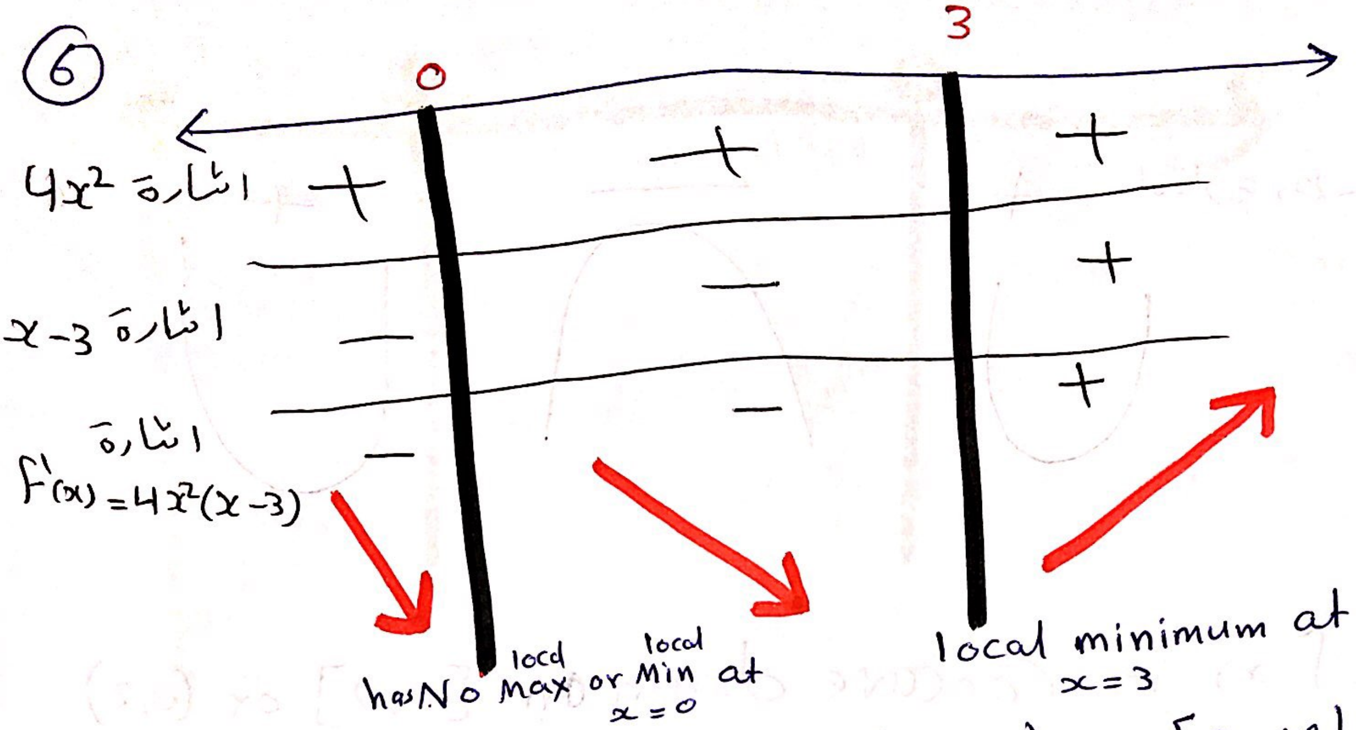
④ the Critical numbers are 0 and 3

5) the critical Points:

$(0, f(0))$ and $(3, f(3))$

$(0, 0)$ and $(3, -27)$

⑥



f(x) is increasing on $(3,\infty)$ or $[3,\infty)$

f(x) is decreasing on $(-\infty,3]$ or $(-\infty,3)$

⑦ $f''(x) = 12x^2 - 24x$

c.n c.n

$f''(0) = 0$ and $f''(3) = 36 > 0$

f(x). has no local Max or Min at $x = 0$

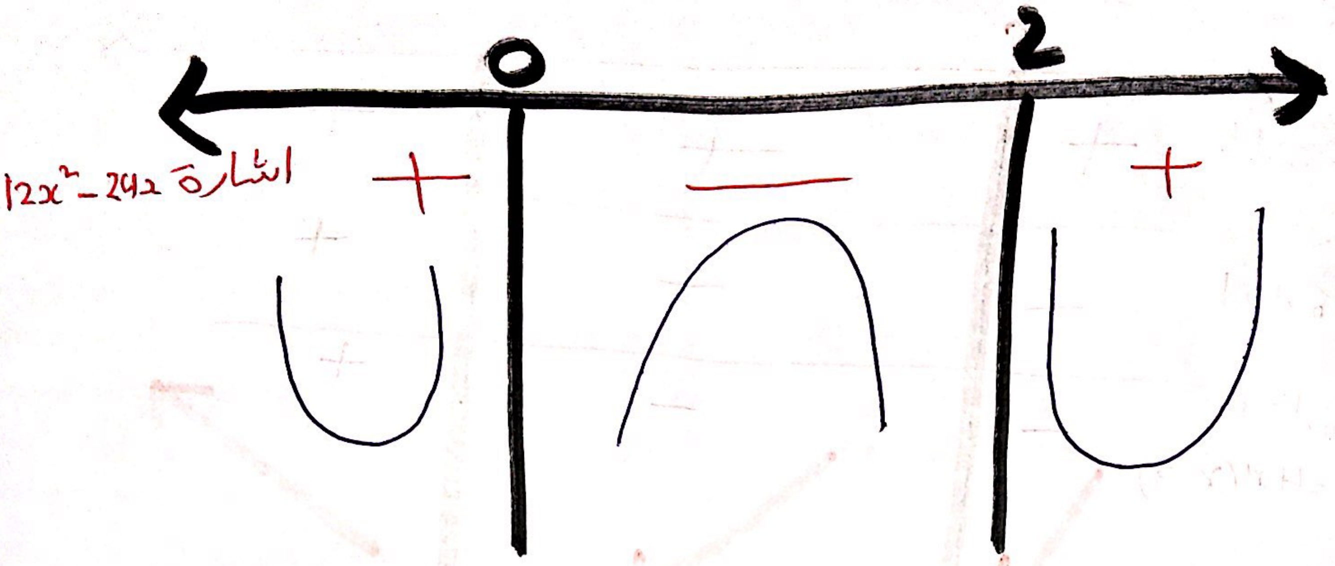
f(x) has local Minimum at $x = 3$

⑧ $f''(x) = 12x^2 - 24x$

$f''(x) = 0 \Longrightarrow 12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0 \in \mathbb{R}$ or $x = 2 \in \mathbb{R}$



$f(x)$ is Concave down on $[0,2]$ or $(0,2)$

$f(x)$ is Concave up on $(-\infty,0] \cup [2,\infty)$

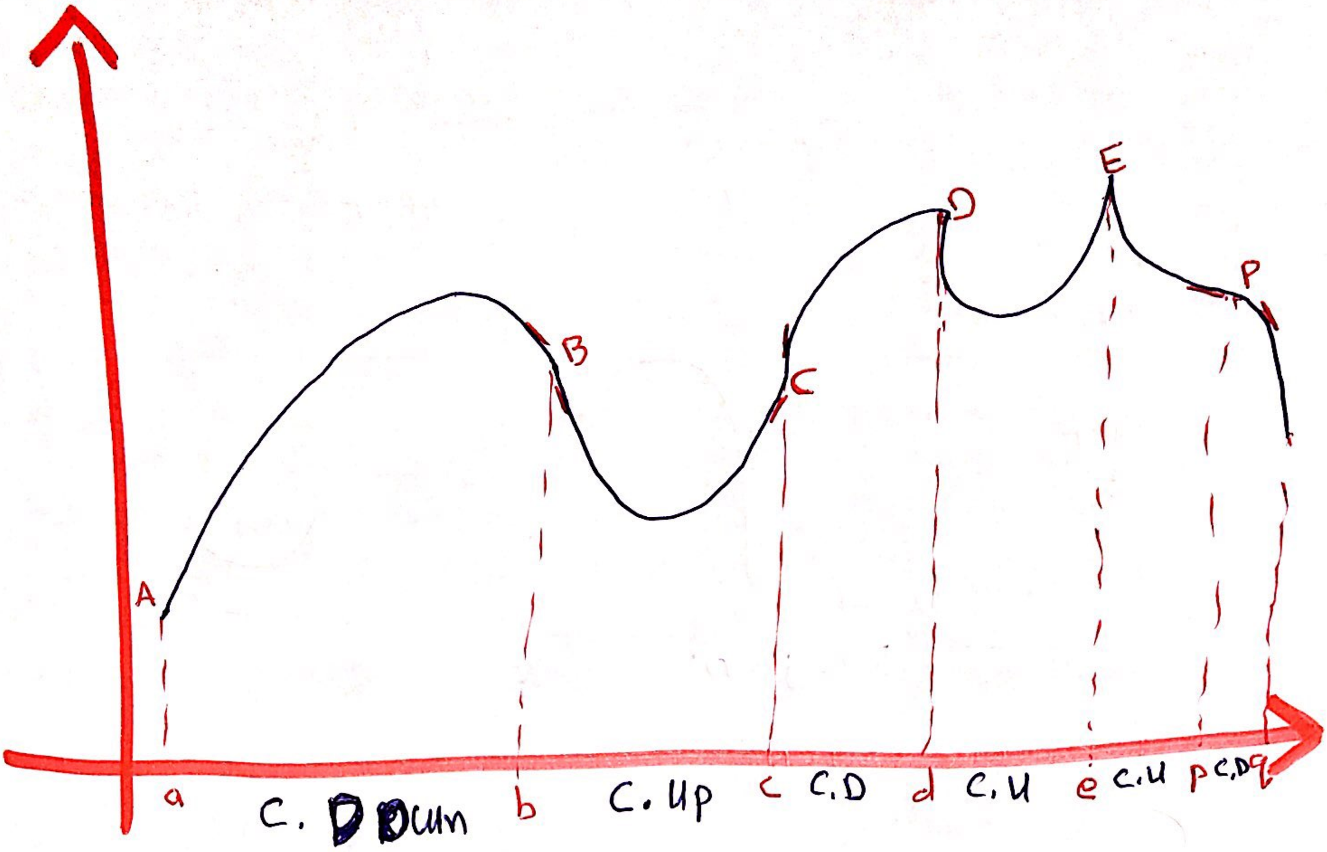
or $(-\infty,0) \cup (2,\infty)$

$f(x)$ has inflection Point at $x = 0$

$(0, f(0)) = (0,0)$

$f(x)$ has inflection Point at $x = 2$

$(2, f(2)) = (2,-16)$



f(x) has inflection point at P and D, C, B

<u>H.W</u> ①, ②, ⑤, ⑥, ⑨, ⑪, ⑯ P. 300 + P 301

Workshop Solutions to Chapter 4 (chapter 3)

1) If $f(x)$ is a differentiable function, then $f'(x) =$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

2) If $f(x) = 4x^2$, then $f'(x) =$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{4(x+h)^2 - 4x^2}{h}$$

3) If $f(x) = x^2 - 3$, then $f'(x) =$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2 - 3] - [x^2 - 3]}{h}$$

4) If $f(x) = \sqrt{x}$, $x \geq 0$, then $f'(x) =$

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

5) If f is a differentiable function at a, then f is a continuous function at a.

6) If f is a continuous function at a, then f is a differentiable function at a.

Solution:

False

7) If $y = x^4 + 5x^2 + 3$, then $y' =$

Solution:

$$y' = 4x^3 + 10x$$

8) If $y = x^4 - 5x^2 + 3$, then $y' =$

Solution:

$$y' = 4x^3 - 10x$$

9) If $y = x^{-5/2}$, then $y' =$

Solution:

$$y' = -\frac{5}{2} x^{-\frac{5}{2}-1} = -\frac{5}{2} x^{-7/2}$$

10) If $y = \frac{1}{3x^3} + 2\sqrt{x} = \frac{1}{3}x^{-3} + 2x^{1/2}$, then $y' =$

Solution:

$$y' = (-3)\left(\frac{1}{3}\right) x^{-3-1} + \left(\frac{1}{2}\right)(2) x^{\frac{1}{2}-1}$$

$$= -x^{-4} + x^{-1/2} = -\frac{1}{x^4} + \frac{1}{x^{1/2}} = -\frac{1}{x^4} + \frac{1}{\sqrt{x}}$$

11) If $y = (x-3)(x-2)$, then $y' =$

Solution:

$$y = (x-3)(x-2) = x^2 - 5x + 6$$

$$y' = 2x - 5$$

12) If $y = (x^3 + 3)(x^2 - 1)$, then $y' =$

Solution:

$$y = (x^3 + 3)(x^2 - 1) = x^5 - x^3 + 3x^2 - 3$$

$$y' = 5x^4 - 3x^2 + 6x$$

13) If $y = \sqrt{x}(2x + 1)$, then $y' =$

Solution:

$$y = \sqrt{x}(2x+1) = 2x\sqrt{x} + \sqrt{x} = 2x^{\frac{3}{2}} + x^{\frac{1}{2}}$$

$$y' = \left(\frac{3}{2}\right)(2) x^{\frac{3}{2}-1} + \left(\frac{1}{2}\right) x^{\frac{1}{2}-1} = 3x^{\frac{1}{2}} + \frac{1}{2} x^{-\frac{1}{2}}$$

$$= 3\sqrt{x} + \frac{1}{2\sqrt{x}}$$

OR

Use the rule $(f.g)' = f'g + fg'$

$$y' = (2)(\sqrt{x}) + \left(\frac{1}{2\sqrt{x}}\right)(2x+1) = 2\sqrt{x} + \frac{2x+1}{2\sqrt{x}}$$

14) If $y = \frac{x+3}{x-2}$, then $y' =$

Solution:

Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$y' = \frac{(1)(x-2) - (x+3)(1)}{(x-2)^2} = \frac{x - 2 - x - 3}{(x-2)^2} = \frac{-5}{(x-2)^2}$$

$$= -\frac{5}{(x-2)^2}$$

15) If $y = \frac{x+3}{x-2}$, then $y'|_{x=4} =$

Solution:

$$y' = \frac{(1)(x-2) - (x+3)(1)}{(x-2)^2} = \frac{x - 2 - x - 3}{(x-2)^2}$$

$$= \frac{-5}{(x-2)^2} = -\frac{5}{(x-2)^2}$$

$$y'|_{x=4} = -\frac{5}{(4-2)^2} = -\frac{5}{4}$$

16) If $y = \frac{x-1}{x+2}$, then $y' =$

Solution:

Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$y' = \frac{(1)(x+2) - (x-1)(1)}{(x+2)^2} = \frac{x + 2 - x + 1}{(x+2)^2} = \frac{3}{(x+2)^2}$$

17) If $y = \sqrt{3x^2 + 6x}$, then $y' =$

Solution:

Use the rule $\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y' = \frac{6x + 6}{2\sqrt{3x^2 + 6x}} = \frac{6(x + 1)}{2\sqrt{3x^2 + 6x}} = \frac{3(x + 1)}{\sqrt{3x^2 + 6x}}$$

18) If $y = \sqrt{3x^2 + 6x}$, then $y'|_{x=1} =$

Solution:

$$y' = \frac{6x + 6}{2\sqrt{3x^2 + 6x}} = \frac{6(x + 1)}{2\sqrt{3x^2 + 6x}} = \frac{3(x + 1)}{\sqrt{3x^2 + 6x}}$$

$$y'|_{x=1} = \frac{3((1) + 1)}{\sqrt{3(1)^2 + 6(1)}} = \frac{6}{\sqrt{9}} = \frac{6}{3} = 2$$

19) The tangent line equation to the curve $y = x^2 + 2$ at the point $(1,3)$ is

Solution:

First, we have to find the slope of the curve which is

$$y' = 2x$$

Thus, the slope at $x = 1$ is

$$y'|_{x=1} = 2(1) = 2$$

Hence, the tangent line equation passing through the point $(1,3)$ with slope $m = 2$ is

$$y - 3 = 2(x - 1)$$
$$y - 3 = 2x - 2$$
$$y = 2x - 2 + 3$$
$$y = 2x + 1$$

20) The tangent line equation to the curve $y = \frac{2x}{x+1}$ at the point $(0,0)$ is

Solution:

First, we have to find the slope of the curve which is

$$y' = \frac{(2)(x + 1) - (2x)(1)}{(x + 1)^2} = \frac{2x + 2 - 2x}{(x + 1)^2} = \frac{2}{(x + 1)^2}$$

Thus, the slope at $x = 0$ is

$$y'|_{x=0} = \frac{2}{(0 + 1)^2} = 2$$

Hence, the tangent line equation passing through the point $(0,0)$ with slope $m = 2$ is

$$y - 0 = (2)(x - 0)$$
$$y = 2x$$

21) The tangent line equation to the curve $y = 3x^2 - 13$ at the point $(2, -1)$ is

Solution:

First, we have to find the slope of the curve which is

$$y' = 6x$$

Thus, the slope at $x = 2$ is

$$y'|_{x=2} = 6(2) = 12$$

Hence, the tangent line equation passing through the point $(2, -1)$ with slope $m = 12$ is

$$y - (-1) = 12(x - 2)$$
$$y + 1 = 12x - 24$$
$$y = 12x - 24 - 1$$
$$y = 12x - 25$$

22) The tangent line equation to the curve $y = 3x^2 + 2x + 5$ at the point $(0,5)$ is

Solution:

First, we have to find the slope of the curve which is

$$y' = 6x + 2$$

Thus, the slope at $x = 2$ is

$$y'|_{x=0} = 6(0) + 2 = 2$$

Hence, the tangent line equation passing through the point $(0,5)$ with slope $m = 2$ is

$$y - 5 = 2(x - 0)$$
$$y - 5 = 2x$$
$$y = 2x + 5$$

23) If $y = xe^x$, then $y' =$

Solution:

Use the rules $(f.g)' = f'g + fg'$ and $(e^u) = e^u.u'$

$$y' = (1)(e^x) + (x)(e^x) = e^x + xe^x = e^x(1 + x)$$

24) If $y = x{-}e^x$, then $y'' =$

Solution:

Use the rules $(f - g)' = f' - g'$ and $(e^u) = e^u.u'$

$$y' = 1 - e^x$$
$$y'' = -e^x$$

25) If $x^2 - y^2 = 4$, then $y' =$

Solution:

$$2x - 2yy' = 0$$
$$-2yy' = -2x$$
$$y' = \frac{-2x}{-2y}$$
$$y' = \frac{x}{y}$$

26) If $x^2 + y^2 = 4$, then $y' =$

Solution:

$$2x + 2yy' = 0$$
$$2yy' = -2x$$
$$y' = \frac{-2x}{2y}$$
$$y' = -\frac{x}{y}$$

27) If $y = \frac{x+1}{x+2}$, then $y' =$

Solution:

Use the rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$y' = \frac{(1)(x + 2) - (x + 1)(1)}{(x + 2)^2} = \frac{x + 2 - x - 1}{(x + 2)^2}$$
$$= \frac{1}{(x + 2)^2}$$

28) If $y = \frac{1}{\sqrt[2]{x^5}} + \sec x$, then $y' =$

Solution:

Use the rules

$(f + g)' = f' + g'$ and $(\sec u)' = \sec u \tan u \,.\, u'$

$$y = \frac{1}{\sqrt[2]{x^5}} + \sec x = x^{-\frac{5}{2}} + \sec x$$

$$y' = \left(-\frac{5}{2}\right)x^{-\frac{5}{2}-1} + \sec x \tan x = -\frac{5}{2}x^{-7/2} + \sec x \tan x$$

29) If $y=\tan^{-1}(x^3)$, then $y'=$

Solution:

Use the rule $(\tan^{-1}u)'=\frac{u'}{1+u^2}$

$$y'=\frac{1}{1+(x^3)^2}.(3x^2)=\frac{3x^2}{1+x^6}$$

30) If $y=\tan x-x$, then $y'=$

Solution:

Use the rules

$(f-g)'=f'-g'$ and $(\tan u)'=\sec^2 u\ .u'$

$$y'=\sec^2 x-1$$

31) If $y=\sec^2 x-1$, then $y'=$

Solution:

Use the rules $(f-g)'=f'-g'$, $(u)^n=n(u)^{n-1}.u'$ and $(\sec u)'=\sec u\tan u\,.u'$

$$y'=2\sec x\,.\ \sec x\tan x=2\sec^2 x\tan x$$

32) If $y=x^{\sin x}$, then $y'=$

Solution:

Use the rule $(\sin u)'=\cos u\ .u'$

$$y=x^{\sin x}$$
$$\ln y=\ln x^{\sin x}$$
$$\ln y=\sin x\,.\ln x$$
$$\frac{y'}{y}=\cos x\,.\ \ln x+\sin x\,.\frac{1}{x}=\cos x\,.\ \ln x+\frac{\sin x}{x}$$
$$y'=y\left(\cos x\,.\ \ln x+\frac{\sin x}{x}\right)=x^{\sin x}\left(\cos x\,.\ \ln x+\frac{\sin x}{x}\right)$$

33) If $y=x^{\cos x}$, then $y'=$

Solution:

Use the rule $(\cos u)'=-\sin u\ .u'$

$$y=x^{\cos x}$$
$$\ln y=\ln x^{\cos x}$$
$$\ln y=\cos x\,.\ln x$$
$$\frac{y'}{y}=-\sin x\,.\ \ln x+\cos x\,.\frac{1}{x}=-\sin x\,.\ \ln x+\frac{\cos x}{x}$$
$$y'=y\left(-\sin x\,.\ \ln x+\frac{\cos x}{x}\right)$$
$$=x^{\cos x}\left(\frac{\cos x}{x}-\sin x\,.\ \ln x\right)$$

34) If $y=(2x^2+\csc x)^9$, then $y'=$

Solution:

Use the rules

$(u)^n=n(u)^{n-1}.u'$ and $(\csc u)'=-\csc u\cot u\,.u'$

$$y'=9(2x^2+\csc x)^8.(4x-\csc x\cot x)$$

35) If $y=\frac{5^x}{\cot x}$, then $y'=$

Solution:

Use the rules

$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$, $(a^u)'=a^u.\ln a\,.u'$

and $(\csc u)'=-\csc u\cot u\,.u'$

$$y'=\frac{(5^x\ln 5)(\cot x)-(5^x)(-\csc^2 x)}{(\cot x)^2}$$
$$=\frac{5^x(\ln 5\cot x+\csc^2 x)}{\cot^2 x}$$

36) If $y=e^{2x}$, then $y^{(6)}=$

Solution:

Use the rule $(e^u)'=e^u.u'$

$$y'=2e^{2x}$$
$$y''=4e^{2x}$$
$$y'''=8e^{2x}$$
$$y^{(4)}=16e^{2x}$$
$$y^{(5)}=32e^{2x}$$
$$y^{(6)}=64e^{2x}$$

37) If $y=x^{-2}e^{\sin x}$, then $y'=$

Solution:

Use the rules $(f.g)'=f'g+fg'$, $(e^u)=e^u.u'$ and $(\sin u)'=\cos u\ .u'$

$$y'=(-2x^{-3})(e^{\sin x})+(x^{-2})(e^{\sin x}.\cos x)$$
$$=-2x^{-3}e^{\sin x}+x^{-2}\cos x\,e^{\sin x}$$
$$=x^{-3}e^{\sin x}(-2+x\cos x)$$
$$=x^{-3}e^{\sin x}(x\cos x-2)$$

38) If $y=5^{\tan x}$, then $y'=$

Solution:

Use the rules

$(a^u)'=a^u.\ln a\,.u'$ and $(\tan u)'=\sec^2 u\,.u'$

$$y'=5^{\tan x}.\ \ln 5\,.\sec^2 x$$

39) If $x^2+y^2=3xy+7$, then $y'=$

Solution:

$$2x+2yy'=3y+3xy'$$
$$2yy'-3xy'=3y-2x$$
$$y'(2y-3x)=3y-2x$$
$$y'=\frac{3y-2x}{2y-3x}$$

40) If $y=\sin^3(4x)$, then $y^{(6)}=$ y'=

Solution:

Use the rules

$(u)^n=n(u)^{n-1}.u'$ and $(\sin u)'=\cos u\,.u'$

$$y'=3\sin^2(4x).\cos(4x).(4)$$
$$=12\sin^2(4x).\cos(4x)$$

41) If $y = 3^x \cot x$, then $y' =$

Solution:

Use the rules $(f.g)' = f'g + fg'$, $(a^u)' = a^u.\ln a\,.u'$ and $(\cot u)' = -\csc^2 u\ .u'$

$$\begin{aligned} y' &= (3^x.\ln 3)(\cot x) + (3^x)(-\csc^2 x) \\ &= 3^x \ln 3\ \cot x - 3^x \csc^2 x \\ &= 3^x(\ln 3\ \cot x - \csc^2 x) \end{aligned}$$

42) If $y = (2x^2 + \sec x)^7$, then $y' =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\sec u)' = \sec u \tan u\,.u'$

$$y' = 7(2x^2 + \sec x)^6.(4x + \sec x \tan x)$$

43) If $f(x) = \cos x$, then $f^{(45)}(x) =$

Solution:

$$\begin{gathered} f'(x) = -\sin x \\ f''(x) = -\cos x \\ f'''(x) = \sin x \\ f^{(4)}(x) = \cos x \end{gathered}$$

Note: $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4.

Hence,

$$\begin{gathered} f^{(44)}(x) = \cos x \\ f^{(45)}(x) = -\sin x \end{gathered}$$

44) If $D^{47}(\sin x) =$

Solution:

$$\begin{gathered} D(\sin x) = \cos x \\ D^2(\sin x) = -\sin x \\ D^3(\sin x) = -\cos x \\ D^4(\sin x) = \sin x \end{gathered}$$

Note: $D^n(\sin x) = \sin x$ whenever n is a multiple of 4.

Hence,

$$\begin{gathered} D^{44}(\sin x) = \sin x \\ D^{45}(\sin x) = \cos x \\ D^{46}(\sin x) = -\sin x \\ D^{47}(\sin x) = -\cos x \end{gathered}$$

45) If $y = x^x$, then $y' =$

Solution:

Use the rule $(\ln u)' = \frac{u'}{u}$

$$\begin{gathered} y = x^x \\ \ln y = \ln x^x \\ \ln y = x \ln x \\ \frac{y'}{y} = (1)(\ln x) + (x)\left(\frac{1}{x}\right) \\ \frac{y'}{y} = \ln x + 1 \\ y' = y(1 + \ln x) = x^x(1 + \ln x) \end{gathered}$$

46) If $f(x) = \frac{\ln x}{x^2}$, then $f'(1) =$

Solution:

Use the rules $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ and $(\ln u)' = \frac{u'}{u}$

$$\begin{aligned} f'(x) &= \frac{\left(\frac{1}{x}\right)(x^2) - (\ln x)(2x)}{(x^2)^2} = \frac{x - 2x\ln x}{x^4} \\ &= \frac{x(1 - 2\ln x)}{x^4} = \frac{1 - 2\ln x}{x^3} \end{aligned}$$

$$\therefore\quad f'(1) = \frac{1 - 2\ln(1)}{(1)^3} = \frac{1 - 2(0)}{1} = 1$$

47) If $y = \cot^{-1}(e^x)$, then $y' =$

Solution:

Use the rules $(\cot^{-1} u)' = -\frac{u'}{1+u^2}$ and $(e^u) = e^u.u'$

$$y' = -\frac{1}{1 + (e^x)^2}\,.e^x = -\frac{e^x}{1 + e^{2x}}$$

48) If $y = \tan^{-1}(e^x)$, then $y' =$

Solution:

Use the rules $(\tan^{-1} u)' = \frac{u'}{1+u^2}$ and $(e^u) = e^u.u'$

$$y' = \frac{1}{1 + (e^x)^2}\,.e^x = \frac{e^x}{1 + e^{2x}}$$

49) If $y = \sin^{-1}(e^x)$, then $y' =$

Solution:

Use the rules $(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$ and $(e^u) = e^u.u'$

$$y' = \frac{1}{\sqrt{1 - (e^x)^2}}\,.e^x = \frac{e^x}{\sqrt{1 - e^{2x}}}$$

50) If $y = \cos^{-1}(e^x)$, then $y' =$

Solution:

Use the rules $(\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$ and $(e^u) = e^u.u'$

$$y' = -\frac{1}{\sqrt{1 - (e^x)^2}}\,.e^x = -\frac{e^x}{\sqrt{1 - e^{2x}}}$$

51) If $y = \cos(2x^3)$, then $y' =$

Solution:

Use the rule $(\cos u)' = -\sin u\ .u'$

$$y' = -\sin(2x^3)\,.(6x^2) = -6x^2 \sin(2x^3)$$

52) If $y = \csc x \cot x$, then $y' =$

Solution:

Use the rules $(f.g)' = f'g + fg'$,

$(\csc u)' = -\csc u \cot u\,.u'$ and $(\cot u)' = -\csc^2 u\ .u'$

$$\begin{aligned} y' &= (-\csc x \cot x)(\cot x) + (\csc x)(-\csc^2 x) \\ &= -\csc x \cot^2 x - \csc^3 x = -\csc x(\cot^2 x + \csc^2 x) \end{aligned}$$

53) If $y = \sqrt{x^2 - 2\sec x}$, then $y' =$

Solution:

Use the rules

$$\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}} \quad \text{and} \quad (\sec u)' = \sec u \tan u \,.\, u'$$

$$y' = \frac{2x - 2\sec x \tan x}{2\sqrt{x^2 - 2\sec x}} = \frac{2(x - \sec x \tan x)}{2\sqrt{x^2 - 2\sec x}}$$

$$= \frac{x - \sec x \tan x}{\sqrt{x^2 - 2\sec x}}$$

54) If $y = (3x^2 + 1)^6$, then $y' =$

Solution:

Use the rule $\quad (u)^n = n(u)^{n-1}.u'$

$$y' = 6(3x^2 + 1)^5.(6x) = 36x(3x^2 + 1)^5$$

55) If $xy + \tan x = 2x^3 + \sin y$, then $y' =$

Solution:

$$[(1)(y) + (x)(y')] + \sec^2 x = 6x^2 + \cos y \,.\, y'$$

$$y + xy' + \sec^2 x = 6x^2 + y' \cos y$$

$$xy' - y' \cos y = 6x^2 - y - \sec^2 x$$

$$y'(x - \cos y) = 6x^2 - y - \sec^2 x$$

$$y' = \frac{6x^2 - y - \sec^2 x}{x - \cos y}$$

56) If $y = x^{-1} \sec x$, then $y' =$

Solution:

Use the rules

$$(f.g)' = f'g + fg' \quad \text{and} \quad (\sec u)' = \sec u \tan u \,.\, u'$$

$$y' = (-x^{-2})(\sec x) + (x^{-1})(\sec x \tan x)$$

$$= x^{-1} \sec x \tan x - x^{-2} \sec x$$

$$= x^{-2} \sec x \,(x \tan x - 1)$$

57) If $y = \sin^{-1}(x^3)$, then $y' =$

Solution:

Use the rule $\quad (\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$

$$y' = \frac{1}{\sqrt{1 - (x^3)^2}} \,.3x^2 = \frac{3x^2}{\sqrt{1 - x^6}}$$

58) If $y = \cos^{-1}(x^3)$, then $y' =$

Solution:

Use the rule $\quad (\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$

$$y' = -\frac{1}{\sqrt{1 - (x^3)^2}} \,.3x^2 = -\frac{3x^2}{\sqrt{1 - x^6}}$$

59) If $y = \sec^{-1}(x^3)$, then $y' =$

Solution:

Use the rule $\quad (\sec^{-1} u)' = \frac{u'}{|u|\sqrt{u^2-1}}$

$$y' = \frac{1}{x^3\sqrt{(x^3)^2 - 1}} \,.3x^2 = \frac{3x^2}{x^3\sqrt{x^6 - 1}} = \frac{3}{x\sqrt{x^6 - 1}}$$

60) If $y = \csc^{-1}(x^3)$, then $y' =$

Solution:

Use the rule $\quad (\csc^{-1} u)' = -\frac{u'}{|u|\sqrt{u^2-1}}$

$$y' = -\frac{1}{x^3\sqrt{(x^3)^2 - 1}} \,.3x^2 = -\frac{3x^2}{x^3\sqrt{x^6 - 1}} = -\frac{3}{x\sqrt{x^6 - 1}}$$

61) If $y = \ln(x^3 - 2\sec x)$, then $y' =$

Solution:

Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\sec u)' = \sec u \tan u \,.\, u'$$

$$y' = \frac{1}{x^3 - 2\sec x} \,.\,(3x^2 - 2\sec x \tan x)$$

$$= \frac{3x^2 - 2\sec x \tan x}{x^3 - 2\sec x}$$

62) If $y = \ln(\cos x)$, then $y' =$

Solution:

Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\cos u)' = -\sin u \,.\, u'$$

$$y' = \frac{1}{\cos x} \,.\,(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x$$

63) If $y = \ln(\sin x)$, then $y' =$

Solution:

Use the rules

$$(\ln u)' = \frac{u'}{u} \quad \text{and} \quad (\sin u)' = \cos u \,.\, u'$$

$$y' = \frac{1}{\sin x} \,.\,(\cos x) = \frac{\cos x}{\sin x} = \cot x$$

64) If $y = \ln\sqrt{3x^2 + 5x}$, then $y' =$

Solution:

Use the rules $\quad (\ln u)' = \frac{u'}{u} \quad$ and $\quad \left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y' = \frac{1}{\sqrt{3x^2 + 5x}} \,.\left(\frac{6x + 5}{2\sqrt{3x^2 + 5x}}\right) = \frac{6x + 5}{2(3x^2 + 5x)}$$

65) If $y = \log_5(x^3 - 2\csc x)$, then $y' =$

Solution:

Use the rules

$$(\log_a u)' = \frac{u'}{u\ln a} \quad \text{and} \quad (\csc u)' = -\csc u \cot u \,.\, u'$$

$$y' = \frac{1}{(x^3 - 2\csc x)(\ln 5)} \,.\, [3x^2 - 2\,(-\csc x \cot x)]$$

$$= \frac{3x^2 + 2\csc x \cot x}{(x^3 - 2\csc x)(\ln 5)}$$

66) If $y = \ln\frac{x-1}{\sqrt{x+2}}$, then $y' =$

Solution:

Use the rules

$$(\ln u)' = \frac{u'}{u} \;,\; \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \quad \text{and} \quad \left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$$

$$y' = \frac{1}{\frac{x-1}{\sqrt{x+2}}} \cdot \left(\frac{(1)\left(\sqrt{x+2}\right) - (x-1)\left(\frac{1}{2\sqrt{x+2}}\right)}{\left(\sqrt{x+2}\right)^2}\right)$$

$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\sqrt{x+2} - \frac{x-1}{2\sqrt{x+2}}}{x+2}\right)$$

$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\frac{2(x+2)-(x-1)}{2\sqrt{x+2}}}{x+2}\right)$$

$$= \frac{\sqrt{x+2}}{x-1} \cdot \left(\frac{\frac{x+5}{2\sqrt{x+2}}}{x+2}\right)$$

$$= \frac{\sqrt{x+2}}{x-1}\left(\frac{x+5}{2(x+2)\sqrt{x+2}}\right)$$

$$= \frac{x+5}{2(x-1)(x+2)}$$

67) If $y = 2x^3 - \sin x$, then $y' =$

Solution:

Use the rule $(\sin u)' = \cos u \,.\, u'$

$$y' = 6x^2 - \cos x$$

68) If $y = x^3 \cos x$, then $y' =$

Solution:

Use the rules

$$(f.g)' = f'g + fg' \quad \text{and} \quad (\cos u)' = -\sin u \,.\, u'$$

$$y' = (3x^2)(\cos x) + (x^3)(-\sin x)$$

$$= 3x^2\cos x - x^3 \sin x$$

69) If $y = x^{\sqrt{x}}$, then $y' =$

Solution:

Use the rule $\left(\sqrt{u}\right)' = \frac{u'}{2\sqrt{u}}$

$$y = x^{\sqrt{x}}$$

$$\ln y = \ln x^{\sqrt{x}}$$

$$\ln y = \sqrt{x}\ln x$$

$$\frac{y'}{y} = \left(\frac{1}{2\sqrt{x}}\right)(\ln x) + \left(\sqrt{x}\right)\left(\frac{1}{x}\right)$$

$$\frac{y'}{y} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{x\ln x + 2x}{2x\sqrt{x}} = \frac{x(\ln x + 2)}{2x\sqrt{x}}$$

$$= \frac{\ln x + 2}{2\sqrt{x}}$$

$$y' = y\left(\frac{\ln x + 2}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right)$$

70) If $y = (\sin x)^x$, then $y' =$

Solution:

Use the rule $(\sin u)' = \cos u \,.\, u'$

$$y = (\sin x)^x$$

$$\ln y = \ln(\sin x)^x$$

$$\ln y = x\ln(\sin x)$$

$$\frac{y'}{y} = (1)(\ln(\sin x)) + (x)\left(\frac{\cos x}{\sin x}\right)$$

$$\frac{y'}{y} = \ln(\sin x) + \frac{x\cos x}{\sin x} = \ln(\sin x) + x\cot x$$

$$y' = y(\ln(\sin x) + x\cot x)$$

$$= (\sin x)^x(\ln(\sin x) + x\cot x)$$

71) If $y = \log_7(x^3 - 2)$, then $y' =$

Solution:

Use the rule $(\log_a u)' = \frac{u'}{u\ln a}$

$$y' = \frac{1}{(x^3-2)(\ln 7)} \,.\, (3x^2) = \frac{3x^2}{(x^3-2)(\ln 7)}$$

72) If $y = \cos(x^5)$, then $y' =$

Solution:

Use the rule $(\cos u)' = -\sin u \,.\, u'$

$$y' = -\sin(x^5)\,.\,(5x^4) = -5x^4\sin(x^5)$$

73) If $y = \sec x \tan x$, then $y' =$

Solution:

$(f.g)' = f'g + fg'$, $(\sec u)' = \sec u \tan u \,.\, u'$ and

$$(\tan u)' = \sec^2 u \,.\, u'$$

$$\begin{aligned} y' &= (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x) \\ &= \sec x \tan^2 x + \sec^3 x = \sec x(\tan^2 x + \sec^2 x) \end{aligned}$$

74) If $D^{99}(\cos x) =$

Solution:

$$D(\cos x) = -\sin x$$
$$D^2(\cos x) = -\cos x$$
$$D^3(\cos x) = \sin x$$
$$D^4(\cos x) = \cos x$$

Note: $D^n(\cos x) = \cos x$ whenever n is a multiple of 4.
Hence,

$$D^{96}(\cos x) = \cos x$$
$$D^{97}(\cos x) = -\sin x$$
$$D^{98}(\cos x) = -\cos x$$
$$D^{99}(\cos x) = \sin x$$

75) If $y = (x + \sec x)^3$, then $y' =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\sec u)' = \sec u \tan u \,.\, u'$

$$y' = 3(x + \sec x)^2.(1 + \sec x \tan x)$$

76) If $x^2 = 5y^2 + \sin y$, then $y' =$

Solution:

$$2x = 10yy' + \cos y \,.\, y'$$
$$y'(10y + \cos y) = 2x$$
$$y' = \frac{2x}{10y + \cos y}$$

77) If $x^2 - 5y^2 + \sin y = 0$, then $y' =$

Solution:

$$2x - 10yy' + \cos y \,.\, y' = 0$$
$$y'(-10y + \cos y) = -2x$$
$$y' = \frac{-2x}{-10y + \cos y} = \frac{2x}{10y - \cos y}$$

78) If $y = \sin x \sec x$, then $y' =$

Solution:

$(f.g)' = f'g + fg'$, $(\sin u)' = \cos u \,.\, u'$ and

$$(\sec u)' = \sec u \tan u \,.\, u'$$

$$\begin{aligned} y' &= (\cos x)(\sec x) + (\sin x)(\sec x \tan x) \\ &= 1 + \sin x \,.\, \frac{1}{\cos x} \,.\, \frac{\sin x}{\cos x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x \\ &= \sec^2 x \end{aligned}$$

79) If $f(x) = \sin^2(x^3 + 1)$, then $f'(x) =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\sin u)' = \cos u \,.\, u'$

$$\begin{aligned} f'(x) &= 2\sin(x^3 + 1) \,.\, (\cos(x^3 + 1)) \,.\, (3x^2) \\ &= 6x^2 \sin(x^3 + 1) \cos(x^3 + 1) \end{aligned}$$

80) If $y = (x + \cot x)^3$, then $y' =$

Solution:

Use the rules

$(u)^n = n(u)^{n-1}.u'$ and $(\cot u)' = -\csc^2 u \,.\, u'$

$$y' = 3(x + \cot x)^2.(1 - \csc^2 x)$$

81) If $y = \tan^{-1}\left(\frac{x}{2}\right)$, then $y' =$

Solution:

Use the rule $(\tan^{-1} u)' = \frac{u'}{1+u^2}$

$$y' = \frac{1}{1 + \left(\frac{x}{2}\right)^2} \,.\, \frac{1}{2} = \frac{1}{2\left(1 + \frac{x^2}{4}\right)} = \frac{1}{2\left(\frac{4 + x^2}{4}\right)} = \frac{2}{4 + x^2}$$

82) If $y = \cot^{-1}\left(\frac{x}{2}\right)$, then $y' =$

Solution:

Use the rule $(\cot^{-1} u)' = -\frac{u'}{1+u^2}$

$$y' = -\frac{1}{1 + \left(\frac{x}{2}\right)^2} \,.\, \frac{1}{2} = -\frac{1}{2\left(1 + \frac{x^2}{4}\right)} = -\frac{1}{2\left(\frac{4 + x^2}{4}\right)}$$
$$= -\frac{2}{4 + x^2}$$

83) If $y = \sin^{-1}\left(\frac{x}{3}\right)$, then $y' =$

Solution:

Use the rule $(\sin^{-1} u)' = \frac{u'}{\sqrt{1-u^2}}$

$$y' = \frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \,.\, \frac{1}{3} = \frac{1}{3\sqrt{1 - \frac{x^2}{9}}} = \frac{1}{3\sqrt{\frac{9 - x^2}{9}}}$$
$$= \frac{1}{\sqrt{9 - x^2}}$$

84) If $y = \cos^{-1}\left(\frac{x}{3}\right)$, then $y' =$

Solution:

Use the rule $(\cos^{-1} u)' = -\frac{u'}{\sqrt{1-u^2}}$

$$y' = -\frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \,.\, \frac{1}{3} = -\frac{1}{3\sqrt{1 - \frac{x^2}{9}}} = -\frac{1}{3\sqrt{\frac{9 - x^2}{9}}}$$
$$= -\frac{1}{\sqrt{9 - x^2}}$$

85) If $D^{99}(\sin x) =$

Solution:

$$D(\sin x) = \cos x$$
$$D^2(\sin x) = -\sin x$$
$$D^3(\sin x) = -\cos x$$
$$D^4(\sin x) = \sin x$$

Note: $D^n(\sin x) = \sin x$ whenever n is a multiple of 4.

Hence,

$$D^{96}(\sin x) = \sin x$$
$$D^{97}(\sin x) = \cos x$$
$$D^{98}(\sin x) = -\sin x$$
$$D^{99}(\sin x) = -\cos x$$

TRAINING FINAL EXAM - MATH 110
SECTIONS (APPENDIX D TO 2.6)

1. The horizontal asymptote(s) of the function $f\left(x\right)=\frac{\sqrt{9x^{2}+2x}}{x-3}$ is (are)

(a) $x=3$

(b) $y=1$

(c) $y=3,y=-3$

(d) $y=-1$

2. $\lim\limits_{x\to 0}\frac{\tan 5x}{\tan 3x}=$

(a) $\frac{3}{5}$

(b) $\frac{5}{3}$

(c) 1

(d) Does not exist

3. $\csc^{-1}\left(\frac{2}{\sqrt{3}}\right)=\frac{\pi}{3}$

(a) True

(b) False

4. $\lim\limits_{x\to 0^{-}}\frac{6x+|x|}{7x}=$

(a) 1

(b) $\frac{7}{6}$

(c) $\frac{6}{7}$

(d) $\frac{5}{7}$

5. The degree measure of $\theta=\frac{5\pi}{12}$ is

(a) 75°

(b) 750°

(c) 150°

(d) 120°

6. If $f(x) = (x+2)^2$, $g(x) = \sqrt{x}$, Then $(g \circ f)(x) =$

(a) $x^2 + 2$

(b) $\sqrt{x+2}$

(c) $x + 2$

(d) $\sqrt{x^2+2}$

7. The function $f(x) = \frac{\sqrt{4-x^2}}{x-2}$ is continuous on

(a) $[-2, 2]$

(b) $[-2, 2)$

(c) $(-\infty, -2] \cup (2, \infty)$

(d) $(-\infty, -2) \cup (2, \infty)$

8. The function $f(x) = x^{\frac{2}{3}} + x^3 + 2x + 1$ is

(a) Algebraic function

(b) Power function

(c) Polynomial function

(d) Exponential function

9. The function $f(x) = x^4 + 5$ is symmetric about origin

(a) True

(b) False

10. The function $f(x) = \left(\frac{1}{4}\right)^x$ is increasing on $\mathbb{R}$

(a) True

(*b*) False

11. $\lim\limits_{x\to\infty} \frac{x-4}{x^2-x-12} =$

(*a*) 0

(*b*) $\frac{1}{3}$

(*c*) 4

(*d*) ∞

12. The vertical asymptote(s) of the function $(f)(x) = \frac{x-4}{x^2-16}$

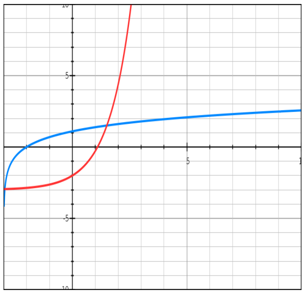
(*a*) $y = -4$

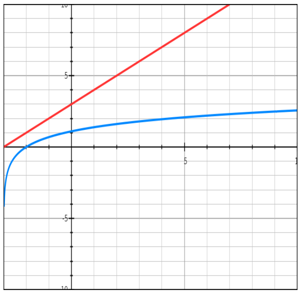
(*b*) $x = -4$

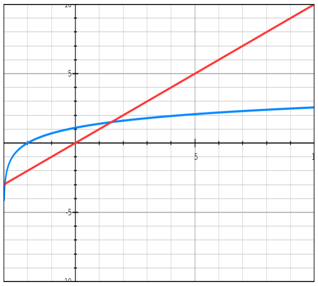
(*c*) $x = 4, x = -4$

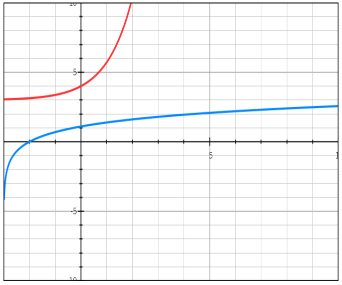
(*d*) $x = 4$

13. Which of the following represents a function and its inverse


(*a*)


(*b*)


(*c*)


(*d*)

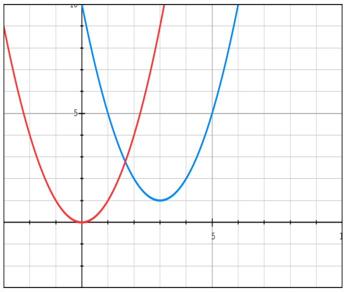
(*a*) a

(b) b

(c) c

(d) d

14. The following figure shows that an equation for new function from old function $f(x) = x^2$



is $f(x) = (x-3)^2 + 1$

(a) True

(b) False

15. The domain of the function $f(x) = \cos^{-1}(3x+4)$ is

(a) $\left[1, \frac{5}{3}\right]$

(b) $\left(1, \frac{5}{3}\right)$

(c) $\left[-\frac{5}{3}, -1\right]$

(d) $\left(-\frac{5}{3}, -1\right)$

16. If $\frac{1}{3}(\cos x + 11) \leq f(x) \leq e^x + 3$, then $\lim_{x \to 0} f(x) =$

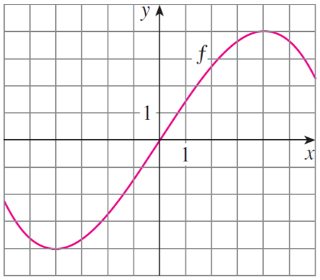
(a) 0

(b) 3

(c) 4

(d) $\frac{1}{3}$

17. The following graph represents one-to-one function



(a) True

(b) False

18. The range of the function $f(x) = e^x - 3$ is

(a) $(3, \infty)$

(b) $(0, \infty)$

(c) $\mathbb{R}$

(d) $(-3, \infty)$

19. $\lim\limits_{x \to \infty} \left(-3x^3 + 2x + 5\right) =$

(a) $+\infty$

(b) -3

(c) 5

(d) $-\infty$

20. The domain of the function $f(x) = \ln(x-1) + \sqrt{x^2+2}$ is

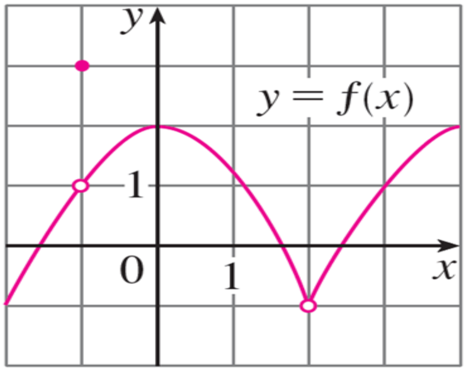
(a) $(0, \infty)$

(b) $(1, \infty)$

(c) $\mathbb{R}$

(d) $(0,1)$

21.



Then $\lim_{x\to -1} f(x) =$

(a) -1

(b) 1

(c) 3

(d) Does not exist

22. $\cot\left(\frac{5\pi}{3}+\pi\right)=\cot\left(\frac{5\pi}{3}\right)$

(a) True

(b) False

23. If $f(x)=\ln(x-5)$, then $f^{-1}(x)=$

(a) e^{x+5}

(b) e^{x-5}

(c) $e^{x}+5$

(d) $e^{x}-5$

FINAL EXAM-MATH 110
FROM SECTION 2.7 TO SECTION 4.3

1. If $f(x) = |x^3 - 25x|$ then $f(x)$ is differentiable at $x =$

   a) $x = 2$ b) $x = 5$ c) $x = 0$ d) $x = -5$

2. If $y = \tan^2(\frac{\pi}{4})$ then $y' = 0$

   a) True b) False

3. The equation of the normal line line of the curve $f(x) = \mathrm{e}^x \mathrm{g}(\mathrm{x})$ where $\mathrm{g}(0) = 2$ and $\mathrm{g}'(0) = 3$ is ……

   a) $y = 2 - \frac{1}{5}x$ b) $y = 5x + 2$

   c) $x = 2 - \frac{1}{5}y$ d) $x = 5y + 2$

4. If $g(x) = 4^{\mathrm{x}-1}$ then $g'(1) =$ ……..

   a) 4 b) 2ln(2) c) ln(2) d) 1

5. If $g(x) = (x^2 + 4)(2x^2 + 3)$ then $g''(x) =$ ……..

   a) $2x^4 + 11x^2 + 12$ b) $48x$

   c) $24x^2 - 22$ d) $8x^3 + 22x$

6. If $f(x) = e^{2x}$ then $f^{(n)}(x) =$ ……..

   a) e^{2x} b) $2e^{2x}$

   c) $2^n e^{2x}$ d) $\frac{e^{2x}}{2^n}$

7. If $f(x) = \dfrac{x^2}{x^2 - 2}$ then $f'(3) =$ ……..

   a) $\frac{-12}{49}$ b) 104

   c) $\frac{3}{2}$ d) -2

8. If $y = x^2(e^x + 5)$ then $y' = e^x(x^2 + 2x) + 5$

a) True **b) False**

9. If $f(x) = cx + \ln(\cos(x))$ *and* $f'\left(\frac{\pi}{4}\right) = 6$ then $c =$ .....

b) 6 **b) 7**

c) 5 **d)$-\frac{13}{2}$**

10. If $f(x) = \sin x$ then $f^{(99)}(x) =$ ....

a) $\cos(x)$ **b) $\sin(x)$**

c) $-\cos(x)$ **d)$-\sin(x)$**

11. If $y = x\sec x + \tan(\sin x)$ then $y' = \cdots$

a) $\sec(x) + \sec^2(\sin(x))$
b) $\sec(x)(1 + x\tan(x)) + \cos(x)\sec^2(\sin(x))$
c) $\sec(x)\tan(x) + \sec^2(\sin x)$
d) $\sec(x) + (\cos\tan(\sin(x))$

12. If $y = \sqrt{x^3 - 27}$ then $y' = \cdots$

a) $\frac{1}{2\sqrt{x^3-27}}$
b) $\frac{3x^2}{\sqrt{x^3-27}}$
c) $\frac{3x^2}{2\sqrt{x^3-27}}$
d) $\frac{1}{\sqrt{x^3-27}}$

13. **If $y = \cos^2 x - \sin^2 x$ then $y' = \cdots$**

- **a)** 1
- **b)** 0
- **c)** $2\sin(2x)$
- **d)** $-2\sin(2x)$

14. **If $h(x) = 10^{2\sqrt{x}}$ then $h'(x) = \cdots$**

- **a)** $10^{2\sqrt{x}} \ln(10)$
- **b)** $\frac{10^{2\sqrt{x}} \ln(10)}{\sqrt{x}}$
- **c)** $\frac{10^{2\sqrt{x}}}{\sqrt{x}}$
- **d)** $\frac{10^{2\sqrt{x}} \ln(10)}{2\sqrt{x}}$

15. **If $h(x) = \sin^{-1}(x)$ then $h''(x) = \frac{x}{\sqrt{(1-x^2)^3}}$**

a) True **b) False**

16. **If $h(x) = x\tan^{-1}\left(\frac{x}{2}\right)$ then $h'(2) =$ ……..**

- **a)** $\frac{\pi}{4}$
- **b)** $\frac{\pi+2}{4}$
- **c)** $\frac{\pi+3}{6}$
- **d)** $\frac{\pi}{6}$

17. If $x^2 + y^2 = 4xy$ then $y' =$ .....

a) $\frac{y+2x}{2y+x}$

b) $\frac{y-2x}{2y-x}$

c) $\frac{2y+x}{y+2x}$

d) $\frac{2y-x}{y-2x}$

18. If $y = \ln(x^2 + y^2)$ then $y' =$ .....

a) $\frac{2x}{x^2+y^2-2}$

b) $\frac{2}{x} + \frac{2}{y}$

c) $\frac{2x+2y}{x^2+y^2}$

d) $\frac{2y}{xy-2x}$

19. If $f(x) = x^{\cos x}$ then $f'(x) =$ .....

a) $-x^{\cos x} \sin(x)\ln(\mathrm{x})$

b) $-\cos(x)x^{\cos x-1}\sin(x)$

c) $\frac{-\cos(x)\sin(x)}{x}$

d) $x^{\cos(x)-1}[\cos(x) - x\ln(x)\sin(x)]$

20. $\frac{d^4}{dx^4}(x^3\ln(x)) =$ .....

a) $\frac{6}{x}$

b) $11 + 6\ln(x)$

c) $x^2 + 3x^2\ln(x)$

d) $5x + 6x\ln(x)$

21. If $f(x) = \ln(\csc(x) - \cot(x))$ then $f'(x) =$ .....

a) $\csc(x)$

b) $-\csc(x)$

c) $-\sin(x)$

d) $\sin(x)$

22. If $f(x) = \log(\sin^3(x))$ then $f'(x) =$ .....

a) $3\tan(x)$

b) $3\cot(x)$

c) $\frac{3\tan(x)}{\ln(10)}$

d) $\frac{3\cot(x)}{\ln(10)}$

23. If $h(x) = \ln\left(xe^{x^2}\right)$ then $h''(2) = \frac{7}{4}$

a) True **b) False**

24. If $h(x) = \log_3(sec(x))$ then $h'''(2) = 4\ln(3)$

a) True **b) False**

25. If $f(x) = \frac{2}{3}x^3 - 8x$ then the critical numbers of $f(x)$ are

a) $x = \pm 2$

b) $x = \pm 4$

c) $x = \pm 8$

d) $x = 0 \;\; and \; x = \pm 2$

26. If $f(x) = x^2 - 6x$ then $f(x)$ has local ………………..

a) minimum $x = 3$

b) minimum $x = -9$

c) maximum $x = 3$

d) maximum $x = 3$

27. If $f(x) = 27x - x^3$ then ……… . is local maximum value

a) 3

b) -3

c) 54

d) -54

28. If $h(x) = 2 - x - x^2$ then $h(x)$ has concave down on R

a) True b) False

29. If $h(x) = 2 - x^2 - x^3$ then $\left(-\frac{1}{3}, h\left(-\frac{1}{3}\right)\right)$ is inflection point of $h(x)$

a) True b) False

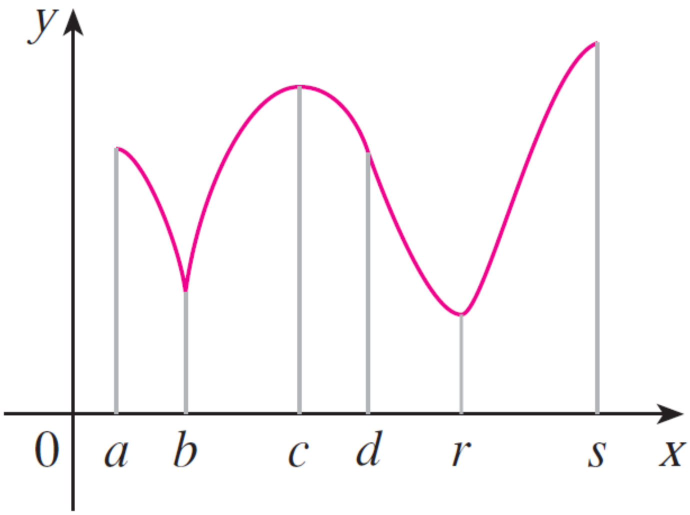
30. If $h(x) = e^x$ then $f(x)$ has no extreme value

a) True b) False

31. If $f(x) = 5x^2 - 20x$ then $f(x)$ has absolute minimum at $x =$ .... on [-1,5]

a) -1
b) 5
c) 2
d) 0

31. If $f(x)$ is a function whose graph is shown



Then f(x) has absolute maximum at $x =$ ……..

f(x) has absolute minimum at $x =$ ……..

f(x) has local minimum at $x =$ ……..

f(x) has local maximum at $x =$ ……..

the critical number of f(x) are ……..

f(x) have concave up on ………………………………………………..
and concave down on ……………………………………..…………

the inflection point of f(x) is ……………………………………………....