324 Stat
Lecture Notes

(3) Mathematical Expectation

(Book: Chapter 4 ,pg 111-137)

Mean of a Random Variable:

Definition:

Let **X** be a random variable with probability distribution **f(x)**.

The mean or expected value of **X** is:

$$\mu = E(X) = \sum_{\forall X} X\ f(X) \qquad if\ \ X\ \ is\ \ \ discrete$$

$$\mu = E(X) = \int_{-\infty}^{\infty} X\ f(X)\, dX \qquad if\ \ X\ \ is\ \ continuous \qquad (1)$$

Properties of the Expectation:

1. $E(a) = \mathbf{a}$, where **a** is a constant
2. $E(aX) = a\,E(X)$
3. $E(aX + b) = a\,E(X) + b$

Ex (1):

Find the expected number of chemists on a committee of **3** selected at random from **4** chemists and **3** biologists.

Find: E (5), E (3x), E (2x-1)

<u>**Solution:**</u>

Let **X** represent the number of chemists on the committee.

The probability distribution of **X** is given by:

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}} \quad , \quad X = 0,1,2,3$$

$$f(0)=\frac{\binom{4}{0}\binom{3}{3}}{\binom{7}{3}}=\frac{1}{35},\qquad f(1)=\frac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}}=\frac{12}{35},$$

$$f(2)=\frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}}=\frac{18}{35},\qquad f(3)=\frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}}=\frac{4}{35}$$

X	0	1	2	3	$\sum$
f(x)	1/35	12/35	18/35	4/35	1
x f(x)	0	12/35	36/35	12/35	60/35=1.71

$$E(X) = \mu_X = \sum xf(x) = 60/35 = 1.71$$

$$E(5) = 5$$

See Ex 4.1 pg 113

$$E(3x) = 3E(x) = 3(60/35) = 5.143$$

$$E(2x - 1) = 2E(x) - 1 = 2(60/35) - 1 = 2.429$$

Ex 4.3 pg114:

Let **X** be a random variable that denotes the life in hours of a certain electronic device. The probability density function is given by:

$$f(x) = \begin{cases} \dfrac{20000}{X^3} & , \quad X > 100 \\ 0 & \quad otherwise \end{cases}$$

Find the expected life of this type of device

Solution:

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{100}^{\infty} X\left(\frac{20000}{X^3}\right) dX$$

$$= \int_{100}^{\infty} \frac{2000}{X^2} dX = 20000 \int_{100}^{\infty} X^{-2} dX$$

$$= 20000\left(\frac{X^{-1}}{-1}\right)_{100} |^{\infty} = 20000[(100)^{-1} - (\infty)^{-1}]$$

$$= \frac{20000}{100} - \frac{20000}{\infty} = 200 - 0 = 200$$

EX 4.4 pg 115:

Suppose that the number of cars **X** that pass through a car wash between **4** P.M. and **5** P.M. on any sunny Friday has the following probability distribution:

X	4	5	6	7	8	9
f(X)	1/12	1/12	1/4	1/4	1/6	1/6

Let $g(x) = 2x - 1$ represent the amount of money in dollars, paid to the attendant by the manager. Find the attendant's expected earning for this particular time period.

Solution:

X	4	5	6	7	8	9	$\sum$
f(X)	1/12	1/12	1/4	1/4	1/6	1/6	1
X f(x)	4/12	5/12	6/4	7/4	8/6	9/6	164/24

$$E(g(x)) = E(2x-1) = 2E(X)-1 =$$

$$= 2\left(\frac{164}{24}\right)-1 = 12.67$$

Ex (4.5 pg 115):

Let **X** be a random variable with density function:

$$f(x) = \begin{cases} \dfrac{X^2}{3}, & -1 < x < 2 \\ 0 & otherwise \end{cases}$$

Find the expected value of g(x) = 4x+3

Solution:

$$E(X)=\int_{-1}^{2} x(\frac{x^2}{3})\,dx=\int_{-1}^{2}(\frac{x^3}{3})dx=(\frac{x^4}{12})_{-1}|^2$$

$$=\frac{1}{12}[2^4-(-1)^4]=\frac{1}{12}(16-1)=\frac{15}{12}$$

$$E(g(x))=E(4x+3)=4E(X)+3=4(\frac{15}{12})+3=8$$

Variance:

Definition:

Let **X** be a random variable with probability distribution **f(x)** and mean μ. The variance of **X** is denoted by V(x) or σ_X^2 :

$$V(x) = \sigma^2 = E(x-\mu)^2 = \sum_{\forall x}(x-\mu)^2 f(x) = E(X^2) - (E(X))^2 \quad if\ x\ is\ discrete\ (2)$$

$$V(x) = \sigma^2 = E(x-\mu)^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = E(X^2) - (E(X))^2\ if\ \ x\ is\ continuous\ (3$$

where:

$$E(x^2) = \begin{cases} \sum x^2 f(x) & if\ \ x\ \ is\ discrete \\ \int_{-\infty}^{\infty} x^2 f(x)dx & if\ \ x\ is\ continuous \end{cases}$$

Properties of the variance:

1. $V(a) = 0$ where **a** is a constant

2. $V(aX) = a^2 V(X)$

3. $V(aX + b) = a^2 V(X) + 0$

The Standard Deviation:

The positive square root of the variance, σ is called the standard deviation of **X** which is given by:

$$\sigma_X = \sqrt{V(x)} = \sqrt{E(x - \mu_X)^2}$$

Ex (4.8 pg 120):

The probability distribution for company **A** is given by:

X	1	2	3
f(x)	0.3	0.4	0.3

and for company **B** is given by:

Y	0	1	2	3	4
f(y)	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company **B** is greater than that of company **A**.

Solution:

X	1	2	3	$\sum$
f(x)	0.3	0.4	0.3	1
x f(x)	0.3	0.8	0.9	2
f(x) x^2	0.3	1.6	2.7	4.6

$$\sigma^2 = E(x^2) - (E(x))^2 = 4.6 - 4 = 0.6, \sigma = .77$$

Y	0	1	2	3	4	$\sum$
f(y)	0.2	0.1	0.3	0.3	0.1	1
Y f(y)	0	0.1.	0.6	0.9	0.4	2
y^2 f(y)	0	0.1	1.2	2.7	1.6	5.6

$$\sigma^2 = E(y^2) - (E(y))^2 = 5.6 - 4 = 1.6, \sigma = 1.26$$

Note that σ_y^2 is greater than σ_x^2.

Ex (4.10 pg 121):

The weekly demand for a drinking-water product, in thousands of liters from a local chain of efficiency stores having the probability density:

$$f(x) = \begin{cases} 2(x-1) \quad , & 1 < X < 2 \\ 0 & otherwise \end{cases}$$

Find the mean and variance of x .

Solution:

$$\mu = \int_1^2 2x(x-1)\ dx = 2\int_1^2 (x^2 - x)\ dx = 2\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_1^2 = 2\left[\left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right]$$

$$= 2\left(\frac{8-6}{3} - \frac{2-3}{6}\right) = 2\left(\frac{2}{3} + \frac{1}{6}\right) = \frac{5}{3}$$

$$E(X^2) = \int_1^2 2x^2(x-1)\,dx = 2\int_1^2 (x^3 - x^2)\,dx = 2\left(\frac{x^4}{4} - \frac{x^3}{3}\right)_1\Big|^2$$

$$= 2\left[\left(4 - \frac{2}{8}\right) - \left(\frac{1}{4} - \frac{1}{3}\right)\right] = 17/6$$

$$\sigma^2 = E(x^2) - (E(x))^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = 1/18$$

Ex 4.18 pg 129:

Let **X** be a random variable having the density function:

$$f(x) = \begin{cases} \frac{x^2}{3} & , \quad -1 < x < 2 \\ 0 & \quad otherwise \end{cases}$$

Find the variance of the random variable **g(x) = 4x+3**.

Solution:

$$V(g(x)) = V(4x+3) = 16\,V(x) = 16[E(x^2)-(E(x))^2]$$

$$E(x) = \int_{-1}^{2} X\frac{X^2}{3}dx = \frac{X^4}{12}\Big|_{-1}^{2} = \frac{16}{12} - \frac{1}{12} = \frac{15}{12} = \frac{5}{4}$$

$$E(x^2) = \int_{-1}^{2} x^2\left(\frac{x^2}{3}\right)dx = \frac{x^5}{15}\Big|_{-1}^{2} = \frac{32}{15} - \frac{(-1)}{15} = \frac{11}{5}$$

$$V(x) = E(x^2) - (E(x))^2 = \frac{11}{5} - \left(\frac{5}{4}\right)^2 = \frac{11}{5} - \frac{25}{16} = \frac{176-125}{80} = 0.6375$$

$$V(g(x)) = V(4x+3) = 16V(x) + 0 = 16(0.6375) = 10.2$$

4.3 Means and Variance of Linear Combinations of Random Variables (pg 128):

The expected value of the sum or difference of two or more functions of a random variable **X** is the sum or difference of the expected values of the functions. That is

$$E(g(x) \pm h(x)) = E(g(x)) \pm E(h(x)) \qquad (6)$$

Ex4.19 pg 129 :

Let **X** be a random variable with probability distribution as follows:

X	0	1	2	3
f(x)	1/3	1/2	0	1/6

Find the expected value of $y=(x-1)^2$.

Solution:

$$E(y) = E(x-1)^2 = E(x^2 - 2x + 1) = E(x^2) - 2E(x) + 1$$

X	0	1	2	3	$\sum$
f(x)	1/3	1/2	0	1/6	1
X f(x)	0	1/2	0	3/6	1
X^2 f(x)	0	1/2	0	9/6	2

$$E(y) = 2 - 2(1) + 1 = 1$$

See ex 4.17 pg 128

Ex 4.20 pg 130:

Find the expected value for $g(x) = x^2+x-2$,

where **X** has the density function:

$$f(x) = \begin{cases} 2(x-1) & , \quad 1 < x < 2 \\ 0 & \quad otherwise \end{cases}$$

Solution:

$$E(x)=\int_1^2 x\,2(x-1)\,dx=2\int_1^2 (x^2-x)dx=2\left(\frac{x^3}{3}-\frac{x^2}{2}\right)_1\Big|^2=$$

$$=2\left(\frac{2}{3}+\frac{1}{6}\right)=\frac{5}{3}$$

$$E(x^2)=\int_1^2 2x^2(x-1)\,dx=2\int_1^2 (x^3-x^2)\,dx=2\left(\frac{x^4}{4}-\frac{x^3}{3}\right)_1\Big|^2$$

$$=2\left[\left(4-\frac{8}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right)\right]=2\left(\frac{12-8}{3}-\frac{3-4}{12}\right)=2\left(\frac{4}{3}+\frac{1}{12}\right)=\frac{17}{6}$$

$$E(x^2+x-2)=E(x^2)+E(x)-2=\frac{17}{6}+\frac{5}{3}-2=\frac{5}{2}$$

4.4 Chebyshev's Theorem (pg 135):

The probability that any random variable **X** will assume a value within **K** standard deviations of the mean μ_X is at least $(1-\frac{1}{K^2})$.

That is:

$$P(\mu - K\sigma < X < \mu + K\sigma) \geq 1 - \frac{1}{K^2} \qquad (7)$$

Ex (4.27 pg 137):

A random variable **X** has a mean $\mu = 8$, a variance $\sigma^2 = 9$ and an unknown probability distribution. Find:

(a) $P(-4 < X < 20)$

(b) $P(|X - 8| \geq 6)$

Solution:

$$(a)\,P(-4 < X < 20) = P[8-(k)(3) < X < 8+(k)(3)] \rightarrow$$

$$8-3k = -4 \rightarrow 8+4 = 3k \rightarrow 12 = 3k \rightarrow k = 4$$

$$P(-4 < X < 20) \geq 1-\frac{1}{16} \rightarrow P(-4 < X < 20) \geq \frac{15}{16}$$

$$(b)\,P(|X-8| \geq 6) = 1-P(|x-8| < 6) = 1-P(-6 < (X-8) < 6)$$

$$= 1-P(-6+8 < X < 6+8) = 1-P(2 < X < 14)$$

$$1-P(2 < X < 14) \geq 1-\frac{1}{k^2} \rightarrow 1-1+\frac{1}{k^2} \geq P(2 < X < 14)$$

$$\rightarrow \frac{1}{k^2} \geq P(2 < X < 14) \rightarrow P(2 < X < 14) \leq \frac{1}{k^2}$$

$$2 = 8-3K \rightarrow 3K = 8-2 = 6 \rightarrow K = 2 \quad or$$

$$14 = 8+3k \rightarrow 14-8 = 3K \rightarrow 6 = 3K \rightarrow K = 2$$

$$P(2 < x < 14) \leq \frac{1}{4}$$