

(Linear Algebra) :

Matrix Algebra

$$\begin{array}{c}
 p, n \geq 1 \\
 A : \mathbb{N}_n \times \mathbb{N}_p \rightarrow K; (i, j) \rightarrow a_{ij} \\
 A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \\
 \begin{array}{c} \cdot (j) \quad (i) \quad a_{ij} \\ n \times p \quad M_{n \times p}(K) \quad K \\ A \quad n \times p \quad A \quad k \\ \cdot k \quad p \quad n \end{array}
 \end{array}$$

$$\begin{array}{c}
 (i) \quad R_i(A) = (a_{i1}, \dots, a_{ip}) \in K^n \quad i = 1, \dots, n \\
 \cdot A \\
 j \quad C_j(A) = (a_{j1}, \dots, a_{jn}) \in K^n \quad j = 1, \dots, p \\
 \cdot A
 \end{array}$$

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$$A = (a_{ij}), B = (b_{ij}) \quad M_{n \times p}(K) \quad A, B$$

$$\forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_p; a_{ij} = b_{ij} \Leftrightarrow A = B \quad \bullet$$

$$M_{n \times p}(K) \quad A + B = (C_{ij}) \quad \bullet$$

$$\forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_p; c_{ij} = a_{ij} + b_{ij}$$

$$: \quad \lambda A \in M_{n \times p}(K) \quad \lambda \in K \quad \bullet$$

$$\lambda A = (d_{ij}); \quad \forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_p; \quad d_{ij} = \lambda a_{ij}$$

$$: \quad \mathbf{0}_k \quad \bullet$$

$$\mathbf{0}_{n \times p} = (f_{ij}); \quad \forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_p; \quad f_{ij} = \mathbf{0}_K$$

$$: \quad 3 \times 4 \quad B \quad A$$

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 5 & 3 & 0 & 1 \\ 9 & 2 & 1 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

$$\Rightarrow A + B = \begin{pmatrix} 2 & 4 & 3 & 7 \\ 7 & 9 & 7 & 9 \\ 18 & 12 & 12 & 18 \end{pmatrix}$$

$$2A = \begin{pmatrix} 2 & 4 & 0 & 6 \\ 10 & 6 & 0 & 2 \\ 18 & 4 & 2 & 12 \end{pmatrix} \quad \& \quad 2A + B = \begin{pmatrix} 3 & 6 & 3 & 10 \\ 12 & 12 & 7 & 10 \\ 27 & 14 & 13 & 24 \end{pmatrix}$$

$$0A = 0B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$: \quad k \in \mathbb{N}_n, l \in \mathbb{N}_p \quad E_{kl} \in M_{n \times p}(K)$$

$$E_{kl} = (\delta_{ik} \delta_{jl}) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

k

$.1$

l

$$\delta_{kl} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases} \quad \delta_{kl}$$

(1)

$n \times p$

$(M_{n \times p}(K), +, \cdot)$

$\cdot (k, l) \in \mathbb{N}_n \times \mathbb{N}_p$

E_{kl}

:

$(M_{n \times p}(K), +, \cdot)$

$\cdot E_{kl}$

$$\forall A \in M_{n,p}, \quad A = \sum_k \sum_l a_{kl} E_{kl}$$

$$\sum_k \sum_l \lambda_{kl} E_{kl} = 0_{n \times p} \quad (\lambda_{kl}) \in K^{n \times p}$$

$$\Rightarrow \begin{pmatrix} \lambda_{11} & \dots & \lambda_{1p} \\ \lambda_{n1} & \dots & \lambda_{np} \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \end{pmatrix}$$

$$\Rightarrow \lambda_{kl} = 0, \forall (k, l) \in \mathbb{N}_n \times \mathbb{N}_p$$

$\cdot n \times p \quad M_{n,p}(K)$

:

: $A \in M_{2 \times 2}(R)$

$$\begin{aligned} A = \begin{pmatrix} 1 & 2 \\ 5 & 9 \end{pmatrix} &= 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 5 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 9 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 5 & 9 \end{pmatrix} = A \end{aligned}$$

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lambda_{11}E_{11} + \lambda_{12}E_{12} + \lambda_{21}E_{21} + \lambda_{22}E_{22} = 0_{22}$$

$$\Rightarrow \lambda_{11} = \lambda_{12} = \lambda_{21} = \lambda_{22} = 0$$

-2

$${}^t A = (\tilde{a}_{ij})$$

$${}^t A \in M_{p \times n}(K)$$

$$A \in M_{n \times p}(K)$$

$$\cdot \forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_p; \tilde{a}_{ij} = a_{ji} \quad A$$

$$\cdot T : M_{n \times p}(K) \rightarrow M_{p \times n}(K); A \rightarrow {}^t A$$

$${}^t A = \begin{pmatrix} 1 & 5 \\ 2 & 9 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 \\ 5 & 9 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 4 \\ 5 & 9 & 0 \end{pmatrix} \Rightarrow {}^t B = \begin{pmatrix} 1 & 5 \\ 2 & 9 \\ 4 & 0 \end{pmatrix}$$

$$) M_{l \times p}(K)$$

$$(\quad)$$

$$\cdot (M_{n \times l}(K)$$

(Matrix Multiplication)

-3

$$B, A \quad . B = (b_{jk}) \in M_{m \times p}(K) \quad A = (a_{ij}) \in M_{n \times m}(K)$$

$$: \quad AB = (c_{ij}) \in M_{n \times p}(K) \quad : \quad AB$$

$$\forall (i, j) \in \mathbb{N}_n \times \mathbb{N}_p; c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$$

$$X = (a, b, c), Y = {}^t(x, y, z) \quad X \in M_{1 \times 3}(\mathbb{R}), Y \in M_{3 \times 1}(\mathbb{R})$$

$$XY = (a \ b \ c) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz \in \mathbb{R}$$

. XY

$$YX = \begin{pmatrix} x \\ y \\ z \end{pmatrix} (a \ b \ c) = \begin{pmatrix} xa & xb & xc \\ ya & yb & yc \\ za & zb & zc \end{pmatrix} \in M_{3 \times 3}(\mathbb{R}) \quad YX$$

XY \neq YX :

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$AB = \begin{pmatrix} (1 \ 2) \begin{pmatrix} a \\ c \end{pmatrix} & (1 \ 2) \begin{pmatrix} b \\ d \end{pmatrix} \\ (3 \ 4) \begin{pmatrix} a \\ c \end{pmatrix} & (3 \ 4) \begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$$

$$BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix} :$$

a, b, c, d

. AB = BA

. AB \neq BA

(Matrix Multiplication)

$$: \quad AB \quad B \in M_{m \times p}(K), \quad A \in M_{n \times m}(K)$$

$$C = AB = \begin{pmatrix} R_1(A)C_1(B) & \cdots & R_1(A)C_p(B) \\ \vdots & \ddots & \vdots \\ R_n(A)C_1(B) & \cdots & R_n(A)C_p(B) \end{pmatrix} \quad (2)$$

$$AB = (AC_1(B) \dots AC_p(B)) \quad B \in M_{m \times p}(K), \quad A \in M_{n \times m}(K)$$

:

$$AC_j(B) = \begin{pmatrix} R_1(A) \\ \vdots \\ R_n(A) \end{pmatrix} C_j(B) = \begin{pmatrix} R_1(A)C_j(B) \\ \vdots \\ R_n(A)C_j(B) \end{pmatrix}; 1 \leq j \leq p.$$

(1)

$$.AB = b_1C_1(A) + \cdots + b_mC_m(A) \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad B \in M_{m \times 1}(k) \quad -1$$

$$AB = a_1R_1(A) + \cdots + a_mR_m(A) \quad A = (a_1, \dots, a_n) \quad A \in M_{1 \times m}(k) \quad -2$$

:

-1

$$\begin{aligned} AB &= \begin{pmatrix} R_1(A) \\ \vdots \\ R_n(A) \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + \cdots + a_{1m}b_m \\ \vdots \\ a_{n1}b_1 + \cdots + a_{nm}b_m \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_1 \\ \vdots \\ a_{n1}b_1 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1m}b_m \\ \vdots \\ a_{nm}b_m \end{pmatrix} \\ &= b_1C_1(A) + \cdots + b_mC_m(A) \end{aligned}$$

-2

(3)

$$: \quad C \in M_{p \times q}(K) \quad B \in M_{m \times p}(K) \quad A \in M_{n \times m}(K)$$

$$. (\quad) A(BC) = (AB)C \quad -1$$

$$\forall \lambda \in k; \lambda(AB) = (\lambda A)B = A(\lambda B) \quad -2$$

$$. (\quad) \lambda(A+B) = \lambda A + \lambda B \quad -3$$

$$0_k A = 0_{n \times m}, \quad A 0_{m \times p} = 0_{n \times p} \quad -4$$

:

$$A(BC) = A(B(C_1(C), \dots, C_q(C))) \quad (2) \quad .C$$

$$: C_j(C) = \begin{pmatrix} c_{1j} \\ \vdots \\ c_{pj} \end{pmatrix} . A(BC) = (A(BC_1(C)), \dots, A(BC_q(C)))$$

$$: (1) . BC_j = c_{1j}C_1(B) + \dots + c_{pj}C_p(B), j = 1, \dots, q$$

$$\begin{aligned} A(BC_j) &= A(c_{1j}C_1(B) + \dots + c_{pj}C_p(B)) \\ &= c_{1j}AC_1(B) + \dots + c_{pj}AC_p(B) \\ &= (AC_1(B) + \dots + AC_p(B))C_j = (AB)C_j \end{aligned}$$

$$. A(BC) = (AB)C$$

(4)

$$B \in M_{m \times p}(K) \quad A \in M_{n \times m}(K)$$

$${}^t(AB) = {}^tB {}^tA \quad -1$$

$${}^t({}^tA) = A \quad -2$$

$$\forall (\lambda, \mu) \in K^2, {}^t(\lambda A + \mu B) = \lambda {}^tA + \mu {}^tB \quad -3$$

(Square Matrices)

.2

$$\begin{array}{l}
 \cdot K \\
 \cdot n \times n \\
 A \in M_n(K) \\
 \forall (i, j) \in \mathbb{N}_n^2, a_{ij} \in K \\
 \cdot \\
 \cdot M_n(K) \\
 A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \\
 \cdot \\
 \cdot (a_{ii})_{i \in \mathbb{N}_n}
 \end{array}$$

$$\begin{array}{l}
 : \\
 I_n \in M_n(K) \\
 I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}
 \end{array}$$

(5)

$$\begin{array}{l}
 \cdot M_n(K) \quad (+), (\times) \\
 \cdot (.) \\
 I_n \in M_n(K) \quad (
 \end{array}
 \quad (M_n(K), +, \times)$$

(6)

$$\begin{array}{l}
 (+), (\times) \quad K \\
 (.) \quad M_n(K)
 \end{array}
 \quad (M_n(K), +, \cdot, \times)$$

$$(.) : K \times E \rightarrow E; (\lambda, x) \rightarrow \lambda x = \lambda x$$

(Matrix Trace)

$$Tr(A) \quad A \quad . \quad A \in M_n(K) \quad :$$

$$.Tr(A) = a_{11} + \dots + a_{nn} = \sum_{i=1}^n a_{ii}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{pmatrix} \quad :$$

$$Tr(A) = 1 + 6 + 11 = 18$$

(7)

$$: \quad A, B \in M_n(K)$$

$$.Tr(A) = Tr(A^t) \quad -1$$

$$\forall \lambda \in K ; Tr(\lambda A) = \lambda Tr(A) \quad -2$$

$$Tr(A + B) = Tr(A) + Tr(B) \quad -3$$

$$.Tr(AB) = Tr(BA) \quad -4$$

:

3 1

$$C = AB$$

4

$$: \quad C_{ii} = \sum_{k=1}^n a_{ik} b_{ki} \quad ; \quad i = 1, \dots, n$$

$$Tr(C) = Tr(AB) = \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki}$$

$$= \sum_{k=1}^n \sum_{i=1}^n b_{ki} a_{ik} = \sum_{k=1}^n C'_{kk}$$

$$Tr(C) = \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} = Tr(C') \quad c'_{kk} = \sum_{i=1}^n b_{ki} a_{ik} \quad C' = BA$$

$$.Tr(AB) = Tr(BA)$$

$M_n(K)$ -3

$M_n(K)$ -1

$AB = BA$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$AB = 0 \not\Rightarrow (A = 0) \vee (B = 0)$ $M_n(K)$ -2

$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 2 & 5 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(+) $M_n(K)$ -3

(x)

$\therefore p \in \mathbb{N} \quad AB = BA \quad \cdot A, B \in M_n(K) \quad \therefore$

$(A+B)^p = \sum_{k=0}^p C_p^k A^k B^{p-k}$

$A^p - B^p = (A-B)(A^{p-1} + A^{p-2}B + \dots + A^{p-1})$

$(I_n - A^p) = (I_n - A)(I_n + A + A^2 + \dots + A^{p-1})$

$\cdot A^p = 0$

A

$\cdot A^p = 0 \quad p \in \mathbb{N}$

$\cdot n \in \mathbb{N}$

A^n

A^2, A^3

$\cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

:

$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} =^t A$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$$\dots \quad A_6 = A_5 \cdot A = {}^t A = I_3 \quad A^5 = A^4 \cdot A = A^2 = {}^t A \quad A^4 = I_3 \cdot A = A$$

$\vdots A^n$

$$A^n = A^{3p} = I_3 \quad ; p = 1, 2, \dots$$

$$A^n = A^{3p-1} = A^2 = {}^t A; \quad p = 1, 2, \dots$$

$$A^n = A^{3p-1} = A^3 = I_3; p = 1, 2, \dots$$

.3

(Diagonal Matrices)

-1

$$\vdots M_n(K)$$

$$D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix} = \text{Diag}(d_1, \dots, d_n); (d_1, \dots, d_n) \in K^n$$

(Scalar Matrices)

-2

$$\vdots M_n(K)$$

$$M = \lambda I_n = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda \end{pmatrix} ; \lambda \in K$$

(8)

$$\cdot M_n(K)$$

$$\cdot (K, +, \times)$$

:

$$\text{Diag}(d_1, \dots, d_n) \text{Diag}(d_1^2, \dots, d_n^2) = \text{Diag}(d_1^3, \dots, d_n^3)$$

$$\cdot (\text{Diag}(d_1, \dots, d_n))^2 = \text{Diag}(d_1^2, \dots, d_n^2)$$

:(Upper Triangular matrix) -i

$$\forall (i, j) \in N_n^2; u_{ij} = 0; i > j \Leftrightarrow U = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}$$

:(Lower Triangular Matrix) -ii

$$L = \begin{pmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix} \quad \forall (i, j) \in N_n^2; l_{ij} = 0; i < j$$

(Symmetric, Skew-symmetric matrices) -4

$$A = {}^t A \quad A \in M_n(K)$$

. $S_n(K)$

$$\cdot A = -{}^t A \quad A \in M_n(K)$$

. $A_n(k)$

:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{pmatrix}$$

$$\forall (i, j) \in N_n^2; a_{ij} = a_{ji} :$$

$${}^t A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} = A \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

(9)

$$\begin{array}{ccc} M_n(K) & S_n(k) & \\ M_n(K) & A_n(K) & \cdot \frac{n(n+1)}{2} \\ & M_n(K) = S_n(K) \oplus A_n(K) & \frac{n(n-1)}{2} \\ & & : \end{array}$$

$$M_n(K) \qquad S_n(K), A_n(K)$$

$$\begin{array}{ccc} \cdot M_n(K) & & (E_{ij})_{(ij) \in N_n^2} \\ \forall (i, j) \in T_n; S_{ij} = \begin{cases} E_{ii}, & i = j \\ E_{ij} + E_{ji} & : i \neq j \end{cases} & T_n = \{(i, j) \in N_n^2; i \leq j\} & S = (S_{ij})_{(ij) \in T_n} \end{array}$$

$$\cdot \dim(S_n(K)) = \text{card}(T_n) = \frac{n(n+1)}{2} \qquad S_n(K) \qquad S$$

$$T'_n = \{(i, j) \in N_n^2; i < j\} \qquad S' = (S'_{i,j}); (i, j) \in T'_n$$

$$A_n(K) \qquad S'_{ij} = E_{ij} - E_{ji} \quad ; 1 \leq i < j \leq n$$

$$M \in S_n(K) \cap A_n(K) \qquad \cdot \dim A_n(K) = \text{Card}(T'_n) = \frac{n(n-1)}{2}$$

$$M = 0_n \qquad M = {}^t M = -{}^t M$$

$$M = \frac{1}{2}(M + {}^t M) + \frac{1}{2}(M - {}^t M) \quad \forall M \in M_n(k);$$

$$\frac{1}{2}(M + {}^t M) \in S_n(K) \qquad \frac{1}{2}(M - {}^t M) \in A_n(K)$$

$$\cdot M_n(K) = S_n(K) \oplus A_n(K)$$

(Invertible Matrix)

-5

$$GL_n(k) \quad A \in M_n(K) \quad AB = BA = I_n \quad B \in M_n(K)$$

(10)

$\cdot I_n$

$(GL_n(k), \times)$

:

$\cdot GL_n(k)$

$$\cdot B = 0_n \quad A \quad \cdot AB = 0_n \quad A, B \in M_n(K)$$

:

$$AA' = A'A = I_n \quad A' \in M_n(K)$$

A

.

$$I_n B = 0_n \quad A'(AB) = A'0_n$$

$$\cdot B = 0_n$$

(11)

:

$$\cdot A \in M_n(K)$$

$$A \in GL_n(k) \text{ -1}$$

$$\exists B \in M_n(K); BA = I_n : \quad A \text{ -2}$$

$$\exists B \in M_n(K); AB = I_n : \quad A \text{ -3}$$

$$\forall X \in M_{n \times 1}(K); AX = 0_{n1} \Rightarrow X = 0_{n1} \text{ -4}$$

$$rg(A) = n \text{ -5}$$

(The matrix of linear map)

.4

$e = (e_1, \dots, e_n) \quad . \quad K$
 E
 $e \quad x \quad x = x_1 e_1 + \dots + x_n e_n \quad x \in E \quad . \quad E$
 $n \times 1$

$$Mat_e(x) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M_{n \times 1}(K)$$

$E \quad p \quad S = (x_1, \dots, x_p)$
 $e \quad S \quad \forall j = 1, \dots, p; x_j = \sum_{i=1}^n x_{ij} e_i$
 $:$
 $n \times p$

$$Mat_e(S) = \begin{pmatrix} x_{11} & \dots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{np} \end{pmatrix} \in M_{n \times p}(K)$$

$. \dim F = n, \dim E = P \quad K \quad F, E$
 $u \in L(E, F) \quad . \quad F \quad f = (f_1, \dots, f_n) \quad E \quad e = (e_1, \dots, e_p)$
 $f, u \quad u \quad .$

$$Mat_{e,f}(u) = \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix} \in M_{n \times p}(k)$$

$Mat_{e,f}(u) \quad \forall j \in N_n; u(e_j) = \sum_{i=1}^n a_{ij} f_i$
 $u(e) = (u(e_1), \dots, u(e_p))$

(12)

$$\begin{aligned}
 F \quad e = (e_1, \dots, e_p) \quad \dim E = p \quad K & \quad E \\
 f = (f_1, \dots, f_n) \quad \dim F = n \quad K & \\
 \phi_{e,f} : L(E, F) \rightarrow M_{n \times p}(K); u \rightarrow \text{Mat}_{e,f}(u) &
 \end{aligned}$$

$$U : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$\begin{aligned}
 e = \{(1,1,0), (0,1,1), (1,0,1)\} \quad \forall (x, y, z) \in \mathbb{R}^3; \quad u(x, y, z) = (x - y, z) \\
 : \quad \mathbb{R}^2 \quad \mathbb{R}^3 \quad f = \{(1,2), (2,1)\}
 \end{aligned}$$

$$u(e_1) = u(1,1,0) = (0,0) = 0\beta_1 - 0\beta_2$$

$$u(e_2) = u(0,1,1) = (-1,1) = \beta_1 - \beta_2$$

$$u(e_3) = u(1,0,1) = (1,1) = \frac{1}{3}\beta_1 - \frac{1}{3}\beta_2$$

$$\text{Mat}_{e,f}(u) = \begin{pmatrix} 0 & 1 & \frac{1}{3} \\ 0 & -1 & \frac{1}{3} \end{pmatrix}$$

(Rank of Matrix) .5

$$\begin{aligned}
 C_1(A), \dots, C_p(A) \quad A \in M_{n \times p}(K) \\
 C = (C_1(A), \dots, C_p(A)) \quad \text{rg}(A) \quad A \\
 \text{rg}(A) = \dim \text{vect}(C)
 \end{aligned}$$

$$C_i \rightleftharpoons C_j; (i, j) \in \mathbb{N}_n^2 \quad -1$$

$$C_i \quad C_j \quad -2$$

$$C_i \leftarrow C_i + \lambda C_j; (i, j) \in \mathbb{N}_n^2, \lambda \in K$$

$$(i, j) \in \mathbb{N}_n^2, \lambda \in K \quad C_i + \lambda C_j$$

: A :

$$A = \begin{matrix} & a_1 & a_2 & a_3 & a_4 & a_5 \\ \begin{bmatrix} 0 & 2 & 5 & -1 & 5 \\ 0 & 0 & 2 & 3 & 4 \\ 4 & 2 & -11 & 11 & 11 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix} \end{matrix}$$

$$\Rightarrow \begin{matrix} a_1, & a_2 - \frac{1}{2}a_1, & a_3 + a_1, & a_4 + a_2 - \frac{1}{2}a_1 + a_3, & a_5 + a_2 \frac{1}{2}a_1 + a_3 \\ \begin{bmatrix} 0 & 2 & 5 & +6 & 12 \\ 0 & 0 & 2 & 5 & 6 \\ 4 & 0 & -7 & 0 & 0 \\ 2 & -1 & -4 & 2 & 0 \end{bmatrix} \end{matrix}$$

:

$$b_1 = a_1 \quad b_2 = a_2 - \frac{1}{2}a_1, \quad b_3 = a_3 + a_1 \quad b_4 = a_4 + a_3 + a_2 - \frac{1}{2}a_1$$

$$b_5 = a_5 + a_3 + a_2 - \frac{1}{2}a_1$$

$$\begin{matrix} b_1 & b_2 & b_3 - b_2 + b_1 & b_4 + 2b_2 & b_5 \\ \begin{bmatrix} 0 & 2 & -3 & 10 & 12 \\ 0 & 0 & 2 & 5 & 6 \\ 4 & 0 & -3 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} b_1 & b_2 & b_3 - b_2 + b_1 & b_4 + 2b_2 & b_5 - \frac{6}{5}(b_4 + 2b_2) \\ \begin{bmatrix} 0 & 2 & -3 & 10 & 0 \\ 0 & 0 & 2 & 5 & 0 \\ 4 & 0 & -3 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$rg(A) = 4$:

(13)

$$\begin{array}{ccc}
K & F, E & u: E \rightarrow F \\
A = \text{Mat}_{e_f}(u) & f = (f_1, \dots, f_n) & e = (e_1, \dots, e_p) \\
& & \text{rg}(u) = \text{rg}(A) \\
& & \vdots
\end{array}$$

$$\text{rg}(u) = \dim \text{Im}(u) = \text{rg}(u(e_1), \dots, u(e_p))$$

$$\varphi_f: M_{n \times 1}(K) \rightarrow F; Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \rightarrow \varphi_f(Y) = \sum_{i=1}^n y_i f_i \quad F \quad f$$

$$\varphi_f(C_j(A)) = u(e_j) \quad ; \quad \forall j \in N_p$$

$$\begin{aligned}
\text{rg}(u) &= \text{rg}[\varphi_f(C_1(A)), \dots, \varphi_f(C_p(A))] \\
&= \text{rg}(C_1(A), \dots, C_p(A)) = \text{rg}(A) \quad \vdots
\end{aligned}$$

(2)

$$\text{rg}(A) = \text{rg}({}^t A) \quad A \in M_{n \times p}(K)$$

.6

$$\begin{array}{ccc}
E & e = (e_1, \dots, e_n) & K \\
f & e & . E \\
& & f = (f_1, \dots, f_n) \\
& & . P_e^f = \text{Mat}_e(f_1, \dots, f_n)
\end{array}$$

(14)

$$\begin{array}{l}
. e = (e_1, \dots, e_n) \quad . k \\
f = (f_1, \dots, f_n) \quad . M_n(K) \quad P = (P_{ij}) \\
. P = (P_{ij}) \quad E \quad f_j = \sum_{i=1}^n P_{ij} \cdot e_i \\
: \\
\forall j \in \mathbb{N}_n; \quad u(e_j) = f_j \quad L(E) \quad u \\
: \quad P = \text{Mat}_e(u) \\
P \Leftrightarrow (n = \text{rg}(P)) \Leftrightarrow (n = \text{rg}(u)) \Leftrightarrow E \quad f \Leftrightarrow E \quad f
\end{array}$$

(15)

$$\begin{array}{l}
. E \quad f, e \quad . x \in E \quad . K \quad E \\
. X = P_e^f X' : \quad X = \text{Mat}_e(x), X' = \text{Mat}_f(x) \\
: \\
\forall x \in E \quad ; \exists (\alpha_1 \dots \alpha_n) \in K^n; \quad x = \alpha_n e_n = \sum_{i=1}^n \alpha_i e_i \\
\forall x \in E \quad ; \exists (\beta_1, \dots, \beta_n) \in K^n; x = \beta_1 f_1 + \dots + \beta_n f_n = \sum_{j=1}^n \beta_j f_j \\
: \\
X = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}, \quad X' = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \\
x = \sum_{j=1}^n \beta_j f_j = \sum_{j=1}^n \beta_j \left(\sum_{i=1}^n P_{ij} e_i \right) = \sum_{i=1}^n \left(\sum_{j=1}^n P_{ij} \beta_j \right) e_i \quad \forall j \in \mathbb{N}_n; f_j = \sum_{i=1}^n p_{ij} e_i \\
\forall i \in \mathbb{N}_n, \alpha_i = \sum_{j=1}^n p_{ij} \beta_j \quad \sum_{i=1}^n \left(\sum_{j=1}^n P_{ij} \cdot \beta_j \right) \cdot e_i = \sum_{i=1}^n \alpha_i \cdot e_i \quad x = \sum_{i=1}^n \alpha_i e_i \\
. X = P_e^f X'
\end{array}$$

$$f_1 = (1,2), \quad f_2 = (1,3) \quad E = \mathbb{R}^2$$

$$. E = \mathbb{R}^2 \quad f = (f_1, f_2) \quad \text{-a}$$

$$. f \quad e = \{(1,0), (0,1)\} \quad \text{-b}$$

$$. f = (f_1, f_2) \quad x \quad . x = (4,1) \in \mathbb{R}^2 \quad \text{-C}$$

:

$$(\lambda, \mu) \in \mathbb{R}^2 \quad \forall x \in \mathbb{R}^2; \quad x = (x_1, x_2) \quad \text{-a}$$

$$\lambda + \mu = x_1, \quad 2\lambda + 3\mu = x_2 \quad \lambda(1,2) + \mu(1,3) = (x_1, x_2) \quad \lambda f_1 + \mu f_2 = (x_1, x_2)$$

$$\Rightarrow \lambda = x_1 - \mu, \quad \lambda = \frac{1}{2}(x_2 - 3\mu) \Rightarrow 2x_1 - 2\mu = x_2 - 3\mu \Rightarrow \mu = x_2 - 2x_1 \in \mathbb{R}$$

$$\Rightarrow \lambda = x_1 - \mu = 3x_1 - x_2 \in \mathfrak{R}$$

$$\lambda + \mu = 0 \quad \forall (\lambda, \mu) \in \mathbb{R}^2; \quad \lambda f_1 + \mu f_2 = 0 \quad . \quad f$$

$$. E \quad f \quad \lambda = \mu = 0 \quad 2\lambda + 3\mu = 0$$

$$\lambda = 1, \mu = 2 \quad f_1 = \lambda e_1 + \mu e_2 \Rightarrow \lambda(1,0) + \mu(0,1) = (1,2) \quad (f_1, f_2) \in E^2 \quad \text{-b}$$

$$. f_2 = 1e + 3e_2 = e_1 + 3e_2$$

$$P_e^f = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} e_1 + 2e_2 \\ e_1 + 3e_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = P_e^f \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$${}^t f = {}^t P_e^f e \Rightarrow f = e P_e^f$$

$$x = (4,1) \in \mathbb{R}^2 \quad (4,1) = 4(1,0) + (0,1) = 4e_1 + 1e_2 = 4e_1 + e_2 \quad \text{-C}$$

$$X = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad X' = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \quad (15) \quad (4,1) = \lambda f_1 + \mu f_2$$

$$X = P X' \Rightarrow \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \Rightarrow \lambda + \mu = 4, \quad 2\lambda + 3\mu = 1$$

$$\lambda = 4 - (-7) = 11 \quad \mu = -7 \quad \lambda = 4 - \mu \Rightarrow 8 - 2\mu + 3\lambda = 1$$

$$X' = {}^t(11, -7) \quad f \quad x = (4, 1)$$

(Similar Matrix) .7

$$A, B \in M_n(K) \\ \exists P \in GL_n(K); B = PAP^{-1}$$

$$A \cong B$$

(16)

$$u: E \rightarrow F \\ A = Mat_e(u), \quad B = Mat_f(u)$$

(17)

$$A \cong B \implies Tr(A) = Tr(B)$$

$$\forall k \in \mathbb{N}^* \implies (A^k \cong B^k) \wedge (A^k = PB^kP^{-1})$$

$$A^{-1} \cong B^{-1}$$

:

$$\begin{aligned}
 \forall n \in \mathbb{N}^*; \quad A^n &= (A_1 + A_2)^n = \sum_{k=1}^n C_n^k A_1^k A_2^{n-k} \\
 &= C_n^0 A_1^0 A_2^n + C_n^1 A_1 A_2^{n-1} + C_n^2 A_1^2 A_2^{n-2} + \dots \\
 &= C_n^0 A_1^0 A_2^n + C_n^1 A_1 A_2^{n-1} + 0_2 + \dots + 0_2 \\
 &= A_2^n + n A_1 A_2^{n-1} = \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} + n \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^{n-1} & 0 \\ 0 & 0^{n-1} \end{pmatrix} \\
 &= \begin{pmatrix} a^n & 0 \\ 0 & a^n \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ nba^{n-1} & 0 \end{pmatrix} \\
 &= \begin{pmatrix} a^n & 0 \\ nba^{n-1} & a^n \end{pmatrix}
 \end{aligned}$$

$$A^2 = \begin{pmatrix} a^2 & 0 \\ 2ba & 0 \end{pmatrix} \quad n = 2$$

$$J = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ Jordan Matrix}$$

$$. n \in \mathbb{N}^* \quad J^n \quad J^2$$

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A \quad \lambda \in \mathbb{R} \quad A = I + \lambda E_{ik} \quad (i, j) \in \mathbb{N}_n^2$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \left(\quad \quad \right)$$

A, B