Workshop Solutions to Sections 3.1 and 3.2

1) $$\lim_{x\to-2}(x^3-2x+1)=(-2)^3-2(-2)+1=-8+4+1=-3$$

2) $$\lim_{x\to2}(3x^2+x-4)=3(2)^2+(2)-4=12+2-4=10$$

3) $$\lim_{x\to1}(x^2+3x-5)^3=((1)^2+3(1)-5)^3=(1+3-5)^3=(-1)^3=-1$$

4) $$\lim_{x\to-2}(2x^3+3x^2+5)=2(-2)^3+3(-2)^2+5=2(-8)+3(4)+5=-16+12+5=1$$

5) $$\lim_{x\to-2}\frac{x^2-2}{x-2}=\frac{(-2)^2-2}{(-2)-2}=\frac{4-2}{-2-2}=\frac{2}{-4}=-\frac{1}{2}$$

6) $$\lim_{x\to2}\frac{x^3+5}{x^2+1}=\frac{(2)^3+5}{(2)^2+1}=\frac{8+5}{4+1}=\frac{13}{5}$$

7) $$\lim_{x\to0}\frac{x^2+3x+5}{x^2-3}=\frac{(0)^2+3(0)+5}{(0)^2-3}=\frac{0+0+5}{0-3}=\frac{5}{-3}=-\frac{5}{3}$$

8) $$\lim_{x\to1}\frac{x-1}{x^2+x-5}=\frac{(1)-1}{(1)^2+(1)-5}=\frac{1-1}{1+1-5}=\frac{0}{-3}=0$$

9) $$\lim_{x\to-1}\sqrt{x^3-10x+7}=\sqrt{(-1)^3-10(-1)+7}=\sqrt{-1+10+7}=\sqrt{16}=4$$

10) $$\lim_{x\to-1}\frac{1-(x+4)^{-2}}{x-2}=\frac{1-\left((-1)+4\right)^{-2}}{(-1)-2}=\frac{1-(-1+4)^{-2}}{-1-2}=\frac{1-(3)^{-2}}{-3}=\frac{1-\frac{1}{3^2}}{-3}=\frac{1-\frac{1}{9}}{-3}=\frac{\frac{8}{9}}{-3}=\frac{8}{9}\times\frac{1}{-3}=\frac{8}{-27}=-\frac{8}{27}$$

11) $$\lim_{x\to-1}\frac{x^3+2x}{8-2x}=\frac{(-1)^3+2(-1)}{8-2(-1)}=\frac{-1-2}{8+2}=\frac{-3}{10}=-\frac{3}{10}$$

12) $$\lim_{x\to4}\frac{x^2-3x}{5+x}=\frac{(4)^2-3(4)}{5+(4)}=\frac{16-12}{5+4}=\frac{4}{9}$$

13) $$\lim_{x\to4}\frac{x^2-4x}{5+x}=\frac{(4)^2-4(4)}{5+(4)}=\frac{16-16}{5+4}=\frac{0}{9}=0$$

15) $$\lim_{x\to0}\frac{x^3-5x^2}{x^2}=\lim_{x\to0}\frac{x^2(x-5)}{x^2}=\lim_{x\to0}(x-5)=(0)-5=-5$$

14) $$\lim_{x\to4}\frac{3^{-1}-(2x-5)^{-1}}{4-x}=\lim_{x\to4}\frac{\frac{1}{3}-\frac{1}{2x-5}}{4-x}=\lim_{x\to4}\frac{\frac{2x-5-3}{3(2x-5)}}{4-x}=\lim_{x\to4}\frac{2x-8}{3(2x-5)(4-x)}=\lim_{x\to4}\frac{2(x-4)}{3(2x-5)(4-x)}=\lim_{x\to4}\frac{-2(4-x)}{3(2x-5)(4-x)}=\lim_{x\to4}\frac{-2}{3(2x-5)}=\frac{-2}{3(2(4)-5)}=\frac{-2}{3(8-5)}=\frac{-2}{9}=-\frac{2}{9}$$

16) $$\lim_{x\to6}\frac{x-6}{x^2-36}=\lim_{x\to6}\frac{x-6}{(x-6)(x+6)}=\lim_{x\to6}\frac{1}{x+6}=\frac{1}{(6)+6}=\frac{1}{12}$$

17) $$\lim_{x\to6}\frac{x^2-36}{x-6}=\lim_{x\to6}\frac{(x-6)(x+6)}{x-6}=\lim_{x\to6}(x+6)=(6)+6=12$$

18) $$\lim_{x\to-6}\frac{x+6}{x^2-36}=\lim_{x\to-6}\frac{x+6}{(x-6)(x+6)}=\lim_{x\to-6}\frac{1}{x-6}=\frac{1}{(-6)-6}=\frac{1}{-12}=-\frac{1}{12}$$

19) $$\lim_{x\to3}\frac{x^3-27}{x-3}=\lim_{x\to3}\frac{(x-3)(x^2+3x+9)}{x-3}=\lim_{x\to3}(x^2+3x+9)=(3)^2+3(3)+9=9+9+9=27$$

20) $$\lim_{x\to3}\frac{x-3}{x^3-27}=\lim_{x\to3}\frac{x-3}{(x-3)(x^2+3x+9)}=\lim_{x\to3}\frac{1}{x^2+3x+9}=\frac{1}{(3)^2+3(3)+9}=\frac{1}{9+9+9}=\frac{1}{27}$$

21) $\lim_{x\to -2}\frac{x+2}{x^3+8}=\lim_{x\to -2}\frac{x+2}{(x+2)(x^2-2x+4)}$
$=\lim_{x\to -2}\frac{1}{x^2-2x+4}$
$=\frac{1}{(-2)^2-2(-2)+4}=\frac{1}{4+4+4}=\frac{1}{12}$

22) $\lim_{x\to -2}\frac{x^3+8}{x+2}=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{x+2}$
$=\lim_{x\to -2}(x^2-2x+4)=(-2)^2-2(-2)+4$
$=4+4+4=12$

23) $\lim_{x\to 4}\frac{x^2-3x-4}{x-4}=\lim_{x\to 4}\frac{(x-4)(x+1)}{x-4}=\lim_{x\to 4}(x+1)$
$=(4)+1=5$

24) $\lim_{x\to 3}\frac{x^2+4x-21}{x^2-8x+15}=\lim_{x\to 3}\frac{(x+7)(x-3)}{(x-5)(x-3)}=\lim_{x\to 3}\frac{x+7}{x-5}$
$=\frac{(3)+7}{(3)-5}=\frac{10}{-2}=-5$

25) $\lim_{x\to 0}\frac{x}{1-(1-x)^2}=\lim_{x\to 0}\frac{x}{1-(1-2x+x^2)}$
$=\lim_{x\to 0}\frac{x}{1-1+2x-x^2}$
$=\lim_{x\to 0}\frac{x}{2x-x^2}=\lim_{x\to 0}\frac{x}{x(2-x)}$
$=\lim_{x\to 0}\frac{1}{2-x}=\frac{1}{2-(0)}=\frac{1}{2}$

26) $\lim_{x\to 2}\frac{\sqrt[3]{x+6}-2}{x-2}=\lim_{x\to 2}\frac{\sqrt[3]{x+6}-2}{(x+6)-8}=\lim_{x\to 2}\frac{\sqrt[3]{x+6}-2}{\left(\sqrt[3]{x+6}\right)^3-8}$
$=\lim_{x\to 2}\frac{\sqrt[3]{x+6}-2}{\left(\sqrt[3]{x+6}-2\right)\left(\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4\right)}$
$=\lim_{x\to 2}\frac{1}{\left(\sqrt[3]{x+6}\right)^2+2\sqrt[3]{x+6}+4}$
$=\frac{1}{\left(\sqrt[3]{(2)+6}\right)^2+2\sqrt[3]{(2)+6}+4}=\frac{1}{4+4+4}=\frac{1}{12}$

27) $\lim_{x\to 0}\frac{\sqrt{x+25}-5}{x}$
$=\lim_{x\to 0}\left[\frac{\sqrt{x+25}-5}{x}\times\frac{\sqrt{x+25}+5}{\sqrt{x+25}+5}\right]$
$=\lim_{x\to 0}\frac{(x+25)-25}{x\left(\sqrt{x+25}+5\right)}$
$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+25}+5\right)}$
$=\lim_{x\to 0}\frac{1}{\sqrt{x+25}+5}=\frac{1}{\sqrt{(0)+25}+5}$
$=\frac{1}{5+5}=\frac{1}{10}$

28) $\lim_{x\to 0}\frac{x}{\sqrt{x+25}-5}=\lim_{x\to 0}\left[\frac{x}{\sqrt{x+25}-5}\times\frac{\sqrt{x+25}+5}{\sqrt{x+25}+5}\right]$
$=\lim_{x\to 0}\frac{x\left(\sqrt{x+25}+5\right)}{(x+25)-25}$
$=\lim_{x\to 0}\frac{x\left(\sqrt{x+25}+5\right)}{x}$
$=\lim_{x\to 0}\left(\sqrt{x+25}+5\right)=\sqrt{(0)+25}+5$
$=5+5=10$

29) $\lim_{x\to 2}\frac{x-2}{2-\sqrt{6-x}}=\lim_{x\to 2}\left[\frac{x-2}{2-\sqrt{6-x}}\times\frac{2+\sqrt{6-x}}{2+\sqrt{6-x}}\right]$
$=\lim_{x\to 2}\frac{(x-2)\left(2+\sqrt{6-x}\right)}{4-(6-x)}$
$=\lim_{x\to 2}\frac{(x-2)\left(2+\sqrt{6-x}\right)}{4-6+x}$
$=\lim_{x\to 2}\frac{(x-2)\left(2+\sqrt{6-x}\right)}{x-2}$
$=\lim_{x\to 2}\left(2+\sqrt{6-x}\right)=2+\sqrt{6-(2)}$
$=2+2=4$

30) $\lim_{x\to 2}\frac{2-\sqrt{6-x}}{x+2}=\frac{2-\sqrt{6-(2)}}{(2)+2}=\frac{2-2}{4}=0$

31) $\lim_{x\to 3}\frac{1-\sqrt{x-2}}{2-\sqrt{x+1}}$
$=\lim_{x\to 3}\left[\frac{1-\sqrt{x-2}}{2-\sqrt{x+1}}\times\frac{1+\sqrt{x-2}}{1+\sqrt{x-2}}\right.$
$\left.\times\frac{2+\sqrt{x+1}}{2+\sqrt{x+1}}\right]$
$=\lim_{x\to 3}\left[\frac{1-(x-2)}{4-(x+1)}\times\frac{2+\sqrt{x+1}}{1+\sqrt{x-2}}\right]$
$=\lim_{x\to 3}\left[\frac{3-x}{3-x}\times\frac{2+\sqrt{x+1}}{1+\sqrt{x-2}}\right]$
$=\lim_{x\to 3}\frac{2+\sqrt{x+1}}{1+\sqrt{x-2}}=\frac{2+\sqrt{(3)+1}}{1+\sqrt{(3)-2}}=\frac{2+2}{1+1}$
$=\frac{4}{2}=2$

32) If $2x \le f(x) \le 3x^2 - 8$, then

$$\lim_{x\to 2} f(x) =$$

Solution:

$$\lim_{x\to 2} 2x = 2(2) = 4$$

and

$$\lim_{x\to 2}(3x^2 - 8) = 3(2)^2 - 8 = 12 - 8 = 4$$

It follows from the Sandwich Theorem that

$$\lim_{x\to 2} f(x) = 4$$

33) $\lim_{x\to 0}\left[x\cos\left(x + \frac{1}{x}\right)\right] =$

We know that the cosine of any angle is between -1 and 1. So,

$$-1 \le \cos\left(x + \frac{1}{x}\right) \le 1$$

Now, multiply throughout by x, we get

$$-x \le x\cos\left(x + \frac{1}{x}\right) \le x$$

But $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(-x) = 0$.
It follows from the Sandwich Theorem that

$$\lim_{x\to 0}\left[x\cos\left(x + \frac{1}{x}\right)\right] = 0$$

34) $\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right] =$

We know that the sine of any angle is between -1 and 1. So,

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

Now, multiply throughout by x, we get

$$-x \le x\sin\left(\frac{1}{x}\right) \le x$$

But $\lim_{x\to 0} x = 0$ and $\lim_{x\to 0}(-x) = 0$.
It follows from the Sandwich Theorem that

$$\lim_{x\to 0}\left[x\sin\left(\frac{1}{x}\right)\right] = 0$$

35) If $\frac{x^2+1}{x-1} \le f(x) \le x - 1$, then

$$\lim_{x\to 0} f(x) =$$

Solution:

$$\lim_{x\to 0}\frac{x^2+1}{x-1} = \frac{(0)^2+1}{(0)-1} = \frac{1}{-1} = -1$$

and

$$\lim_{x\to 0}(x - 1) = (0) - 1 = -1$$

It follows from the Sandwich Theorem that

$$\lim_{x\to 0} f(x) = -1$$

36) If $4(x-1) \le f(x) \le x^3 + x - 2$, then

$$\lim_{x\to 1} f(x) =$$

Solution:

$$\lim_{x\to 1}\big(4(x-1)\big) = 4\big((1)-1\big) = 4\times 0 = 0$$

and

$$\lim_{x\to 1}(x^3 + x - 2) = (1)^3 + (1) - 2 = 1 + 1 - 2 = 0$$

It follows from the Sandwich Theorem that

$$\lim_{x\to 1} f(x) = 0$$

37) If

$$\lim_{x\to 3}\frac{f(x)+4}{x-1} = 3,$$

then

$$\lim_{x\to 3} f(x) =$$

Solution:

$$\lim_{x\to 3}\frac{f(x)+4}{x-1} = \frac{\lim_{x\to 3}(f(x)+4)}{\lim_{x\to 3}(x-1)} = \frac{\lim_{x\to 3} f(x) + \lim_{x\to 3}(4)}{\lim_{x\to 3}(x) - \lim_{x\to 3}(1)}$$

$$= \frac{\lim_{x\to 3} f(x) + 4}{3-1} = \frac{\lim_{x\to 3} f(x) + 4}{2}$$

Now

$$\frac{\lim_{x\to 3} f(x) + 4}{2} = 3$$

$$\lim_{x\to 3} f(x) + 4 = 6 \quad\Leftrightarrow\quad \lim_{x\to 3} f(x) = 2$$

38) $\lim_{x\to2}\dfrac{2^{-1}-(3x-4)^{-1}}{2-x}$

$$=\lim_{x\to2}\frac{\frac{1}{2}-\frac{1}{3x-4}}{2-x}$$
$$=\lim_{x\to2}\frac{\frac{3x-4-2}{2(3x-4)}}{2-x}$$
$$=\lim_{x\to2}\frac{\frac{3x-6}{2(3x-4)}}{2-x}$$
$$=\lim_{x\to2}\frac{\frac{3(x-2)}{2(3x-4)}}{2-x}$$
$$=\lim_{x\to2}\frac{3(x-2)}{2(3x-4)(2-x)}$$
$$=\lim_{x\to2}\frac{-3(2-x)}{2(3x-4)(2-x)}=\lim_{x\to2}\frac{-3}{2(3x-4)}$$
$$=\frac{-3}{2(3(2)-4)}=\frac{-3}{2\times2}=-\frac{3}{4}$$

39) $\lim_{x\to0}\dfrac{(x+1)^3-1}{x}=\lim_{x\to0}\dfrac{(x^3+3x^2+3x+1)-1}{x}$

$$=\lim_{x\to0}\frac{x^3+3x^2+3x}{x}$$
$$=\lim_{x\to0}\frac{x(x^2+3x+3)}{x}=\lim_{x\to0}(x^2+3x+3)$$
$$=(0)^2+3(0)+3=3$$

40) If

$$\lim_{x\to1}\frac{f(x)+3x}{x^2-5f(x)}=1,$$

then

$$\lim_{x\to1}f(x)=$$

<u>Solution:</u>

$$\lim_{x\to1}\frac{f(x)+3x}{x^2-5f(x)}=\frac{\lim_{x\to1}(f(x)+3x)}{\lim_{x\to1}(x^2-5f(x))}$$
$$=\frac{\lim_{x\to1}f(x)+\lim_{x\to1}(3x)}{\lim_{x\to1}(x^2)-\lim_{x\to1}(5f(x))}$$
$$=\frac{\lim_{x\to1}f(x)+3(1)}{(1)^2-5\lim_{x\to1}f(x)}=\frac{\lim_{x\to1}f(x)+3}{1-5\lim_{x\to1}f(x)}$$

Now

$$\frac{\lim_{x\to1}f(x)+3}{1-5\lim_{x\to1}f(x)}=1$$

$$\lim_{x\to1}f(x)+3=(1)\left(1-5\lim_{x\to1}f(x)\right)$$
$$\Leftrightarrow\quad\lim_{x\to1}f(x)+3=1-5\lim_{x\to1}f(x)$$
$$\Leftrightarrow\quad\lim_{x\to1}f(x)+5\lim_{x\to1}f(x)=1-3$$
$$\Leftrightarrow\quad 6\lim_{x\to1}f(x)=-2$$
$$\Leftrightarrow\quad\lim_{x\to1}f(x)=\frac{-2}{6}=-\frac{1}{3}$$

41) $\lim_{x\to4}\dfrac{x^2-6x+8}{x^2+x-20}$

$$=\lim_{x\to4}\frac{(x-2)(x-4)}{(x-4)(x+5)}$$
$$=\lim_{x\to4}\frac{x-2}{x+5}=\frac{(4)-2}{(4)+5}=\frac{2}{9}$$

42) $\lim_{x\to-2}\dfrac{x^3+8}{x^2-x-6}$

$$=\lim_{x\to-2}\frac{(x+2)(x^2-2x+4)}{(x-3)(x+2)}$$
$$=\lim_{x\to-2}\frac{x^2-2x+4}{x-3}=\frac{(-2)^2-2(-2)+4}{(-2)-3}$$
$$=\frac{4+4+4}{-5}=\frac{12}{-5}=-\frac{12}{5}$$

43) $\lim_{x\to1}\left[\dfrac{x^2-2}{x+4}+x^2-2x\right]=\dfrac{(1)^2-2}{(1)+4}+(1)^2-2(1)$

$$=\frac{1-2}{1+4}+1-2=\frac{-1}{5}-1=\frac{-1-5}{5}=-\frac{6}{5}$$

44) $$\lim_{x\to-2}\frac{4x^2+6x-4}{2x^2-8}$$
$$=\lim_{x\to-2}\frac{2(2x^2+3x-2)}{2(x^2-4)}$$
$$=\lim_{x\to-2}\frac{2x^2+3x-2}{x^2-4}$$
$$=\lim_{x\to-2}\frac{(2x-1)(x+2)}{(x-2)(x+2)}$$
$$=\lim_{x\to-2}\frac{2x-1}{x-2}=\frac{2(-2)-1}{(-2)-2}=\frac{-4-1}{-2-2}$$
$$=\frac{-5}{-4}=\frac{5}{4}$$

45) $$\lim_{x\to-1}\frac{x^2-2x-3}{x^5-x^3}$$
$$=\lim_{x\to-1}\frac{(x-3)(x+1)}{x^3(x^2-1)}$$
$$=\lim_{x\to-1}\frac{(x-3)(x+1)}{x^3(x-1)(x+1)}$$
$$=\lim_{x\to-1}\frac{x-3}{x^3(x-1)}=\frac{(-1)-3}{(-1)^3\big((-1)-1\big)}$$
$$=\frac{-1-3}{(-1)(-2)}=\frac{-4}{2}=-2$$

46) $$\lim_{x\to3}\frac{\sqrt{2x+1}(x^2-9)}{(2x+3)(x-3)}$$
$$=\lim_{x\to3}\frac{\sqrt{2x+1}(x-3)(x+3)}{(2x+3)(x-3)}$$
$$=\lim_{x\to3}\frac{\sqrt{2x+1}(x+3)}{2x+3}=\frac{\sqrt{2(3)+1}\big((3)+3\big)}{2(3)+3}$$
$$=\frac{6\sqrt{7}}{9}=\frac{2\sqrt{7}}{3}$$

47) $$\lim_{x\to1}\frac{\sqrt{3-2x}-1}{x-1}=\lim_{x\to1}\left[\frac{\sqrt{3-2x}-1}{x-1}\times\frac{\sqrt{3-2x}+1}{\sqrt{3-2x}+1}\right]$$
$$=\lim_{x\to1}\frac{(3-2x)-1}{(x-1)\big(\sqrt{3-2x}+1\big)}$$
$$=\lim_{x\to1}\frac{2-2x}{(x-1)\big(\sqrt{3-2x}+1\big)}$$
$$=\lim_{x\to1}\frac{2(1-x)}{(x-1)\big(\sqrt{3-2x}+1\big)}=$$
$$=\lim_{x\to1}\frac{-2(x-1)}{(x-1)\big(\sqrt{3-2x}+1\big)}=$$
$$=\lim_{x\to1}\frac{-2}{\sqrt{3-2x}+1}=\frac{-2}{\sqrt{3-2(1)}+1}$$
$$=\frac{-2}{\sqrt{3-2}+1}=\frac{-2}{2}=-1$$

48) $$\lim_{x\to0}\frac{(x+1)^2-1}{x}=\lim_{x\to0}\frac{(x^2+2x+1)-1}{x}$$
$$=\lim_{x\to0}\frac{x^2+2x}{x}=\lim_{x\to0}\frac{x(x+2)}{x}$$
$$=\lim_{x\to0}(x+2)=(0)+2=2$$

49) $$\lim_{x\to1}\frac{\sqrt{2x+2}-2}{\sqrt{3x-2}-1}$$
$$=\lim_{x\to1}\left[\frac{\sqrt{2x+2}-2}{\sqrt{3x-2}-1}\times\frac{\sqrt{2x+2}+2}{\sqrt{2x+2}+2}\times\frac{\sqrt{3x-2}+1}{\sqrt{3x-2}+1}\right]$$
$$=\lim_{x\to1}\left[\frac{(2x+2)-4}{(3x-2)-1}\times\frac{\sqrt{3x-2}+1}{\sqrt{2x+2}+2}\right]$$
$$=\lim_{x\to1}\left[\frac{2x-2}{3x-3}\times\frac{\sqrt{3x-2}+1}{\sqrt{2x+2}+2}\right]$$
$$=\lim_{x\to1}\left[\frac{2(x-1)}{3(x-1)}\times\frac{\sqrt{3x-2}+1}{\sqrt{2x+2}+2}\right]$$
$$=\lim_{x\to1}\left[\frac{2}{3}\times\frac{\sqrt{3x-2}+1}{\sqrt{2x+2}+2}\right]=\frac{2}{3}\times\frac{\sqrt{3(1)-2}+1}{\sqrt{2(1)+2}+2}$$
$$=\frac{2}{3}\times\frac{\sqrt{1}+1}{\sqrt{4}+2}=\frac{2}{3}\times\frac{2}{4}=\frac{1}{3}$$

50) $$\lim_{x\to 2}\frac{3-\sqrt{2x+5}}{x-2}$$
$$=\lim_{x\to 2}\left[\frac{3-\sqrt{2x+5}}{x-2}\times\frac{3+\sqrt{2x+5}}{3+\sqrt{2x+5}}\right]$$
$$=\lim_{x\to 2}\frac{9-(2x+5)}{(x-2)\left(3+\sqrt{2x+5}\right)}$$
$$=\lim_{x\to 2}\frac{4-2x}{(x-2)\left(3+\sqrt{2x+5}\right)}$$
$$=\lim_{x\to 2}\frac{2(2-x)}{(x-2)\left(3+\sqrt{2x+5}\right)}$$
$$=\lim_{x\to 2}\frac{-2(x-2)}{(x-2)\left(3+\sqrt{2x+5}\right)}$$
$$=\lim_{x\to 2}\frac{-2}{3+\sqrt{2x+5}}=\frac{-2}{3+\sqrt{2(2)+5}}$$
$$=\frac{-2}{3+\sqrt{9}}=\frac{-2}{6}=-\frac{1}{3}$$

51) $$\lim_{x\to -1}\frac{x^2+3x+2}{x^2+1}=\frac{(-1)^2+3(-1)+2}{(-1)^2+1}=\frac{1-3+2}{1+1}$$
$$=\frac{0}{2}=0$$

52) If
$$\lim_{x\to k}f(x)=-\frac{1}{2}$$
and
$$\lim_{x\to k}g(x)=\frac{2}{3}$$
Then
$$\lim_{x\to k}\frac{f(x)}{g(x)}=\frac{-\frac{1}{2}}{\frac{2}{3}}=-\frac{1}{2}\times\frac{3}{2}=-\frac{3}{4}$$

53) $$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\lim_{x\to 0}\left[\frac{\sqrt{x+4}-2}{x}\times\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right]$$
$$=\lim_{x\to 0}\frac{(x+4)-4}{x\left(\sqrt{x+4}+2\right)}$$
$$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+4}+2\right)}$$
$$=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}=\frac{1}{\sqrt{(0)+4}+2}$$
$$=\frac{1}{\sqrt{4}+2}=\frac{1}{4}$$

54) $$\lim_{x\to -1}\frac{x^2-5x-6}{x+1}=\lim_{x\to -1}\frac{(x-6)(x+1)}{x+1}=\lim_{x\to -1}(x-6)$$
$$=(-1)-6=-7$$

55) $$\lim_{x\to 0}\frac{(x+3)^{-1}-3^{-1}}{x}=\lim_{x\to 0}\frac{\frac{1}{x+3}-\frac{1}{3}}{x}=\lim_{x\to 0}\frac{\frac{3-(x+3)}{3(x+3)}}{x}$$
$$=\lim_{x\to 0}\frac{-x}{3x(x+3)}=\lim_{x\to 0}\frac{-1}{3(x+3)}$$
$$=\frac{-1}{3\left((0)+3\right)}=\frac{-1}{9}=-\frac{1}{9}$$

56) If
$$\lim_{x\to 1}f(x)=3$$
$$\lim_{x\to 1}g(x)=-4$$
and
$$\lim_{x\to 1}h(x)=-1$$
then
$$\lim_{x\to 1}\left[\frac{5f(x)}{2g(x)}+h(x)\right]=\frac{\lim_{x\to 1}5f(x)}{\lim_{x\to 1}2g(x)}+\lim_{x\to 1}h(x)$$
$$=\frac{5\lim_{x\to 1}f(x)}{2\lim_{x\to 1}g(x)}+\lim_{x\to 1}h(x)$$
$$=\frac{5(3)}{2(-4)}+(-1)=\frac{15}{-8}-1=-\frac{15}{8}-1$$
$$=\frac{-15-8}{8}=-\frac{23}{8}$$

57) If
$$\lim_{x\to 1}g(x)=-4$$
and
$$\lim_{x\to 1}h(x)=-1$$
then
$$\lim_{x\to 1}\sqrt{g(x)h(x)}=\sqrt{\left[\lim_{x\to 1}g(x)\right]\left[\lim_{x\to 1}h(x)\right]}=\sqrt{(-4)(-1)}$$
$$=\sqrt{4}=2$$

58) If
$$\lim_{x\to 1}f(x)=3$$
$$\lim_{x\to 1}g(x)=-4$$
and
$$\lim_{x\to 1}h(x)=-1$$
then
$$\lim_{x\to 1}[2f(x)g(x)h(x)]=2\left[\lim_{x\to 1}f(x)\right]\left[\lim_{x\to 1}g(x)\right]\left[\lim_{x\to 1}h(x)\right]$$
$$=2(3)(-4)(-1)=24$$

Workshop Solutions to Section 3.3

1) If $f(x) = \begin{cases} 2x+3; & x \geq -2 \\ 2x+5; & x < -2 \end{cases}$ then

$$\lim_{x \to (-2)^-} f(x) =$$

Solution:

$$\lim_{x \to (-2)^-} f(x) = \lim_{x \to (-2)^-} (2x+5) = 2(-2)+5 = -4+5 = 1$$

2) If $f(x) = \begin{cases} 2x+3; & x \geq -2 \\ 2x+5; & x < -2 \end{cases}$ then

$$\lim_{x \to (-2)^+} f(x) =$$

Solution:

$$\lim_{x \to (-2)^+} f(x) = \lim_{x \to (-2)^+} (2x+3) = 2(-2)+3 = -4+3 = -1$$

3) If $f(x) = \begin{cases} 2x+3; & x \geq -2 \\ 2x+5; & x < -2 \end{cases}$ then

$$\lim_{x \to -2} f(x) =$$

Solution:

$\lim_{x \to -2} f(x)$ does not exist because

$$\lim_{x \to (-2)^-} f(x) \neq \lim_{x \to (-2)^+} f(x)$$

4) If $f(x) = \begin{cases} x^2-2x+3; & x \geq 3 \\ x^3-3x-12; & x < 3 \end{cases}$ then

$$\lim_{x \to 3} f(x) =$$

Solution:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^3-3x-12) = (3)^3 - 3(3) - 12 = 27 - 9 - 12 = 6$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2-2x+3) = (3)^2 - 2(3) + 3 = 9 - 6 + 3 = 6$$

$$\therefore \lim_{x \to 3} f(x) = 6$$

5) If $f(x) = \begin{cases} x^2-7x; & x < 1 \\ 5; & 1 \leq x \leq 3 \\ 3x+1; & x > 3 \end{cases}$ then

$$\lim_{x \to 1^-} f(x) =$$

Solution:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2-7x) = (1)^2 - 7(1) = 1 - 7 = -6$$

6) If $f(x) = \begin{cases} x^2-7x; & x < 1 \\ 5; & 1 \leq x \leq 3 \\ 3x+1; & x > 3 \end{cases}$ then

$$\lim_{x \to 1^+} f(x) =$$

Solution:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5) = 5$$

7) If $f(x) = \begin{cases} x^2-7x; & x < 1 \\ 5; & 1 \leq x \leq 3 \\ 3x+1; & x > 3 \end{cases}$ then

$$\lim_{x \to 3^-} f(x) =$$

Solution:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (5) = 5$$

8) If $f(x) = \begin{cases} x^2-7x; & x < 1 \\ 5; & 1 \leq x \leq 3 \\ 3x+1; & x > 3 \end{cases}$ then

$$\lim_{x \to 3^+} f(x) =$$

Solution:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (3x+1) = 3(3)+1 = 9+1 = 10$$

9) If $f(x) = \begin{cases} \frac{x^2+x-6}{x^2-4}; & x^2-4 > 0 \\ \frac{x^2+x-6}{4-x^2}; & x^2-4 < 0 \end{cases}$ then

$$\lim_{x \to 2^+} f(x) =$$

Solution:

$$f(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4}; & x^2-4 > 0 \\ \dfrac{x^2+x-6}{4-x^2}; & x^2-4 < 0 \end{cases}$$

$$= \begin{cases} \dfrac{x^2+x-6}{x^2-4}; & x^2 > 4 \\ \dfrac{x^2+x-6}{-(x^2-4)}; & x^2 < 4 \end{cases}$$

$$= \begin{cases} \dfrac{(x+3)(x-2)}{(x-2)(x+2)}; & |x| > 4 \\ \dfrac{(x+3)(x-2)}{-(x-2)(x+2)}; & |x| < 4 \end{cases}$$

$$= \begin{cases} \dfrac{x+3}{x+2}; & x > 2 \text{ or } x < -2 \\ -\dfrac{x+3}{x+2}; & -2 < x < 2 \end{cases} \quad \text{then}$$

$$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\frac{x+3}{x+2}\right) = \frac{(2)+3}{(2)+2} = \frac{5}{4}$$

10) If $f(x) = \begin{cases} \frac{x^2+x-6}{x^2-4}; & x^2-4 > 0 \\ \frac{x^2+x-6}{4-x^2}; & x^2-4 < 0 \end{cases}$ then

$$\lim_{x \to 2^-} f(x) =$$

Solution:

$$f(x) = \begin{cases} \dfrac{x^2+x-6}{x^2-4}; & x^2-4 > 0 \\ \dfrac{x^2+x-6}{4-x^2}; & x^2-4 < 0 \end{cases}$$

$$= \begin{cases} \dfrac{x^2+x-6}{x^2-4}; & x^2 > 4 \\ \dfrac{x^2+x-6}{-(x^2-4)}; & x^2 < 4 \end{cases}$$

$$= \begin{cases} \dfrac{(x+3)(x-2)}{(x-2)(x+2)}; & |x| > 4 \\ \dfrac{(x+3)(x-2)}{-(x-2)(x+2)}; & |x| < 4 \end{cases}$$

$$= \begin{cases} \dfrac{x+3}{x+2}; & x > 2 \text{ or } x < -2 \\ -\dfrac{x+3}{x+2}; & -2 < x < 2 \end{cases} \quad \text{then}$$

$$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \left(-\frac{x+3}{x+2}\right) = -\frac{(2)+3}{(2)+2} = -\frac{5}{4}$$

11)

$$\lim_{x\to a^-}\frac{|x-a|}{x-a}=$$

Solution:

$$f(x)=\frac{|x-a|}{x-a}=\begin{cases}\frac{x-a}{x-a} & ;\ x-a>0\\ \frac{-(x-a)}{x-a}; & x-a<0\end{cases}=\begin{cases}1; & x>a\\ -1; & x<a\end{cases}$$

$$\therefore\quad \lim_{x\to a^-}\frac{|x-a|}{x-a}=\lim_{x\to a^-}\frac{-(x-a)}{x-a}=\lim_{x\to a^-}(-1)=-1$$

12)

$$\lim_{x\to a^+}\frac{|x-a|}{x-a}=$$

Solution:

$$f(x)=\frac{|x-a|}{x-a}=\begin{cases}\frac{x-a}{x-a} & ;\ x-a>0\\ \frac{-(x-a)}{x-a}; & x-a<0\end{cases}=\begin{cases}1; & x>a\\ -1; & x<a\end{cases}$$

$$\therefore\quad \lim_{x\to a^+}\frac{|x-a|}{x-a}=\lim_{x\to a^+}\frac{(x-a)}{x-a}=\lim_{x\to a^+}(1)=1$$

13)

$$\lim_{x\to a}\frac{|x-a|}{x-a}=$$

Solution:

$\lim_{x\to a}\frac{|x-a|}{x-a}$ does not exist because

$$\lim_{x\to a^-}\frac{|x-a|}{x-a}\neq\lim_{x\to a^+}\frac{|x-a|}{x-a}$$

It is clearly obvious from questions (11) and (12) above.

14)

$$\lim_{x\to a^+}\frac{|a-x|}{x-a}=$$

Solution:

$$f(x)=\frac{|a-x|}{x-a}=\begin{cases}\frac{a-x}{x-a} & ;\ a-x>0\\ \frac{-(a-x)}{x-a}; & a-x<0\end{cases}$$

$$=\begin{cases}\frac{-(x-a)}{x-a} & ;\ a>x\\ \frac{(x-a)}{x-a} & ;\ a<x\end{cases}=\begin{cases}-1; & x<a\\ 1; & x>a\end{cases}$$

$$\therefore\quad \lim_{x\to a^+}\frac{|a-x|}{x-a}=\lim_{x\to a^+}(1)=1$$

15)

$$\lim_{x\to a^-}\frac{|a-x|}{x-a}=$$

Solution:

$$f(x)=\frac{|a-x|}{x-a}=\begin{cases}\frac{a-x}{x-a} & ;\ a-x>0\\ \frac{-(a-x)}{x-a}; & a-x<0\end{cases}$$

$$=\begin{cases}\frac{-(x-a)}{x-a} & ;\ a>x\\ \frac{(x-a)}{x-a} & ;\ a<x\end{cases}=\begin{cases}-1; & x<a\\ 1; & x>a\end{cases}$$

$$\therefore\quad \lim_{x\to a^-}\frac{|a-x|}{x-a}=\lim_{x\to a^-}(-1)=-1$$

16)

$$\lim_{x\to a}\frac{|a-x|}{x-a}=$$

Solution:

$\lim_{x\to a}\frac{|a-x|}{x-a}$ does not exist because

$$\lim_{x\to a^-}\frac{|a-x|}{x-a}\neq\lim_{x\to a^+}\frac{|a-x|}{x-a}$$

It is clearly obvious from questions (14) and (15) above.

17)

$$\lim_{x\to(-a)^-}\frac{|x+a|}{x+a}=$$

Solution:

$$f(x)=\frac{|x+a|}{x+a}=\begin{cases}\frac{x+a}{x+a} & ;\ x+a>0\\ \frac{-(x+a)}{x+a}; & x+a<0\end{cases}=\begin{cases}1; & x>-a\\ -1; & x<-a\end{cases}$$

$$\therefore\quad \lim_{x\to(-a)^-}\frac{|x+a|}{x+a}=\lim_{x\to(-a)^-}(-1)=-1$$

18)

$$\lim_{x\to(-a)^+}\frac{|x+a|}{x+a}=$$

Solution:

$$f(x)=\frac{|x+a|}{x+a}=\begin{cases}\frac{x+a}{x+a} & ;\ x+a>0\\ \frac{-(x+a)}{x+a}; & x+a<0\end{cases}=\begin{cases}1; & x>-a\\ -1; & x<-a\end{cases}$$

$$\therefore\quad \lim_{x\to(-a)^+}\frac{|x+a|}{x+a}=\lim_{x\to(-a)^+}(1)=1$$

19)

$$\lim_{x\to -a}\frac{|x+a|}{x+a}=$$

Solution:

$\lim_{x\to -a}\frac{|x+a|}{x+a}$ does not exist because

$$\lim_{x\to(-a)^-}\frac{|x+a|}{x+a}\neq\lim_{x\to(-a)^+}\frac{|x+a|}{x+a}$$

It is clearly obvious from questions (17) and (18) above.

20)

$$\lim_{x\to 0^+}\frac{2x-|x|}{x^2+|x|}=$$

Solution:

$$f(x)=\frac{2x-|x|}{x^2+|x|}=\begin{cases}\dfrac{2x-(x)}{x^2+(x)} & ; \ x>0\\ \dfrac{2x-(-x)}{x^2+(-x)} & ; \ x<0\end{cases}$$

$$=\begin{cases}\dfrac{2x-x}{x^2+x} & ; \ x>0\\ \dfrac{2x+x}{x^2-x} & ; \ x<0\end{cases}=\begin{cases}\dfrac{x}{x^2+x} & ; \ x>0\\ \dfrac{3x}{x^2-x} & ; \ x<0\end{cases}$$

$$=\begin{cases}\dfrac{x}{x(x+1)} & ; \ x>0\\ \dfrac{3x}{x(x-1)} & ; \ x<0\end{cases}$$

$$=\begin{cases}\dfrac{1}{x+1} & ; \ x>0\\ \dfrac{3}{x-1} & ; \ x<0\end{cases}$$

$$\therefore \quad \lim_{x\to 0^+}\frac{2x-|x|}{x^2+|x|}=\lim_{x\to 0^+}\frac{1}{x+1}=\frac{1}{0+1}=1$$

21)

$$\lim_{x\to 0^-}\frac{2x-|x|}{x^2+|x|}=$$

Solution:

$$f(x)=\frac{2x-|x|}{x^2+|x|}=\begin{cases}\dfrac{2x-(x)}{x^2+(x)} & ; \ x>0\\ \dfrac{2x-(-x)}{x^2+(-x)} & ; \ x<0\end{cases}$$

$$=\begin{cases}\dfrac{2x-x}{x^2+x} & ; \ x>0\\ \dfrac{2x+x}{x^2-x} & ; \ x<0\end{cases}=\begin{cases}\dfrac{x}{x^2+x} & ; \ x>0\\ \dfrac{3x}{x^2-x} & ; \ x<0\end{cases}$$

$$=\begin{cases}\dfrac{x}{x(x+1)} & ; \ x>0\\ \dfrac{3x}{x(x-1)} & ; \ x<0\end{cases}$$

$$=\begin{cases}\dfrac{1}{x+1} & ; \ x>0\\ \dfrac{3}{x-1} & ; \ x<0\end{cases}$$

$$\therefore \quad \lim_{x\to 0^-}\frac{2x-|x|}{x^2+|x|}=\lim_{x\to 0^-}\frac{3}{x-1}=\frac{3}{0-1}=-3$$

22)

$$\lim_{x\to 0}\frac{2x-|x|}{x^2+|x|}=$$

Solution:

$\lim\limits_{x\to 0}\dfrac{2x-|x|}{x^2+|x|}$ does not exist because

$$\lim_{x\to 0^-}\frac{2x-|x|}{x^2+|x|}\neq\lim_{x\to 0^+}\frac{2x-|x|}{x^2+|x|}$$

It is clearly obvious from questions (20) and (21) above.

23)

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos^2 x-\sin^2 x}=$$

Solution:

$$\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{\cos^2 x-\sin^2 x}=\lim_{x\to\frac{\pi}{4}}\frac{\cos x-\sin x}{(\cos x-\sin x)(\cos x+\sin x)}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{1}{\cos x+\sin x}=\frac{1}{\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)}$$

$$=\frac{1}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}=\frac{1}{\frac{2}{\sqrt{2}}}=\frac{\sqrt{2}}{2}$$

24)

$$\lim_{x\to 0}\frac{\cos^2 x+2\cos x-3}{2\cos^2 x-\cos x-1}=$$

Solution:

$$\lim_{x\to 0}\frac{\cos^2 x+2\cos x-3}{2\cos^2 x-\cos x-1}=\lim_{x\to 0}\frac{(\cos x+3)(\cos x-1)}{(2\cos x+1)(\cos x-1)}$$

$$=\lim_{x\to 0}\frac{\cos x+3}{2\cos x+1}=\frac{\cos(0)+3}{2\cos(0)+1}$$

$$=\frac{1+3}{2(1)+1}=\frac{4}{3}$$

25)

$$\lim_{x\to 0}(\sin^2 x+3\tan x-4)=$$

Solution:

$$\lim_{x\to 0}(\sin^2 x+3\tan x-4)=\sin^2(0)+3\tan(0)-4$$

$$=0+3(0)-4=-4$$

26) If $m\neq 0$, then

$$\lim_{x\to 0}\frac{\sin(nx)}{mx}=$$

Solution:

$$\lim_{x\to 0}\frac{\sin(nx)}{mx}=\frac{n}{m}\lim_{x\to 0}\frac{\sin(nx)}{nx}=\frac{n}{m}(1)=\frac{n}{m}$$

27) If $m\neq 0$, then

$$\lim_{x\to 0}\frac{\tan(nx)}{mx}=$$

Solution:

$$\lim_{x\to 0}\frac{\tan(nx)}{mx}=\frac{n}{m}\lim_{x\to 0}\frac{\tan(nx)}{nx}=\frac{n}{m}(1)=\frac{n}{m}$$

28) If $m\neq 0$, then

$$\lim_{x\to 0}\frac{nx}{\sin(mx)}=$$

Solution:

$$\lim_{x\to 0}\frac{nx}{\sin(mx)}=\frac{n}{m}\lim_{x\to 0}\frac{mx}{\sin(mx)}=\frac{n}{m}(1)=\frac{n}{m}$$

29) If $m\neq 0$, then

$$\lim_{x\to 0}\frac{nx}{\tan(mx)}=$$

Solution:

$$\lim_{x\to 0}\frac{nx}{\tan(mx)}=\frac{n}{m}\lim_{x\to 0}\frac{mx}{\tan(mx)}=\frac{n}{m}(1)=\frac{n}{m}$$

30) If $m \neq 0$, then

$$\lim_{x\to 0}\frac{\sin(nx)}{\sin(mx)} =$$

Solution:

$$\lim_{x\to 0}\frac{\sin(nx)}{\sin(mx)} = \frac{n}{m}\left(\lim_{x\to 0}\frac{\sin(nx)}{nx}\right)\left(\lim_{x\to 0}\frac{mx}{\sin(mx)}\right)$$
$$= \frac{n}{m}(1)(1) = \frac{n}{m}$$

31) If $m \neq 0$, then

$$\lim_{x\to 0}\frac{\sin(nx)}{\tan(mx)} =$$

Solution:

$$\lim_{x\to 0}\frac{\sin(nx)}{\tan(mx)} = \frac{n}{m}\left(\lim_{x\to 0}\frac{\sin(nx)}{nx}\right)\left(\lim_{x\to 0}\frac{mx}{\tan(mx)}\right)$$
$$= \frac{n}{m}(1)(1) = \frac{n}{m}$$

32) If $m \neq 0$, then

$$\lim_{x\to 0}\frac{\tan(nx)}{\tan(mx)} =$$

Solution:

$$\lim_{x\to 0}\frac{\tan(nx)}{\tan(mx)} = \frac{n}{m}\left(\lim_{x\to 0}\frac{\tan(nx)}{nx}\right)\left(\lim_{x\to 0}\frac{mx}{\tan(mx)}\right)$$
$$= \frac{n}{m}(1)(1) = \frac{n}{m}$$

33) If $m \neq 0$, then

$$\lim_{x\to 0}\frac{\tan(nx)}{\sin(mx)} =$$

Solution:

$$\lim_{x\to 0}\frac{\tan(nx)}{\sin(mx)} = \frac{n}{m}\left(\lim_{x\to 0}\frac{\tan(nx)}{nx}\right)\left(\lim_{x\to 0}\frac{mx}{\sin(mx)}\right)$$
$$= \frac{n}{m}(1)(1) = \frac{n}{m}$$

34)

$$\lim_{x\to 0}\frac{\sin(1-\cos x)}{1-\cos x} =$$

Solution:

$$\lim_{x\to 0}\frac{\sin(1-\cos x)}{1-\cos x} = 1$$

35)

$$\lim_{x\to 0}\frac{\sin(\sin(2x))}{\sin(2x)} =$$

Solution:

$$\lim_{x\to 0}\frac{\sin(\sin(2x))}{\sin(2x)} = 1$$

36)

$$\lim_{x\to 0}\frac{1-\cos(2x)}{x^2} =$$

Solution:

$$\lim_{x\to 0}\frac{1-\cos(2x)}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$
$$= 2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$$

37)

$$\lim_{x\to\infty}\sqrt{\frac{1}{x^2}-\frac{3}{x}+4} =$$

Solution:

$$\lim_{x\to\infty}\sqrt{\frac{1}{x^2}-\frac{3}{x}+4} = \sqrt{\lim_{x\to\infty}\left(\frac{1}{x^2}-\frac{3}{x}+4\right)} = \sqrt{0-0+4}$$
$$= 2$$

38)

$$\lim_{x\to\infty}\left(\frac{1}{x^{2/5}}+2\right) =$$

Solution:

$$\lim_{x\to-\infty}\left(\frac{1}{x^{2/5}}+2\right) = 0+2 = 2$$

39)

$$\lim_{x\to\infty}\frac{3x+15}{9x^2+4x-13} =$$

Solution:

$$\lim_{x\to\infty}\frac{3x+15}{9x^2+4x-13} = \lim_{x\to\infty}\frac{\frac{3x}{x^2}+\frac{15}{x^2}}{\frac{9x^2}{x^2}+\frac{4x}{x^2}-\frac{13}{x^2}}$$
$$= \lim_{x\to\infty}\frac{\frac{3}{x}+\frac{15}{x^2}}{9+\frac{4}{x}-\frac{13}{x^2}} = \frac{0+0}{9+0+0} = 0$$

40)

$$\lim_{x\to\infty}\frac{3x^2-8x+15}{9x^2+4x-13} =$$

Solution:

$$\lim_{x\to\infty}\frac{3x^2-8x+15}{9x^2+4x-13} = \lim_{x\to\infty}\frac{\frac{3x^2}{x^2}-\frac{8x}{x^2}+\frac{15}{x^2}}{\frac{9x^2}{x^2}+\frac{4x}{x^2}-\frac{13}{x^2}}$$
$$= \lim_{x\to\infty}\frac{3-\frac{8}{x}+\frac{15}{x^2}}{9+\frac{4}{x}-\frac{13}{x^2}} = \frac{3-0+0}{9+0+0} = \frac{1}{3}$$

41)

$$\lim_{x\to-\infty}\frac{3x^2-8x+15}{9x^2+4x-13} =$$

Solution:

$$\lim_{x\to-\infty}\frac{3x^2-8x+15}{9x^2+4x-13} = \lim_{x\to-\infty}\frac{\frac{3x^2}{-x^2}-\frac{8x}{-x^2}+\frac{15}{-x^2}}{\frac{9x^2}{-x^2}+\frac{4x}{-x^2}-\frac{13}{-x^2}}$$
$$= \lim_{x\to-\infty}\frac{-3+\frac{8}{x}-\frac{15}{x^2}}{-9-\frac{4}{x}+\frac{13}{x^2}} = \frac{-3+0-0}{-9-0+0} = \frac{1}{3}$$

42)

$$\lim_{x\to\infty}\frac{3x^5-8x+15}{9x^2+4x-13}=$$

Solution:

$$\lim_{x\to\infty}\frac{3x^5-8x+15}{9x^2+4x-13}=\lim_{x\to\infty}\frac{\frac{3x^5}{x^2}-\frac{8x}{x^2}+\frac{15}{x^2}}{\frac{9x^2}{x^2}+\frac{4x}{x^2}-\frac{13}{x^2}}$$

$$=\lim_{x\to\infty}\frac{3x^3-\frac{8}{x}+\frac{15}{x^2}}{9+\frac{4}{x}-\frac{13}{x^2}}=\frac{3(\infty)-0+0}{9+0+0}=\infty$$

43)

$$\lim_{x\to-\infty}\frac{3x^5-8x+15}{9x^2+4x-13}=$$

Solution:

$$\lim_{x\to-\infty}\frac{3x^5-8x+15}{9x^2+4x-13}=\lim_{x\to-\infty}\frac{\frac{3x^5}{-x^2}-\frac{8x}{-x^2}+\frac{15}{-x^2}}{\frac{9x^2}{-x^2}+\frac{4x}{-x^2}-\frac{13}{-x^2}}$$

$$=\lim_{x\to-\infty}\frac{-3x^3+\frac{8}{x}-\frac{15}{x^2}}{-9-\frac{4}{x}+\frac{13}{x^2}}=\frac{-3(-\infty)+0-0}{-9-0+0}=-\infty$$

44)

$$\lim_{x\to\infty}\left(\sqrt{x^2-3x+7}-x\right)=$$

Solution:

$$\lim_{x\to\infty}\left(\sqrt{x^2-3x+7}-x\right)$$

$$=\lim_{x\to\infty}\left[\left(\sqrt{x^2-3x+7}-x\right)\times\frac{\left(\sqrt{x^2-3x+7}+x\right)}{\left(\sqrt{x^2-3x+7}+x\right)}\right]$$

$$=\lim_{x\to\infty}\left(\frac{(x^2-3x+7)-x^2}{\sqrt{x^2-3x+7}+x}\right)=\lim_{x\to\infty}\left(\frac{-3x+7}{\sqrt{x^2-3x+7}+x}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{-3x}{x}+\frac{7}{x}}{\frac{\sqrt{x^2-3x+7}}{x}+\frac{x}{x}}$$

$$=\lim_{x\to\infty}\frac{-3+\frac{7}{x}}{\sqrt{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{7}{x^2}}+1}$$

$$=\lim_{x\to\infty}\frac{-3+\frac{7}{x}}{\sqrt{1-\frac{3}{x}+\frac{7}{x^2}}+1}$$

$$=\frac{-3+0}{\sqrt{1-0+0}+1}=\frac{-3}{1+1}=-\frac{3}{2}$$

45)

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=$$

Solution:

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$$

$$=\lim_{x\to\infty}\left[\left(\sqrt{x^2+x}-x\right)\times\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right]$$

$$=\lim_{x\to\infty}\left(\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}\right)$$

$$=\lim_{x\to\infty}\left(\frac{x}{\sqrt{x^2+x}+x}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{x}{x}}{\frac{\sqrt{x^2+x}}{x}+\frac{x}{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}}+1}$$

$$=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{1+1}$$

$$=\frac{1}{2}$$

46)

$$\lim_{x\to\infty}(x^2-5x+4)=$$

Solution:

$$\lim_{x\to\infty}(x^2-5x+4)=\lim_{x\to\infty}x^2\left(\frac{x^2}{x^2}-\frac{5x}{x^2}+\frac{4}{x^2}\right)$$

$$=\lim_{x\to\infty}x^2\left(1-\frac{5}{x}+\frac{4}{x^2}\right)=(\infty)^2(1-0+0)=\infty$$

OR

$$\lim_{x\to\infty}(x^2-5x+4)=\lim_{x\to\infty}(x^2)=\infty$$

47)

$$\lim_{x\to-\infty}(x^4-2x^3+9)=$$

Solution:

$$\lim_{x\to-\infty}(x^4-2x^3+9)=\lim_{x\to-\infty}x^4\left(\frac{x^4}{x^4}-\frac{2x^3}{x^4}+\frac{9}{x^4}\right)$$

$$=\lim_{x\to-\infty}x^4\left(1-\frac{2}{x}+\frac{9}{x^4}\right)=(-\infty)^4(1-0+0)=\infty$$

OR

$$\lim_{x\to-\infty}(x^4-2x^3+9)=\lim_{x\to-\infty}(x^4)=\infty$$

48)

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=$$

Solution:

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=\lim_{x\to-\infty}\frac{\frac{\sqrt{3x^2-8}}{-x}+\frac{2}{-x}}{\frac{x}{-x}+\frac{5}{-x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{\frac{3x^2-8}{x^2}}-\frac{2}{x}}{-1-\frac{5}{x}}=\lim_{x\to-\infty}\frac{\sqrt{\frac{3x^2}{x^2}-\frac{8}{x^2}}-\frac{2}{x}}{-1-\frac{5}{x}}$$

$$=\lim_{x\to-\infty}\frac{\sqrt{3-\frac{8}{x^2}}-\frac{2}{x}}{-1-\frac{5}{x}}=\frac{\sqrt{3-0}-0}{-1-0}=-\sqrt{3}$$

49)

$$\lim_{x\to\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=$$

Solution:

$$\lim_{x\to\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=\lim_{x\to\infty}\frac{\frac{\sqrt{3x^2-8}}{x}+\frac{2}{x}}{\frac{x}{x}+\frac{5}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{\frac{3x^2-8}{x^2}}+\frac{2}{x}}{1+\frac{5}{x}}=\lim_{x\to\infty}\frac{\sqrt{\frac{3x^2}{x^2}-\frac{8}{x^2}}+\frac{2}{x}}{1+\frac{5}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{3-\frac{8}{x^2}}+\frac{2}{x}}{1+\frac{5}{x}}=\frac{\sqrt{3-0}+0}{1+0}=\sqrt{3}$$

50) The horizontal asymptotes of

$$f(x)=\frac{\sqrt{3x^2-8}+2}{x+5}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty}\frac{\sqrt{3x^2-8}+2}{x+5}$$

It is clear from the previous questions (48) and (49) that

$$\lim_{x\to\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=\sqrt{3}$$

and

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2-8}+2}{x+5}=-\sqrt{3}$$

Thus, the horizontal asymptotes are

$$y=\pm\sqrt{3}$$

51) The horizontal asymptote of

$$f(x)=\frac{1-x}{2x+1}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty}\frac{1-x}{2x+1}$$

$$\lim_{x\to\infty}\frac{1-x}{2x+1}=\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{x}{x}}{\frac{2x}{x}+\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}-1}{2+\frac{1}{x}}=\frac{0-1}{2+0}=-\frac{1}{2}$$

$$\lim_{x\to-\infty}\frac{1-x}{2x+1}=\lim_{x\to-\infty}\frac{\frac{1}{-x}-\frac{x}{-x}}{\frac{2x}{-x}+\frac{1}{-x}}=\lim_{x\to-\infty}\frac{\frac{1}{-x}+1}{-2-\frac{1}{x}}=\frac{0+1}{-2-0}$$

$$=-\frac{1}{2}$$

Thus, the horizontal asymptote is

$$y=-\frac{1}{2}$$

52) The horizontal asymptote of

$$f(x)=\frac{7x^2+5}{3x^2+2}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty}\frac{7x^2+5}{3x^2+2}$$

$$\lim_{x\to\infty}\frac{7x^2+5}{3x^2+2}=\lim_{x\to\infty}\frac{\frac{7x^2}{x^2}+\frac{5}{x^2}}{\frac{3x^2}{x^2}+\frac{2}{x^2}}=\lim_{x\to\infty}\frac{7+\frac{5}{x^2}}{3+\frac{2}{x^2}}=\frac{7+0}{3+0}=\frac{7}{3}$$

$$\lim_{x\to-\infty}\frac{7x^2+5}{3x^2+2}=\lim_{x\to-\infty}\frac{\frac{7x^2}{-x^2}+\frac{5}{-x^2}}{\frac{3x^2}{-x^2}+\frac{2}{-x^2}}$$

$$=\lim_{x\to-\infty}\frac{-7-\frac{5}{x^2}}{-3-\frac{2}{x^2}}=\frac{-7-0}{-3-0}=\frac{7}{3}$$

Thus, the horizontal asymptote is

$$y=\frac{7}{3}$$

53) The horizontal asymptote of

$$f(x) = \frac{\sqrt{x^2+2x-3}}{2x+7}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty} \frac{\sqrt{x^2+2x-3}}{2x+7}$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x-3}}{2x+7} = \lim_{x\to\infty} \frac{\frac{\sqrt{x^2+2x-3}}{x}}{\frac{2x}{x}+\frac{7}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+2x-3}{x^2}}}{2+\frac{7}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}-\frac{3}{x^2}}}{2+\frac{7}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{2}{x}-\frac{3}{x^2}}}{2+\frac{7}{x}} = \frac{\sqrt{1+0-0}}{2+0} = \frac{1}{2}$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2x-3}}{2x+7} = \lim_{x\to-\infty} \frac{\frac{\sqrt{x^2+2x-3}}{-x}}{\frac{2x}{-x}+\frac{7}{-x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\frac{x^2+2x-3}{x^2}}}{-2-\frac{7}{x}} = \lim_{x\to-\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}-\frac{3}{x^2}}}{-2-\frac{7}{x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{1+\frac{2}{x}-\frac{3}{x^2}}}{-2-\frac{7}{x}} = \frac{\sqrt{1+0-0}}{-2-0} = -\frac{1}{2}$$

Thus, the horizontal asymptotes are

$$y = \pm\frac{1}{2}$$

54) The horizontal asymptote of

$$f(x) = \frac{\sqrt{2x-3}}{2x^2+7x-1}$$

Solution:

First, we have to find

$$\lim_{x\to\pm\infty} \frac{\sqrt{2x-3}}{2x^2+7x-1}$$

$$\lim_{x\to\infty} \frac{\sqrt{2x-3}}{2x^2+7x-1} = \lim_{x\to\infty} \frac{\frac{\sqrt{2x-3}}{x^2}}{\frac{2x^2}{x^2}+\frac{7x}{x^2}-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{2x-3}{x^4}}}{2+\frac{7}{x}-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\sqrt{\frac{2x}{x^4}-\frac{3}{x^4}}}{2+\frac{7}{x}-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{2}{x^3}-\frac{3}{x^4}}}{2+\frac{7}{x}-\frac{1}{x^2}} = \frac{\sqrt{0-0}}{2+0-0} = \frac{0}{2} = 0$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x-3}}{2x^2+7x-1} = \lim_{x\to-\infty} \frac{\frac{\sqrt{2x-3}}{-x^2}}{\frac{2x^2}{-x^2}+\frac{7x}{-x^2}-\frac{1}{-x^2}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\frac{2x-3}{x^4}}}{-2-\frac{7}{x}+\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\sqrt{\frac{2x}{x^4}-\frac{3}{x^4}}}{-2-\frac{7}{x}+\frac{1}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\frac{2}{x^3}-\frac{3}{x^4}}}{-2-\frac{7}{x}+\frac{1}{x^2}} = \frac{\sqrt{0-0}}{-2-0+0} = \frac{0}{-2} = 0$$

Thus, the horizontal asymptote is

$$y = 0$$

55)

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-8}+3}{x+1} =$$

Solution:

$$\lim_{x\to-\infty} \frac{\sqrt{4x^2-8}+3}{x+1} = \lim_{x\to-\infty} \frac{\frac{\sqrt{4x^2-8}}{-x}+\frac{3}{-x}}{\frac{x}{-x}+\frac{1}{-x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\frac{4x^2-8}{x^2}}-\frac{3}{x}}{-1-\frac{1}{x}} = \lim_{x\to-\infty} \frac{\sqrt{\frac{4x^2}{x^2}-\frac{8}{x^2}}-\frac{3}{x}}{-1-\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{4-\frac{8}{x^2}}-\frac{3}{x}}{-1-\frac{1}{x}} = \frac{\sqrt{4-0}-0}{-1-0} = -2$$

56)

$$\lim_{x\to\infty} \frac{\sqrt{4x^2-8}+3}{x+1} =$$

Solution:

$$\lim_{x\to\infty} \frac{\sqrt{4x^2-8}+3}{x+1} = \lim_{x\to\infty} \frac{\frac{\sqrt{4x^2-8}}{x}+\frac{3}{x}}{\frac{x}{x}+\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\frac{4x^2-8}{x^2}}+\frac{3}{x}}{1+\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{4x^2}{x^2}-\frac{8}{x^2}}+\frac{3}{x}}{1+\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{4-\frac{8}{x^2}}+\frac{3}{x}}{1+\frac{1}{x}} = \frac{\sqrt{4-0}+0}{1+0} = 2$$

Workshop Solutions to Sections 3.4 and 3.5

1) $\lim\limits_{x\to 3^+} \dfrac{2}{x-3} =$

Solution:

If $x \to 3^+$, then $x > 3 \implies x-3>0$

$$\therefore \lim_{x\to 3^+} \frac{2}{x-3} = \infty$$

2) $\lim\limits_{x\to 3^-} \dfrac{2}{x-3} =$

Solution:

If $x \to 3^-$, then $x < 3 \implies x-3<0$

$$\therefore \lim_{x\to 3^-} \frac{2}{x-3} = -\infty$$

3) $\lim\limits_{x\to 3^+} \dfrac{-2}{x-3} =$

Solution:

If $x \to 3^+$, then $x > 3 \implies x-3>0$

$$\therefore \lim_{x\to 3^+} \frac{-2}{x-3} = -\infty$$

4) $\lim\limits_{x\to 3^-} \dfrac{-2}{x-3} =$

Solution:

If $x \to 3^-$, then $x < 3 \implies x-3<0$

$$\therefore \lim_{x\to 3^-} \frac{2}{x-3} = \infty$$

5) $\lim\limits_{x\to -3^+} \dfrac{2}{x+3} =$

Solution:

If $x \to -3^+$, then $x > -3 \implies x+3>0$

$$\therefore \lim_{x\to -3^+} \frac{2}{x+3} = \infty$$

6) $\lim\limits_{x\to -3^-} \dfrac{2}{x+3} =$

Solution:

If $x \to -3^-$, then $x < -3 \implies x+3<0$

$$\therefore \lim_{x\to -3^-} \frac{2}{x+3} = -\infty$$

7) $\lim\limits_{x\to 2^+} \dfrac{3x-1}{x-2} =$

Solution:

If $x \to 2^+$, then $x > 2 \implies x-2>0$ and $3x-1>0$

$$\therefore \lim_{x\to 2^+} \frac{3x-1}{x-2} = \infty$$

8) $\lim\limits_{x\to 2^-} \dfrac{3x-1}{x-2} =$

Solution:

If $x \to 2^-$, then $x < 2 \implies x-2<0$ and $3x-1>0$

$$\therefore \lim_{x\to 2^-} \frac{3x-1}{x-2} = -\infty$$

9) $\lim\limits_{x\to -2^+} \dfrac{1-x}{(x+2)^2} =$

Solution:

If $x \to -2^+$, then $x > -2$

$\implies 1-x>0$ and $(x+2)^2>0$

$$\therefore \lim_{x\to -2^+} \frac{1-x}{(x+2)^2} = \infty$$

10) $\lim\limits_{x\to -2^-} \dfrac{1-x}{(x+2)^2} =$

Solution:

If $x \to -2^-$, then $x < -2$

$\implies 1-x>0$ and $(x+2)^2>0$

$$\therefore \lim_{x\to -2^+} \frac{1-x}{(x+2)^2} = \infty$$

11) $\lim\limits_{x\to -2^+} \dfrac{x-1}{(x+2)^2} =$

Solution:

If $x \to -2^+$, then $x > -2$

$\implies x-1<0$ and $(x+2)^2>0$

$$\therefore \lim_{x\to -2^+} \frac{x-1}{(x+2)^2} = -\infty$$

12) $\lim\limits_{x\to -2^-} \dfrac{x-1}{(x+2)^2} =$

Solution:

If $x \to -2^-$, then $x < -2$

$\implies x-1<0$ and $(x+2)^2>0$

$$\therefore \lim_{x\to -2^-} \frac{x-1}{(x+2)^2} = -\infty$$

13) $\lim\limits_{x\to 2^+} \dfrac{6x-1}{x^2-4} =$

Solution:

If $x \to 2^+$, then $x^2 > 4$

$\implies x^2-4>0$ and $6x-1>0$

$$\therefore \lim_{x\to 2^+} \frac{6x-1}{x^2-4} = \infty$$

14) $\lim\limits_{x\to 2^-} \dfrac{6x-1}{x^2-4} =$

Solution:

If $x \to 2^-$, then $x^2 < 4$

$\implies x^2-4<0$ and $6x-1>0$

$$\therefore \lim_{x\to 2^+} \frac{6x-1}{x^2-4} = -\infty$$

15) $\lim_{x \to -2^+} \frac{6x-1}{x^2-4} =$

Solution:

If $x \to -2^+$, then $x^2 < 4$

$$\Longrightarrow x^2 - 4 < 0 \text{ and } 6x - 1 < 0$$

$$\therefore \lim_{x \to 2^+} \frac{6x-1}{x^2-4} = \infty$$

16) $\lim_{x \to -2^-} \frac{6x-1}{x^2-4} =$

Solution:

If $x \to -2^-$, then $x^2 > 4$

$$\Longrightarrow x^2 - 4 > 0 \text{ and } 6x - 1 < 0$$

$$\therefore \lim_{x \to 2^+} \frac{6x-1}{x^2-4} = -\infty$$

17) $\lim_{x \to -2^-} \frac{6x-1}{x^2-x-6} =$

Solution:

$$f(x) = \frac{6x-1}{x^2-x-6} = \frac{6x-1}{(x-3)(x+2)}$$

If $x \to -2^-$, then $x < -2$

$$\Longrightarrow x - 3 < 0 \,,\ x + 2 < 0 \text{ and } 6x - 1 < 0$$

$$\therefore \lim_{x \to -2^-} \frac{6x-1}{x^2-x-6} = -\infty$$

18) $\lim_{x \to -2^+} \frac{6x-1}{x^2-x-6} =$

Solution:

$$f(x) = \frac{6x-1}{x^2-x-6} = \frac{6x-1}{(x-3)(x+2)}$$

If $x \to -2^+$, then $x > -2$

$$\Longrightarrow x - 3 < 0 \,,\ x + 2 > 0 \text{ and } 6x - 1 < 0$$

$$\therefore \lim_{x \to -2^+} \frac{6x-1}{x^2-x-6} = \infty$$

19) $\lim_{x \to 3^+} \frac{-1}{x^2-x-6} =$

Solution:

$$f(x) = \frac{-1}{x^2-x-6} = \frac{-1}{(x-3)(x+2)}$$

If $x \to 3^+$, then $x > 3$

$$\Longrightarrow x - 3 > 0 \,,\ x + 2 > 0 \text{ and } -1 < 0$$

$$\therefore \lim_{x \to 3^+} \frac{-1}{x^2-x-6} = -\infty$$

20) $\lim_{x \to 3^-} \frac{-1}{x^2-x-6} =$

Solution:

$$f(x) = \frac{-1}{x^2-x-6} = \frac{-1}{(x-3)(x+2)}$$

If $x \to 3^-$, then $x < 3$

$$\Longrightarrow x - 3 < 0 \,,\ x + 2 > 0 \text{ and } -1 < 0$$

$$\therefore \lim_{x \to 3^-} \frac{-1}{x^2-x-6} = \infty$$

21) $\lim_{x \to (\pi/2)^+} \tan x =$

Solution:

$$\lim_{x \to (\pi/2)^+} \tan x = -\infty$$

22) $\lim_{x \to (\pi/2)^-} \tan x =$

Solution:

$$\lim_{x \to (\pi/2)^-} \tan x = \infty$$

23) The vertical asymptote of $f(x) = \frac{1-x}{2x+1}$ is

Solution:

We see that the function $f(x)$ is not defined when $2x + 1 = 0 \implies x = -\frac{1}{2}$. Since

$$\lim_{x \to \left(-\frac{1}{2}\right)^+} \frac{1-x}{2x+1} = \infty$$

and

$$\lim_{x \to \left(-\frac{1}{2}\right)^-} \frac{1-x}{2x+1} = -\infty$$

then, $x = -\frac{1}{2}$ is a vertical asymptote.

24) The vertical asymptote of $f(x) = \frac{3-x}{x^2-4}$ is

Solution:

We see that the function $f(x)$ is not defined when $x^2 - 4 = 0 \implies x = \pm 2$. Since

$$\lim_{x \to 2^+} \frac{3-x}{x^2-4} = \infty \,, \qquad \lim_{x \to 2^-} \frac{3-x}{x^2-4} = -\infty$$

and

$$\lim_{x \to -2^+} \frac{3-x}{x^2-4} = -\infty \,, \qquad \lim_{x \to -2^-} \frac{3-x}{x^2-4} = \infty$$

then, $x = \pm 2$ are vertical asymptotes.

25) The vertical asymptote of $f(x) = \frac{3-x}{x^2-x-6}$ is

Solution:

$$f(x) = \frac{3-x}{x^2-x-6} = \frac{3-x}{(x-3)(x+2)} = \frac{-(x-3)}{(x-3)(x+2)} = -\frac{1}{x+2}$$

We see that the function $f(x)$ is not defined when $x^2 - x - 6 = 0 \implies (x-3)(x+2) = 0$ $\implies x = 3$ or $x = -2$. Since

$$\lim_{x\to 3} \frac{3-x}{x^2-x-6} = \lim_{x\to 3} \frac{3-x}{(x-3)(x+2)} = \lim_{x\to 3} \frac{-(x-3)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{-1}{x+2} = -\frac{1}{5}$$

then, $x = 3$ is a removable discontinuity.

$$\lim_{x\to -2^+} \frac{3-x}{x^2-x-6} = \lim_{x\to -2^+} \frac{3-x}{(x-3)(x+2)} = \infty$$

and

$$\lim_{x\to -2^-} \frac{3-x}{x^2-x-6} = \lim_{x\to -2^-} \frac{3-x}{(x-3)(x+2)} = -\infty$$

then, $x = -2$ is a vertical asymptote only.

26) The vertical asymptote of $f(x) = \frac{7-x}{x^2-5x+6}$ is

Solution:

$$f(x) = \frac{7-x}{x^2-5x+6} = \frac{7-x}{(x-3)(x-2)}$$

We see that the function $f(x)$ is not defined when $x - 3 = 0$ or $x - 2 = 0 \implies x = 3$ or $x = 2$. Since

$$\lim_{x\to 3^+} \frac{7-x}{x^2-5x+6} = \lim_{x\to 3^+} \frac{7-x}{(x-3)(x-2)} = \infty$$

$$\lim_{x\to 3^-} \frac{7-x}{x^2-5x+6} = \lim_{x\to 3^-} \frac{7-x}{(x-3)(x-2)} = -\infty$$

and

$$\lim_{x\to 2^+} \frac{7-x}{x^2-5x+6} = \lim_{x\to 2^+} \frac{7-x}{(x-3)(x-2)} = -\infty$$

$$\lim_{x\to 2^-} \frac{7-x}{x^2-5x+6} = \lim_{x\to 2^-} \frac{7-x}{(x-3)(x-2)} = \infty$$

then, $x = 3$ and $x = 2$ are vertical asymptotes.

27) The vertical asymptote of $f(x) = \frac{x-7}{x^2+5x+6}$ is

Solution:

$$f(x) = \frac{x-7}{x^2+5x+6} = \frac{x-7}{(x+3)(x+2)}$$

We see that the function $f(x)$ is not defined when $x + 3 = 0$ or $x + 2 = 0 \implies x = -3$ or $x = -2$. Since

$$\lim_{x\to -3^+} \frac{x-7}{x^2+5x+6} = \lim_{x\to -3^+} \frac{x-7}{(x+3)(x+2)} = \infty$$

$$\lim_{x\to -3^-} \frac{x-7}{x^2+5x+6} = \lim_{x\to -3^-} \frac{x-7}{(x+3)(x+2)} = -\infty$$

and

$$\lim_{x\to -2^+} \frac{x-7}{x^2+5x+6} = \lim_{x\to -2^+} \frac{x-7}{(x+3)(x+2)} = -\infty$$

$$\lim_{x\to -2^-} \frac{x-7}{x^2+5x+6} = \lim_{x\to -2^-} \frac{x-7}{(x+3)(x+2)} = \infty$$

then, $x = -3$ and $x = -2$ are vertical asymptotes.

28) The vertical asymptote of $f(x) = \frac{x-7}{x^2+3x}$ is

Solution:

$$f(x) = \frac{x-7}{x^2+3x} = \frac{x-7}{x(x+3)}$$

We see that the function $f(x)$ is not defined when $x = 0$ or $x + 3 = 0 \implies x = 0$ or $x = -3$. Since

$$\lim_{x\to -3^+} \frac{x-7}{x^2+3x} = \lim_{x\to -3^+} \frac{x-7}{x(x+3)} = \infty$$

$$\lim_{x\to -3^-} \frac{x-7}{x^2+3x} = \lim_{x\to -3^-} \frac{x-7}{x(x+3)} = -\infty$$

and

$$\lim_{x\to 0^+} \frac{x-7}{x^2+3x} = \lim_{x\to 0^+} \frac{x-7}{x(x+3)} = -\infty$$

$$\lim_{x\to 0^-} \frac{x-7}{x^2+3x} = \lim_{x\to 0^-} \frac{x-7}{x(x+3)} = \infty$$

then, $x = -3$ and $x = 0$ are vertical asymptotes.

29) The vertical asymptote of $f(x) = \frac{x-7}{x^2-3x}$ is

Solution:

$$f(x) = \frac{x-7}{x^2-3x} = \frac{x-7}{x(x-3)}$$

We see that the function $f(x)$ is not defined when $x = 0$ or $x - 3 = 0 \implies x = 0$ or $x = 3$. Since

$$\lim_{x\to 3^+} \frac{x-7}{x^2-3x} = \lim_{x\to 3^+} \frac{x-7}{x(x-3)} = -\infty$$

$$\lim_{x\to 3^-} \frac{x-7}{x^2-3x} = \lim_{x\to 3^-} \frac{x-7}{x(x-3)} = \infty$$

and

$$\lim_{x\to 0^+} \frac{x-7}{x^2-3x} = \lim_{x\to 0^+} \frac{x-7}{x(x-3)} = \infty$$

$$\lim_{x\to 0^-} \frac{x-7}{x^2-3x} = \lim_{x\to 0^-} \frac{x-7}{x(x-3)} = -\infty$$

then, $x = 3$ and $x = 0$ are vertical asymptotes.

30) The vertical asymptotes of $f(x) = \frac{2x^2+1}{x^2-9}$ are

Solution:

$$f(x) = \frac{2x^2+1}{x^2-9} = \frac{2x^2+1}{(x+3)(x-3)}$$

We see that the function $f(x)$ is not defined when $x^2 - 9 = 0 \implies x = \pm 3$. Since

$$\lim_{x\to 3^+} \frac{2x^2+1}{x^2-9} = \lim_{x\to 3^+} \frac{2x^2+1}{(x+3)(x-3)} = \infty$$

$$\lim_{x\to 3^-} \frac{2x^2+1}{x^2-9} = \lim_{x\to 3^-} \frac{2x^2+1}{(x+3)(x-3)} = -\infty$$

and

$$\lim_{x\to -3^+} \frac{2x^2+1}{x^2-9} = \lim_{x\to -3^+} \frac{2x^2+1}{(x+3)(x-3)} = -\infty$$

$$\lim_{x\to -3^-} \frac{2x^2+1}{x^2-9} = \lim_{x\to -3^-} \frac{2x^2+1}{(x+3)(x-3)} = \infty$$

then, $x = \pm 3$ are vertical asymptotes.

31) The function $f(x)=\frac{x+1}{x^2-9}$ is continuous at $a=2$ because

$$1-\quad f(2)=\frac{(2)+1}{(2)^2-9}=\frac{3}{-5}=-\frac{3}{5}$$

$$2-\quad \lim_{x\to 3^-}\frac{x+1}{x^2-9}=\lim_{x\to 2}\frac{(2)+1}{(2)^2-9}=\frac{3}{-5}=-\frac{3}{5}$$

$$3-\quad \lim_{x\to 2}\frac{x+1}{x^2-9}=f(2)$$

OR

We know that $D_f=\mathbb{R}\setminus\{\pm 3\}$, so $\{2\}\in D_f$.

Note: Any function is continuous on its domain.

32) The function $f(x)=\frac{x+1}{x^2-9}$ is discontinuous at $a=\pm 3$ because we know that $D_f=\mathbb{R}\setminus\{\pm 3\}$, so $\{\pm 3\}\notin D_f$.

33) The function $f(x)=\frac{x+1}{x^2-9}$ is discontinuous at ± 3 because $\{\pm 3\}\notin D_f$.

34) The function $f(x)=\frac{x+1}{x^2-9}$ is continuous on its domain which is $D_f=\mathbb{R}\setminus\{\pm 3\}$.

35) The function $f(x)=\begin{cases}\frac{\sin 3x}{x}, & x\neq 0\\ 3, & x=0\end{cases}$ is continuous at $a=0$ because

1- $f(0)=3$

2- $\lim_{x\to 0}\frac{\sin 3x}{x}=3\lim_{x\to 0}\frac{\sin 3x}{3x}=3(1)=3$

3- $\lim_{x\to 0}f(x)=f(0)$

36) The function $f(x)=\begin{cases}\frac{\sin 3x}{x}, & x\neq 0\\ 5, & x=0\end{cases}$ is discontinuous at $a=0$ because

1- $f(0)=5$

2- $\lim_{x\to 0}\frac{\sin 3x}{x}=3\lim_{x\to 0}\frac{\sin 3x}{3x}=3(1)=3$

3- $\lim_{x\to 0}f(x)\neq f(0)$

37) The function $f(x)=\begin{cases}\frac{2x^2-3x+1}{x-1}, & x\neq 1\\ 7, & x=1\end{cases}$ is discontinuous at $a=1$ because

1- $f(1)=7$

2- $\lim_{x\to 1}\frac{2x^2-3x+1}{x-1}=\lim_{x\to 1}\frac{(2x-1)(x-1)}{x-1}=\lim_{x\to 1}(2x-1)=1$

3- $\lim_{x\to 1}f(x)\neq f(1)$

38) The function $f(x)=\begin{cases}\frac{2x^2-3x+1}{x-1}, & x\neq 1\\ 1, & x=1\end{cases}$ is continuous at $a=1$ because

1- $f(1)=1$

2- $\lim_{x\to 1}\frac{2x^2-3x+1}{x-1}=\lim_{x\to 1}\frac{(2x-1)(x-1)}{x-1}=\lim_{x\to 1}(2x-1)=1$

3- $\lim_{x\to 1}f(x)=f(1)$

39) The function $f(x)=\frac{x^2-x-2}{x-2}$ is discontinuous at $a=2$ because $\{2\}\notin D_f$.

40) The function $f(x)=\begin{cases}2x+3, & x>2\\ 3x+1, & x\leq 2\end{cases}$ is continuous at $a=2$ because

1- $f(2)=3(2)+1=7$

2- $\lim_{x\to 2^+}(2x+3)=2(2)+3=7$

$\lim_{x\to 2^-}(3x+1)=3(2)+1=7$

$\therefore\ \lim_{x\to 2}f(x)=7$

3- $\lim_{x\to 2}f(x)=f(2)$

41) The function $f(x)=\frac{x+3}{\sqrt{x^2-4}}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x^2-4>0\implies x^2>4\implies\sqrt{x^2}>\sqrt{4}$$
$$\implies |x|>2\iff x>2\ \text{or}\ x<-2$$

Hence,

$D_f=(-\infty,-2)\cup(2,\infty)$.

42) The function $f(x)=\sqrt{x^2-4}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x^2-4\geq 0\implies x^2\geq 4\implies\sqrt{x^2}\geq\sqrt{4}$$
$$\implies |x|\geq 2\iff x\geq 2\ \text{or}\ x\leq -2$$

Hence,

$D_f=(-\infty,-2]\cup[2,\infty)$.

43) The function $f(x)=\sqrt{4-x^2}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$4-x^2\geq 0\implies -x^2\geq -4\implies x^2\leq 4$$
$$\implies\sqrt{x^2}\leq\sqrt{4}\implies |x|\leq 2\iff -2\leq x\leq 2$$

Hence,

$D_f=[-2,2]$.

44) The function $f(x)=\frac{x+3}{\sqrt{4-x^2}}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$4-x^2>0\implies -x^2>-4\implies x^2<4$$
$$\implies\sqrt{x^2}<\sqrt{4}\implies |x|<2\iff -2<x<2$$

Hence,

$D_f=(-2,2)$.

45) The function $f(x)=\frac{x+1}{x^2-4}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x^2-4\neq 0\implies x^2\neq 4\implies x\neq\pm 2$$

Hence,

$D_f=\mathbb{R}\setminus\{\pm 2\}$

$=(-\infty,-2)\cup(-2,2)\cup(2,\infty)=\{x\in\mathbb{R}: x\neq\pm 2\}$.

46) The function $f(x) = \log_2(x+2)$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x+2>0 \implies x>-2$$

Hence,

$D_f = (-2, \infty)$.

47) The function $f(x) = \sqrt{x-1} + \sqrt{x+4}$ is continuous on its domain where $f(x)$ is defined, we mean that

$$x-1 \geq 0 \text{ and } x+4 \geq 0 \implies x \geq 1 \ \cap \ x \geq -4$$

Hence,

$D_f = [1, \infty)$.

48) The function $f(x) = 5^x$ is continuous on its domain .

Hence,

$D_f = \mathbb{R} = (-\infty, \infty)$.

49) The function $f(x) = e^x$ is continuous on its domain .

Hence,

$D_f = \mathbb{R} = (-\infty, \infty)$.

50) The function $f(x) = \sin^{-1}(3x-5)$ is continuous on its domain where $f(x)$ is defined, we mean that

$$-1 \leq 3x-5 \leq 1 \iff 4 \leq 3x \leq 6 \iff \frac{4}{3} \leq x \leq 2 .$$

Hence,

$D_f = \left[\frac{4}{3}, 2\right]$.

51) The function $f(x) = \cos^{-1}(3x+5)$ is continuous on its domain where $f(x)$ is defined, we mean that

$$-1 \leq 3x+5 \leq 1 \iff -6 \leq 3x \leq -4 \iff -2 \leq x \leq -\frac{4}{3} .$$

Hence,

$D_f = \left[-2, -\frac{4}{3}\right]$.

52) The number c that makes $f(x) = \begin{cases} c+x\,, & x>2 \\ 2x-c, & x \leq 2 \end{cases}$ is continuous at $x=2$ is

Solution:

$\lim_{x\to 2} f(x)$ exists if

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$
$$\lim_{x\to 2^+}(c+x) = \lim_{x\to 2^-}(2x-c)$$
$$c+2 = 4-c$$
$$c+c = 4-2$$
$$2c = 2$$
$$c = 1$$

53) The number c that makes $f(x) = \begin{cases} cx^2-2x+1, & x \leq -1 \\ 3x+2 \quad , & x>-1 \end{cases}$ is continuous at -1 is

Solution:

$\lim_{x\to -1} f(x)$ exists if

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x)$$
$$\lim_{x\to -1^+}(3x+2\ ) = \lim_{x\to -1^-}(cx^2-2x+1)$$
$$3(-1)+2 = c(-1)^2-2(-1)+1$$
$$-1 = c+3$$
$$c = -1-3$$
$$c = -4$$

54) The number c that makes

$f(x) = \begin{cases} \frac{\sin cx}{x} + 2x - 1, & x<0 \\ 3x+4 \quad , & x \geq 0 \end{cases}$ is continuous at 0 is

Solution:

$\lim_{x\to 0} f(x)$ exists if

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$
$$\lim_{x\to 0^+}(3x+4) = \lim_{x\to 0^-}\left(\frac{\sin cx}{x} + 2x - 1\right)$$
$$3(0)+4 = c(1)+2(0)-1$$
$$4 = c-1$$
$$c = 4+1$$
$$c = 5$$

55) The value c that makes $f(x) = \begin{cases} cx^2+2x, & x \leq 2 \\ x^3-cx\ , & x>2 \end{cases}$ is continuous at 2 is

Solution:

$\lim_{x\to 2} f(x)$ exists if

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$
$$\lim_{x\to 2^+}(x^3-cx\ ) = \lim_{x\to 2^-}(cx^2+2x)$$
$$(2)^3-c(2) = c(2)^2+2(2)$$
$$8-2c = 4c+4$$
$$-2c-4c = 4-8$$
$$-6c = -4$$
$$c = \frac{-4}{-6}$$
$$c = \frac{2}{3}$$

56) The number c that makes $f(x) = \begin{cases} c^2x^2-1, & x \leq 3 \\ x+5 \quad , & x>3 \end{cases}$ is continuous at 3 is

Solution:

$\lim_{x\to 3} f(x)$ exists if

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$$
$$\lim_{x\to 3^+}(x+5) = \lim_{x\to 3^-}(c^2x^2-1)$$
$$(3)+5 = c^2(3)^2-1$$
$$8 = 9c^2-1$$
$$9c^2 = 8+1$$
$$c^2 = 1$$
$$c = \pm 1$$

57) The number c that makes $f(x) = \begin{cases} x-2\,, & x>5 \\ cx-3, & x \leq 5 \end{cases}$ is continuous at 5 is

Solution:

$\lim_{x\to 5} f(x)$ exists if

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^-} f(x)$$
$$\lim_{x\to 5^+}(x-2) = \lim_{x\to 5^-}(cx-3)$$
$$(5)-2 = c(5)-3$$
$$3 = 5c-3$$
$$5c = 3+3$$
$$5c = 6$$
$$c = \frac{6}{5}$$

58) The number c that makes $f(x) = \begin{cases} x+3\,, & x > -1 \\ 2x - c, & x \le -1 \end{cases}$ is continuous at -1 is Solution: $\lim\limits_{x \to -1} f(x)$ exists if $$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$$ $$\lim_{x \to -1^+} (x+3) = \lim_{x \to -1^-} (2x - c)$$ $$(-1) + 3 = 2(-1) - c$$ $$2 = -2 - c$$ $$c = -2 - 2$$ $$c = -4$$	

THOMAS' CALCULUS

MEDIA UPGRADE

Chapter 2

Limits and Continuity

2.1

Rates of Change and Limits

TABLE 2.1 Average speeds over short time intervals

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

DEFINITION **Average Rate of Change over an Interval**

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

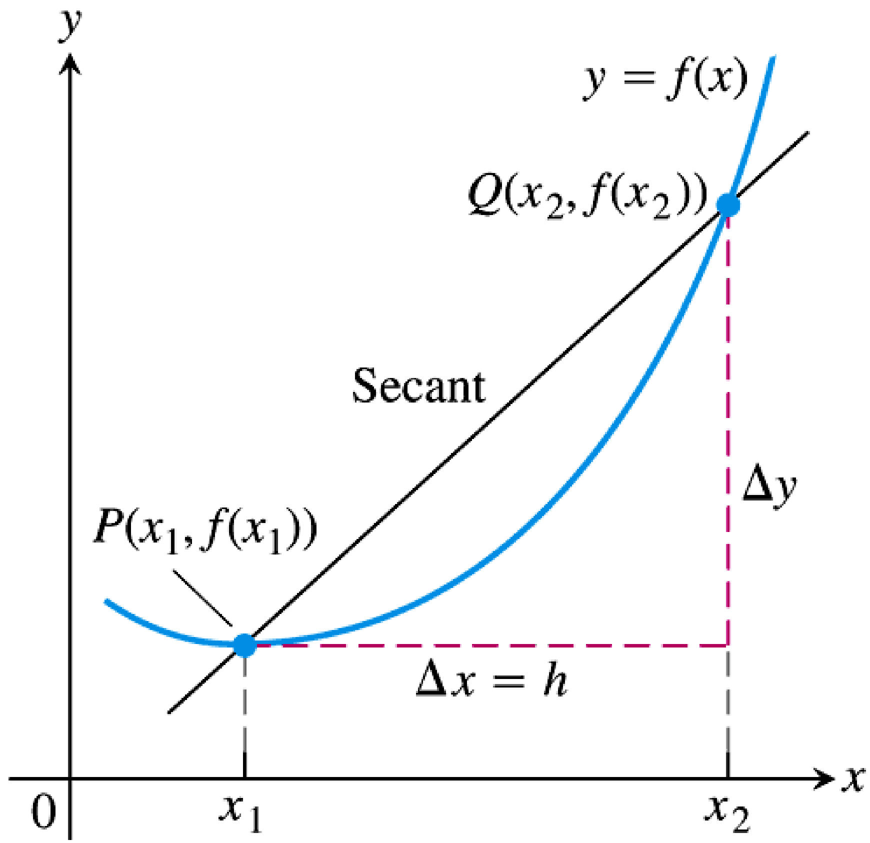


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y / \Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

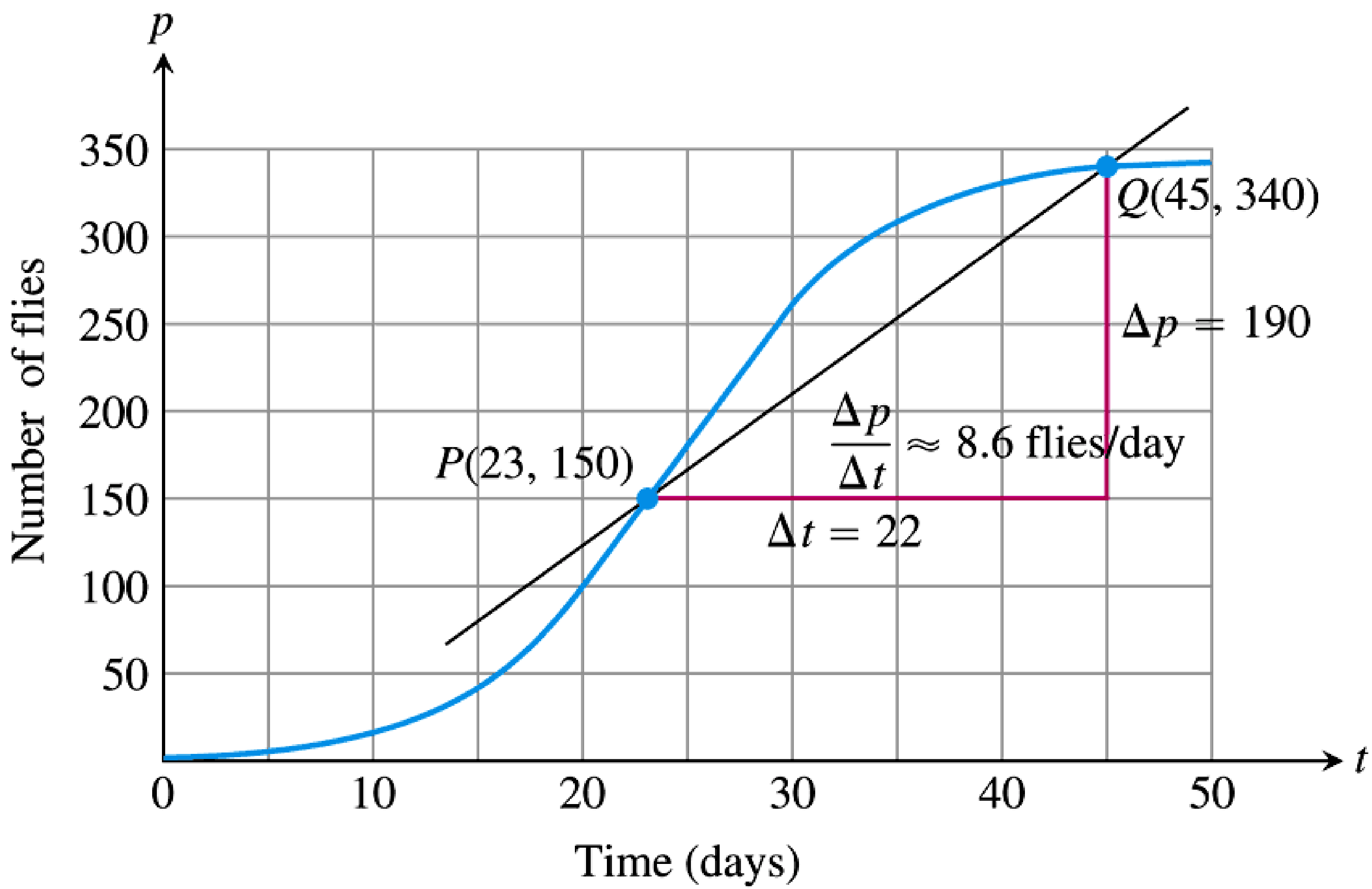


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p / \Delta t$ of the secant line.

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$

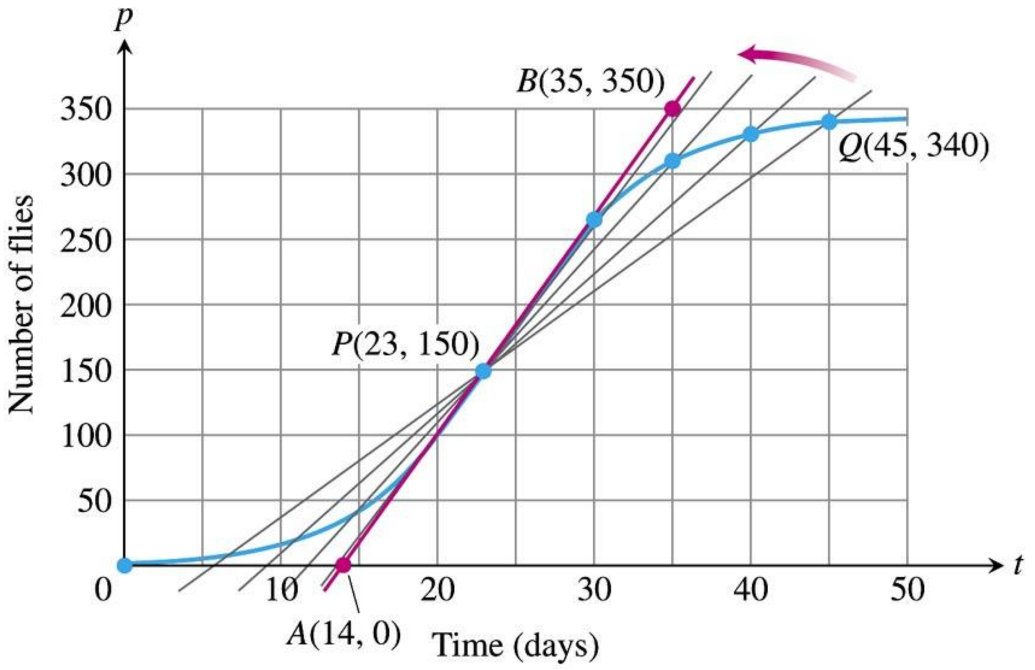
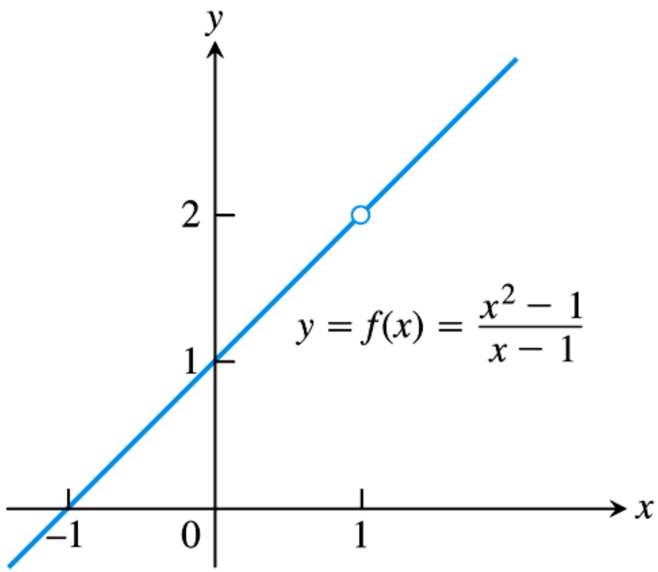


FIGURE 2.3 The positions and slopes of four secants through the point P on the fruit fly graph (Example 4).

TABLE 2.2 The closer x gets to 1, the closer $f(x) = (x^2 - 1)/(x - 1)$ seems to get to 2

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001



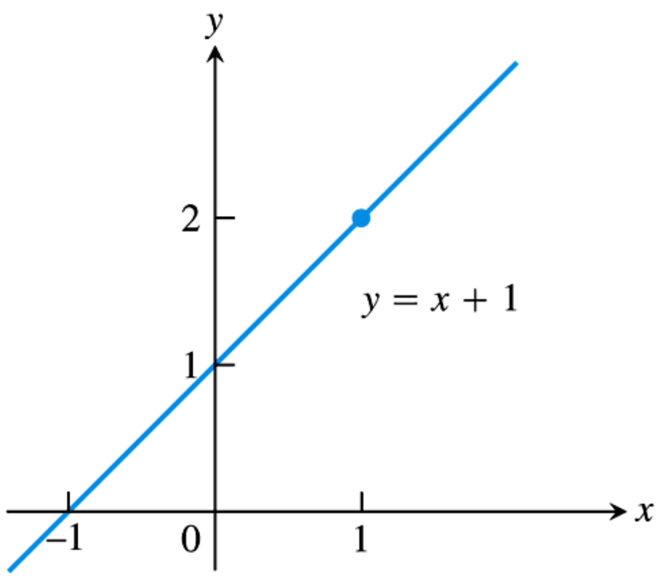
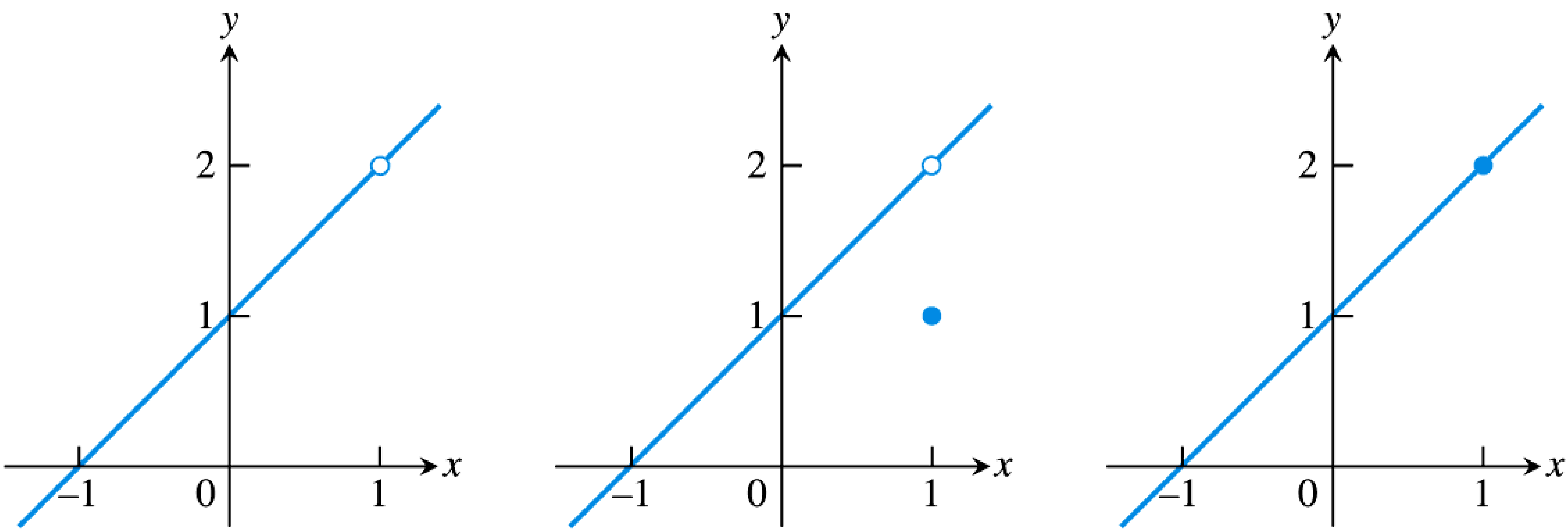


FIGURE 2.4 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 5).

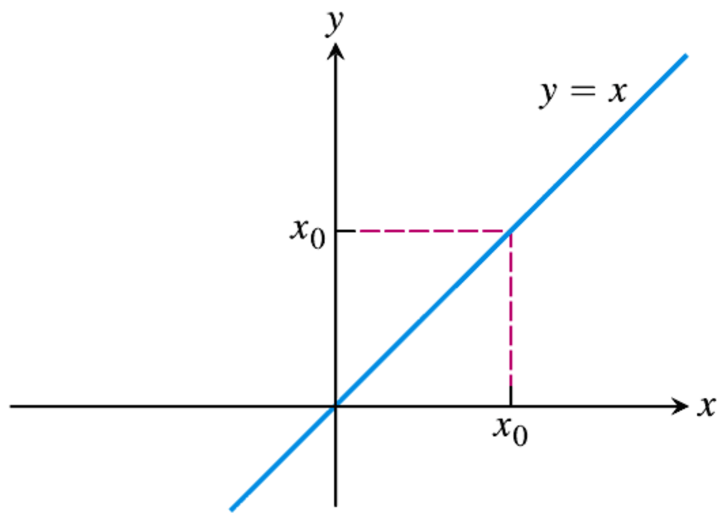


(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

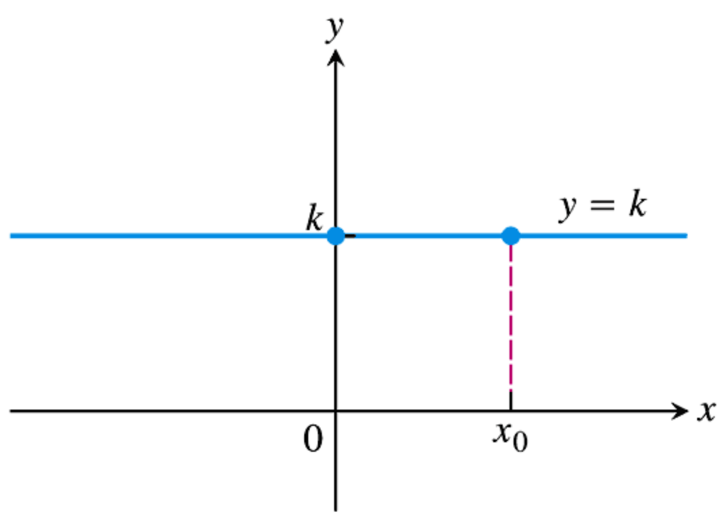
(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

FIGURE 2.5 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 6).

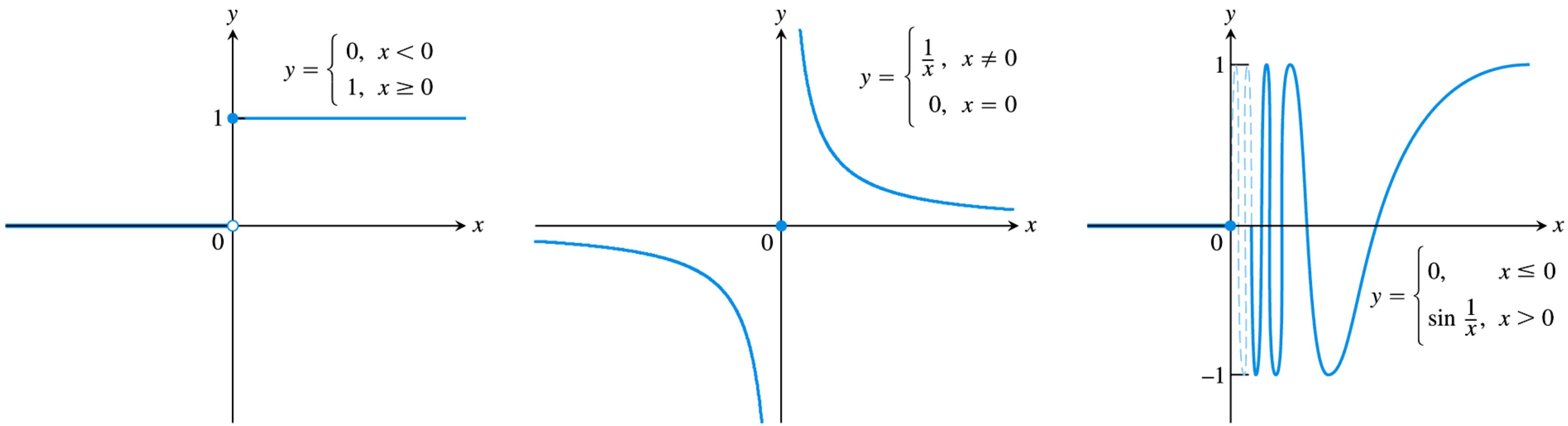


(a) Identity function



(b) Constant function

FIGURE 2.6 The functions in Example 8.



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

FIGURE 2.7 None of these functions has a limit as x approaches 0 (Example 9).

2.2

Calculating Limits Using the Limits Laws

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x \to c} (f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x \to c} (f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

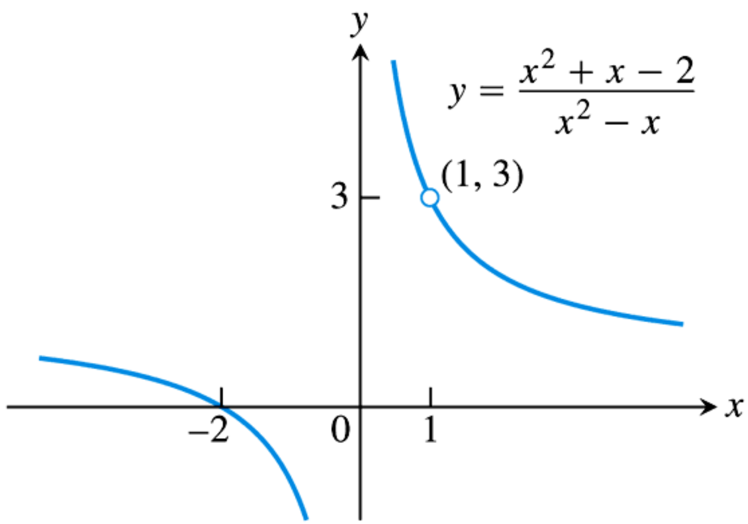
THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

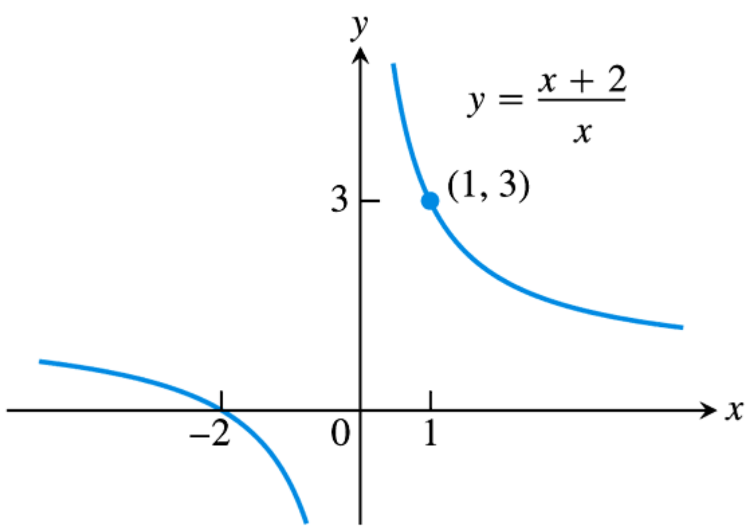
$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Identifying Common Factors

It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.



(a)



(b)

FIGURE 2.8 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 3).

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

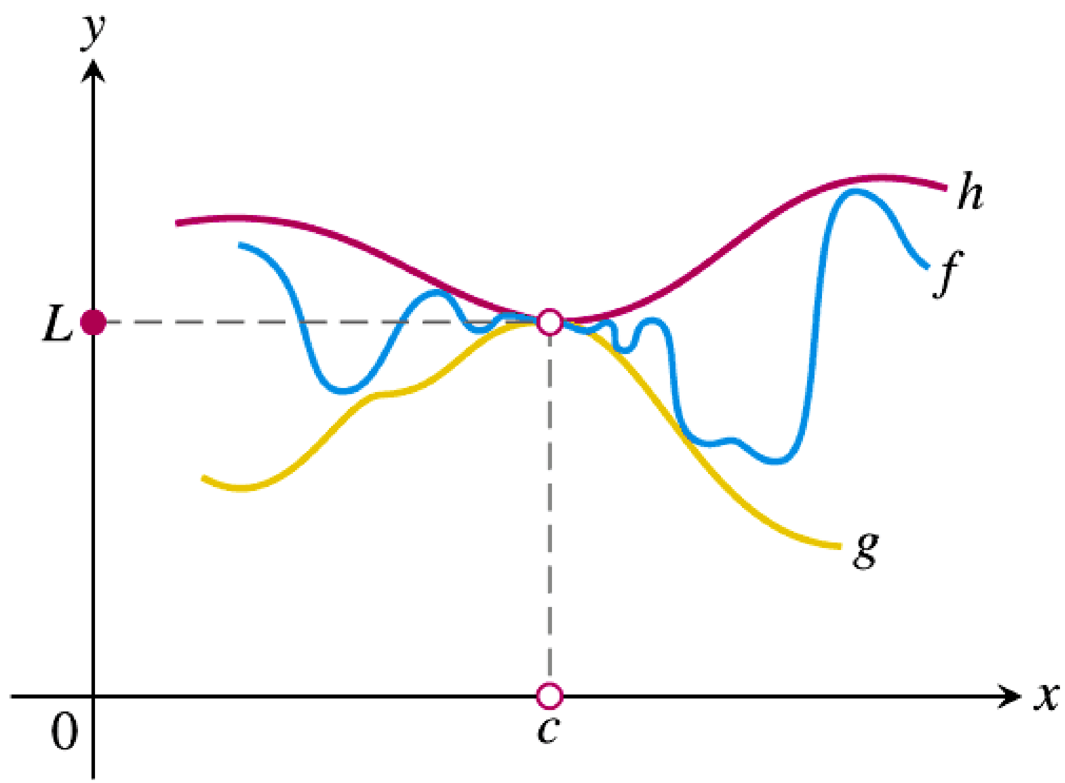


FIGURE 2.9 The graph of f is sandwiched between the graphs of g and h.

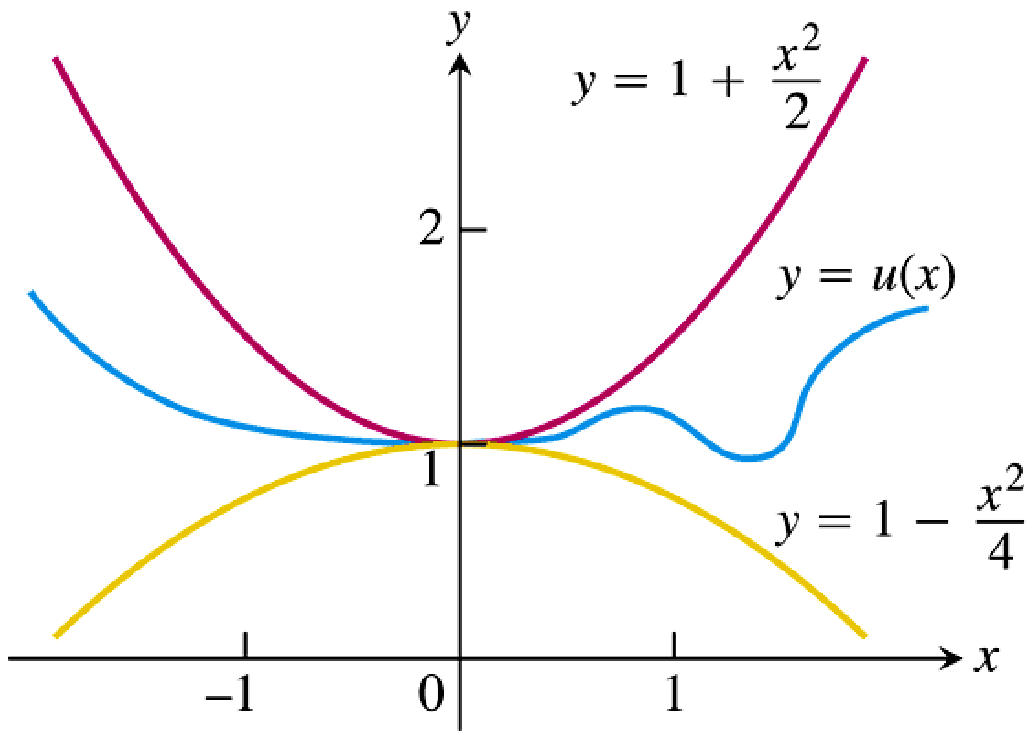


FIGURE 2.10 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 5).

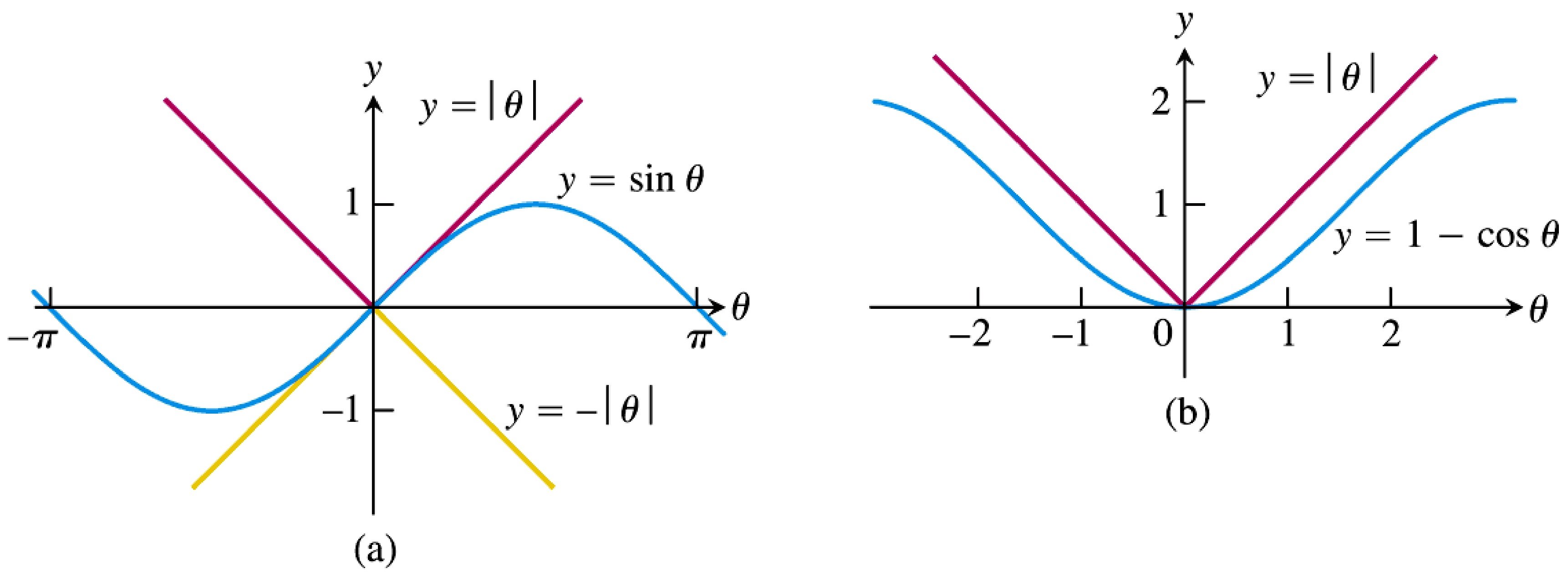


FIGURE 2.11 The Sandwich Theorem confirms that (a) $\lim_{\theta \to 0} \sin \theta = 0$ and (b) $\lim_{\theta \to 0} (1 - \cos \theta) = 0$ (Example 6).

THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

2.3

The Precise Definition of a Limit

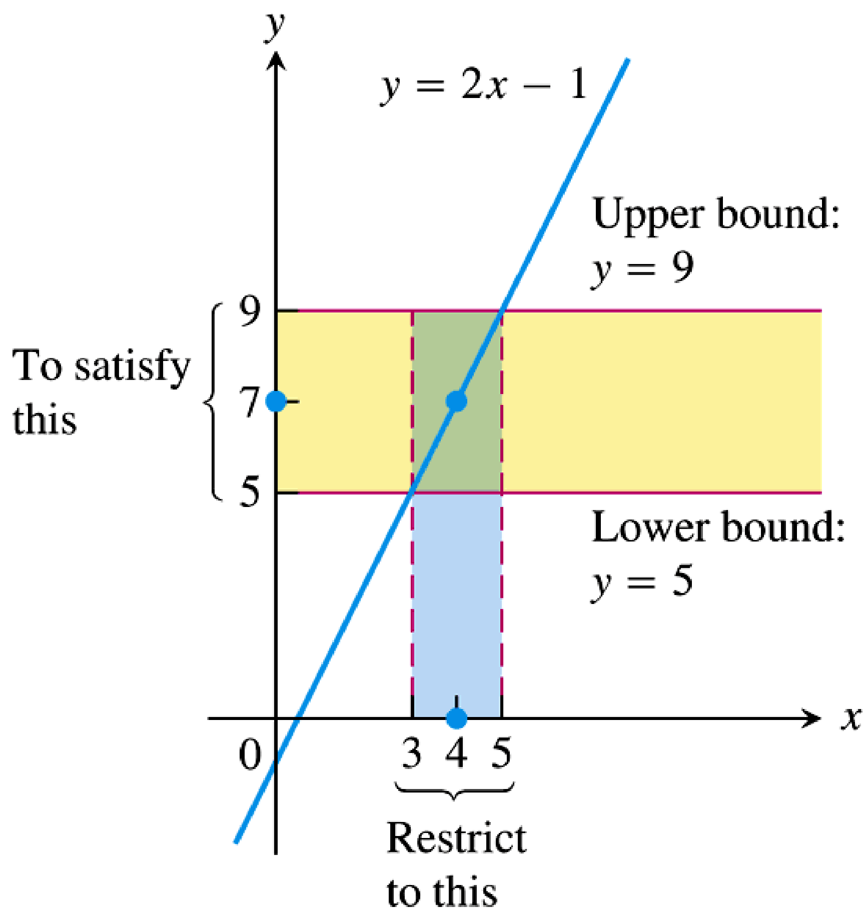


FIGURE 2.12 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

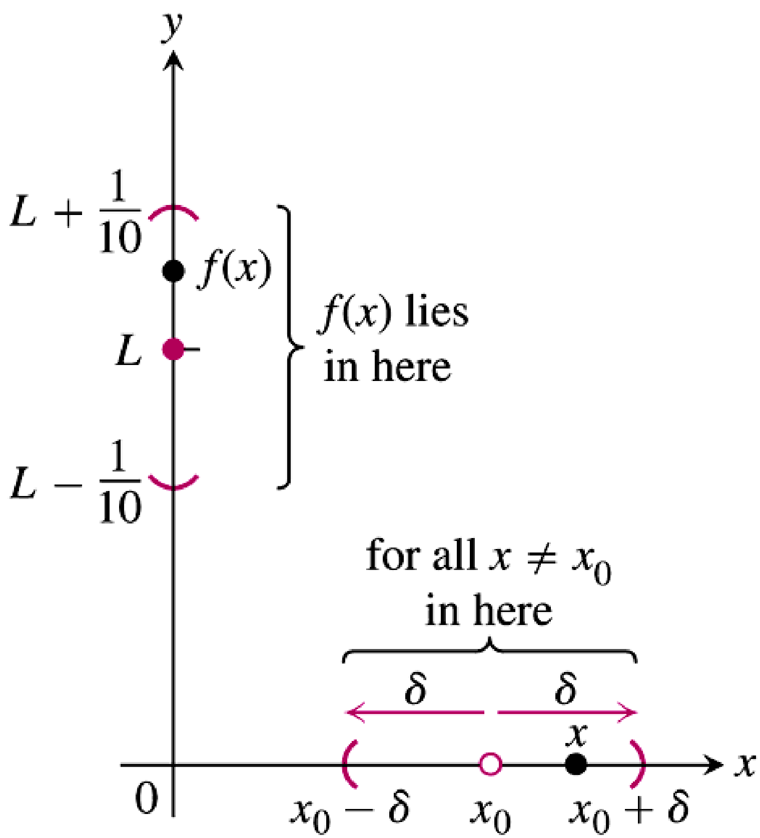


FIGURE 2.13 How should we define $\delta > 0$ so that keeping x within the interval $(x_0 - \delta, x_0 + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

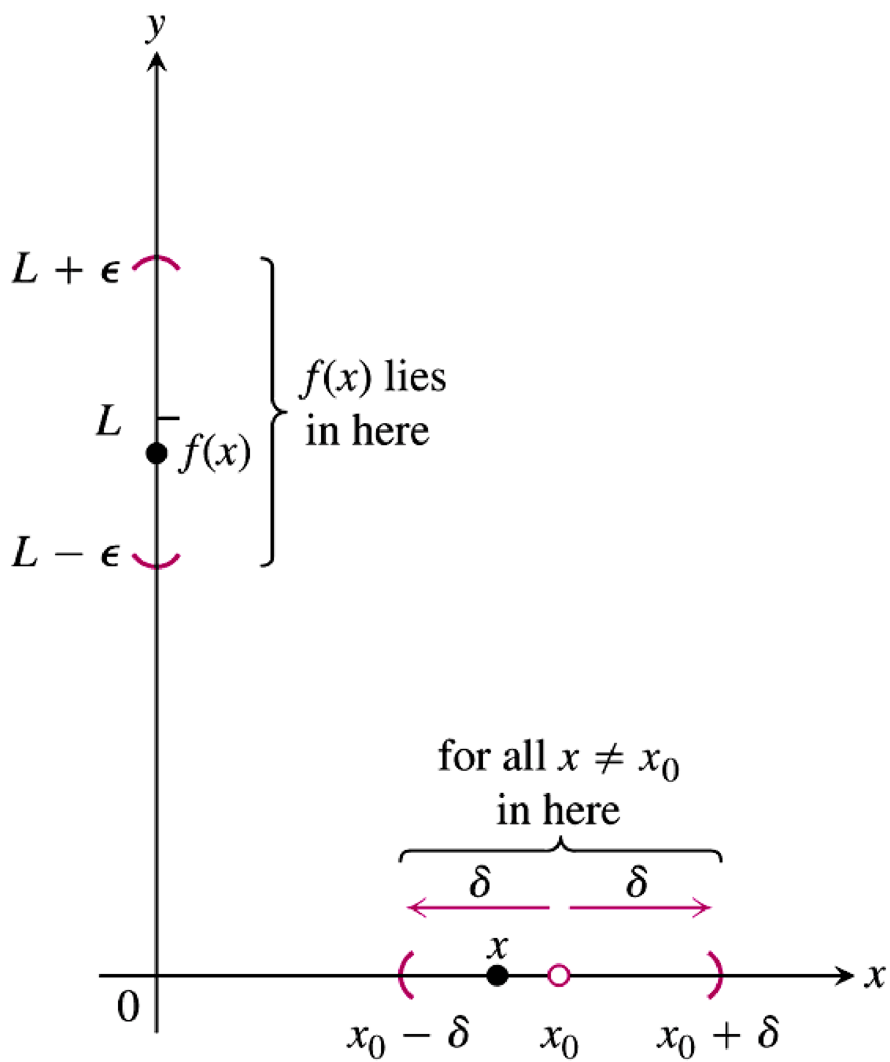


FIGURE 2.14 The relation of δ and ϵ in the definition of limit.

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

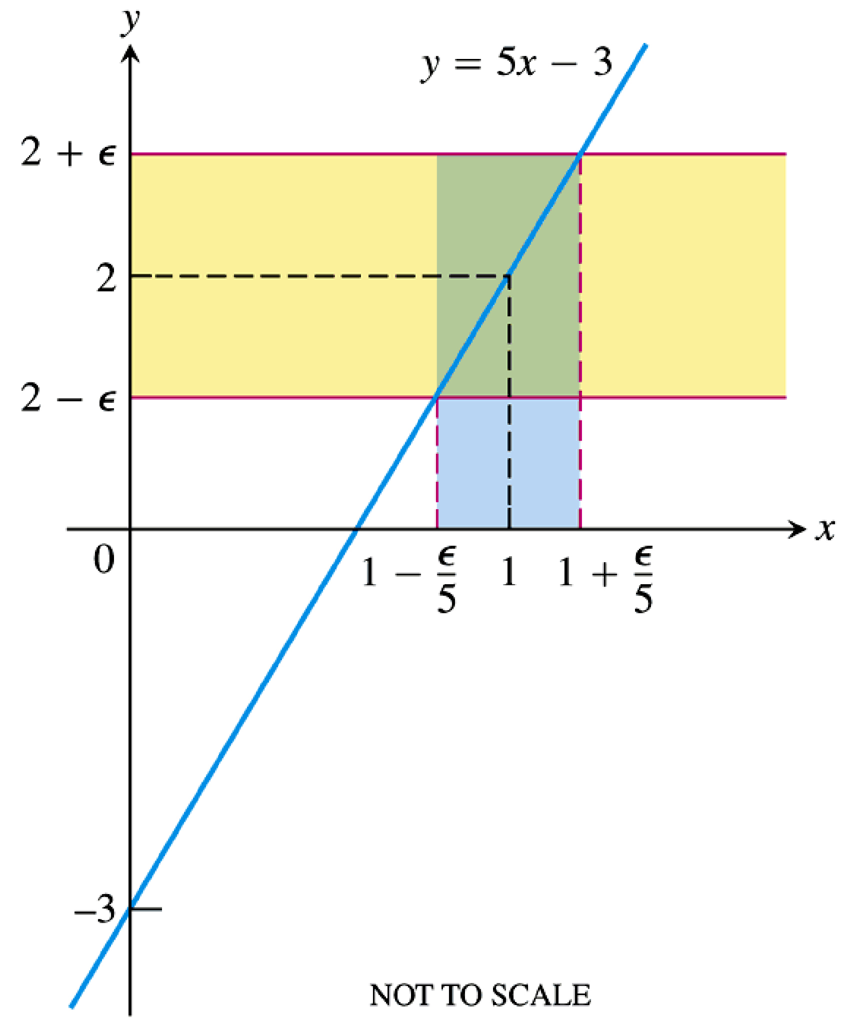


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

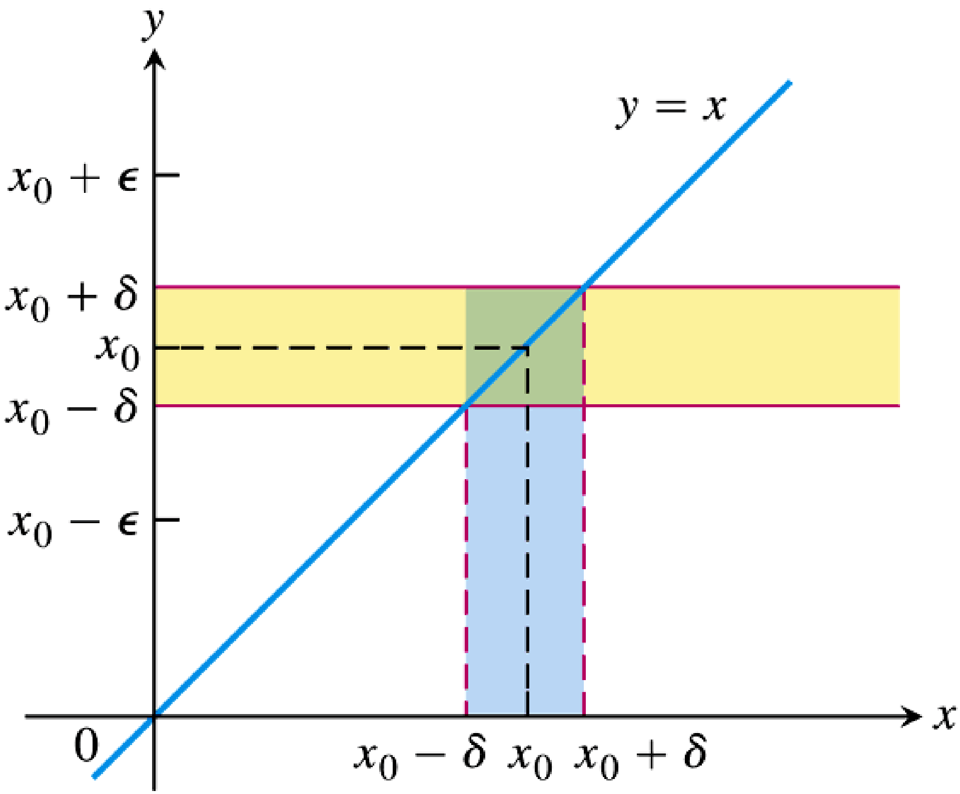


FIGURE 2.16 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \leq \epsilon$ (Example 3a).

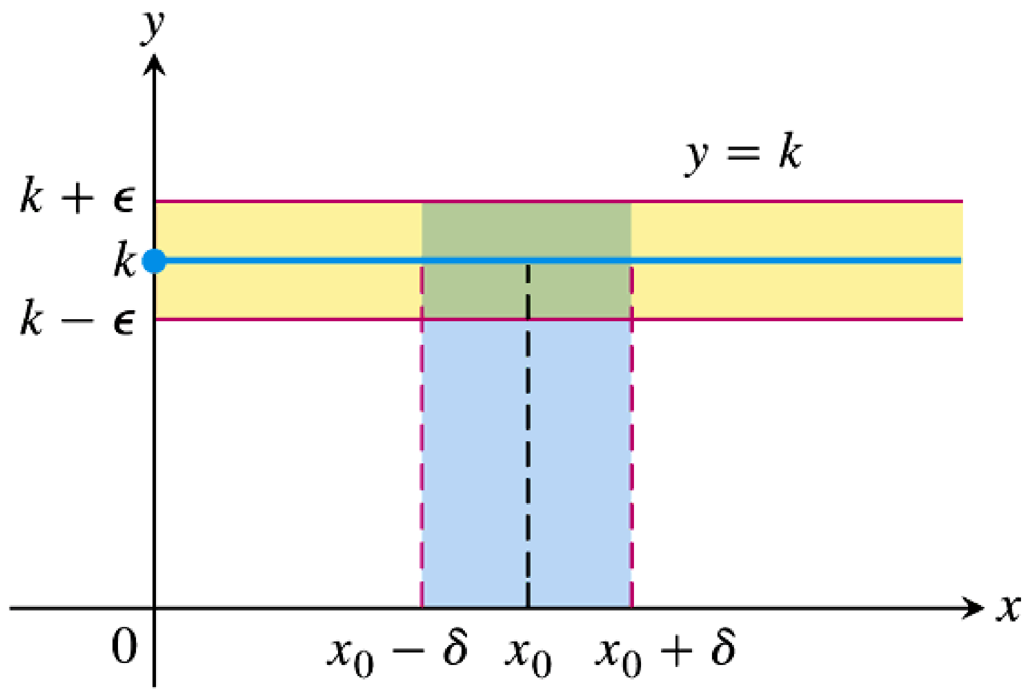


FIGURE 2.17 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).

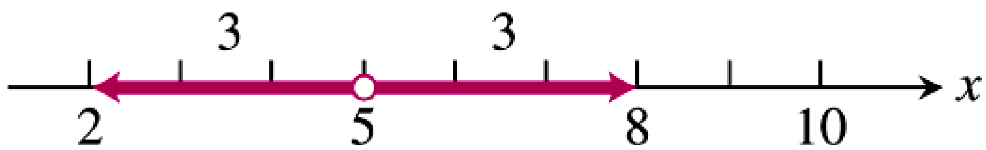


FIGURE 2.18 An open interval of radius 3 about $x_0 = 5$ will lie inside the open interval (2, 10).

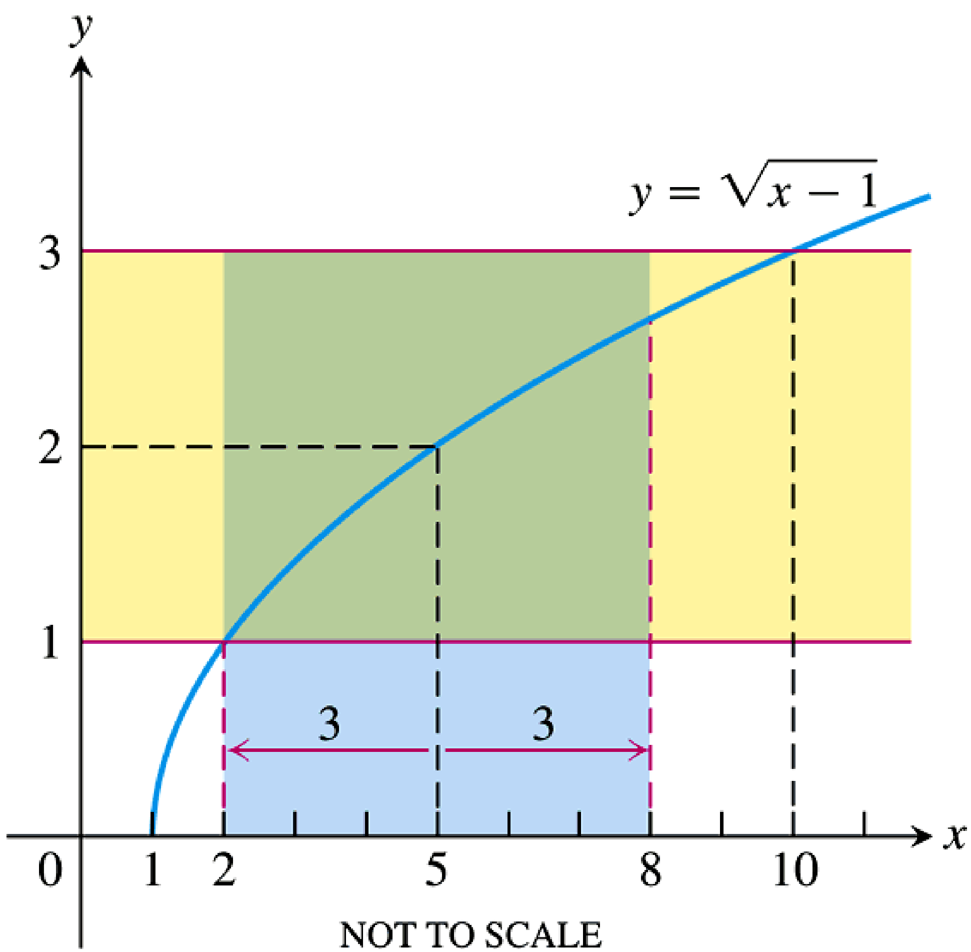


FIGURE 2.19 The function and intervals in Example 4.

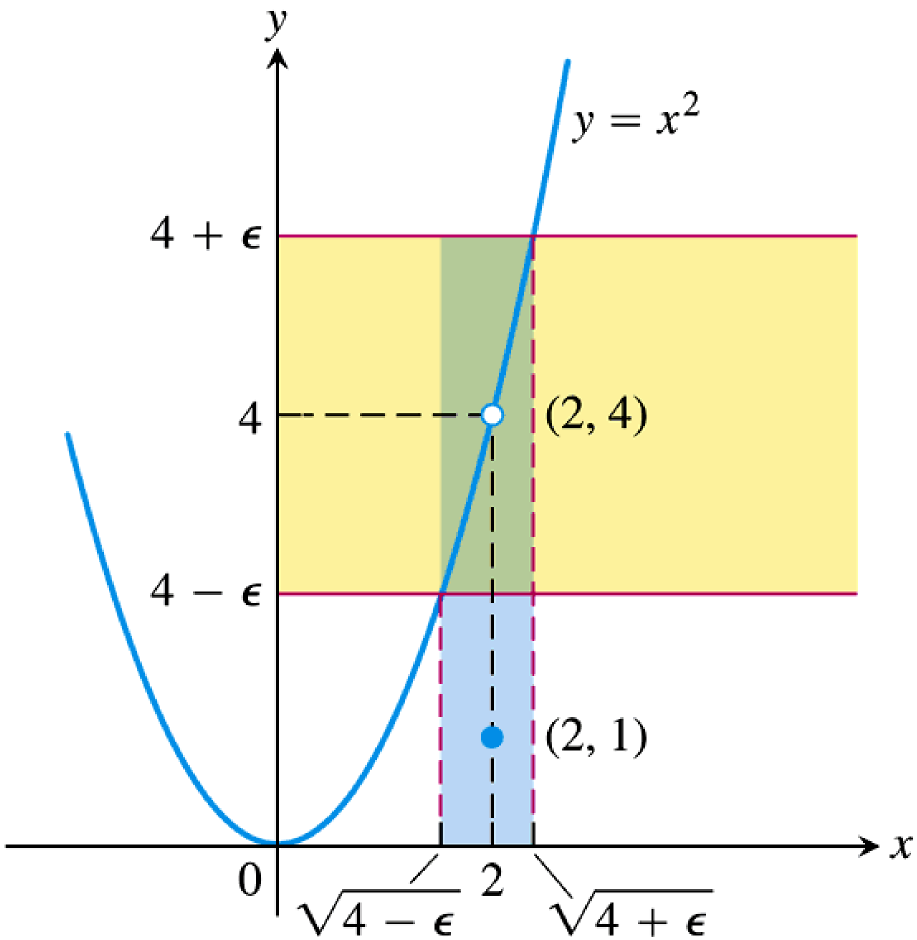


FIGURE 2.20 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \epsilon$.

2.4

One-Sided Limits and Limits at Infinity

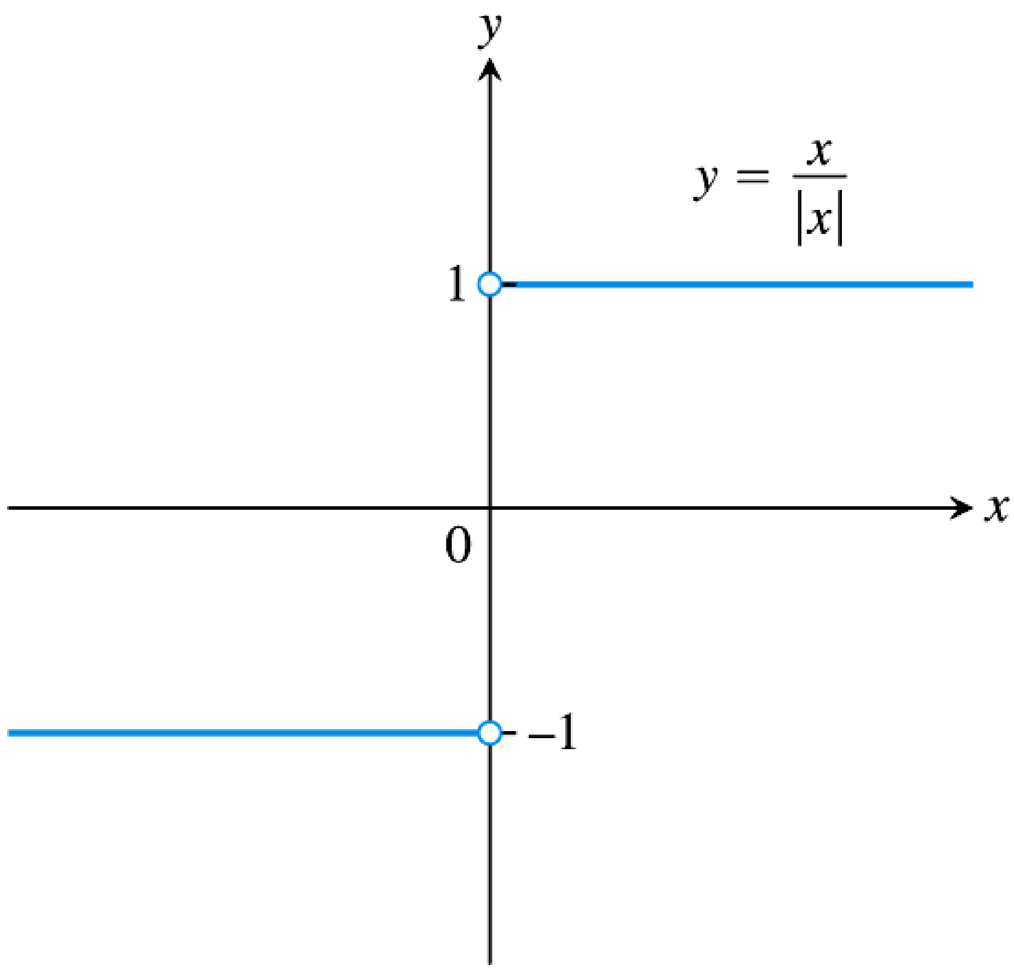


FIGURE 2.21 Different right-hand and left-hand limits at the origin.

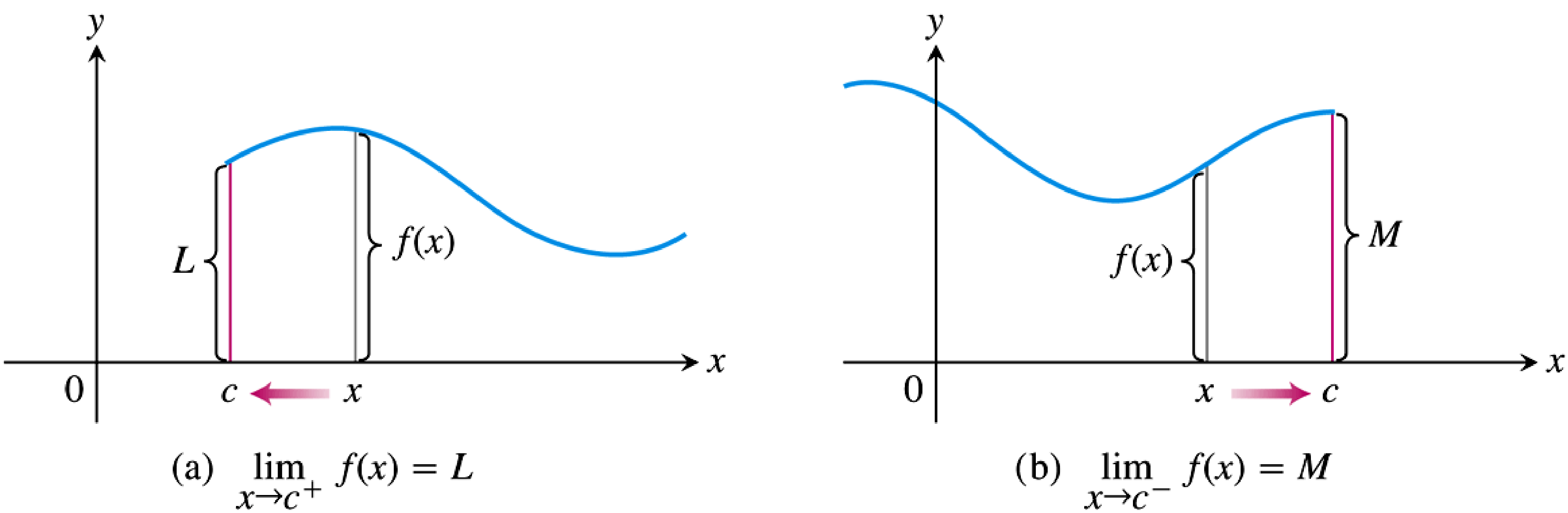


FIGURE 2.22 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

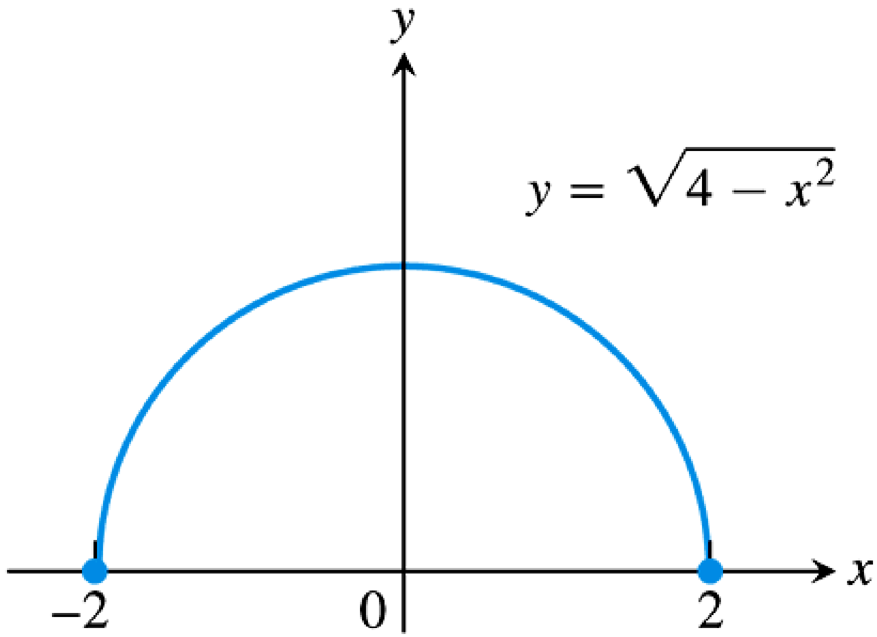


FIGURE 2.23 $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

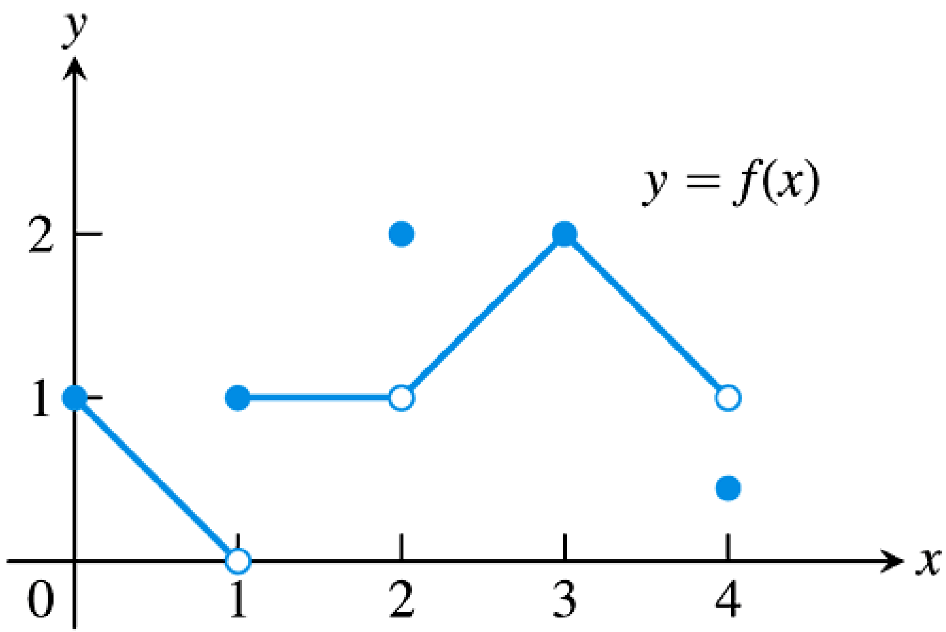


FIGURE 2.24 Graph of the function in Example 2.

DEFINITIONS Right-Hand, Left-Hand Limits

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

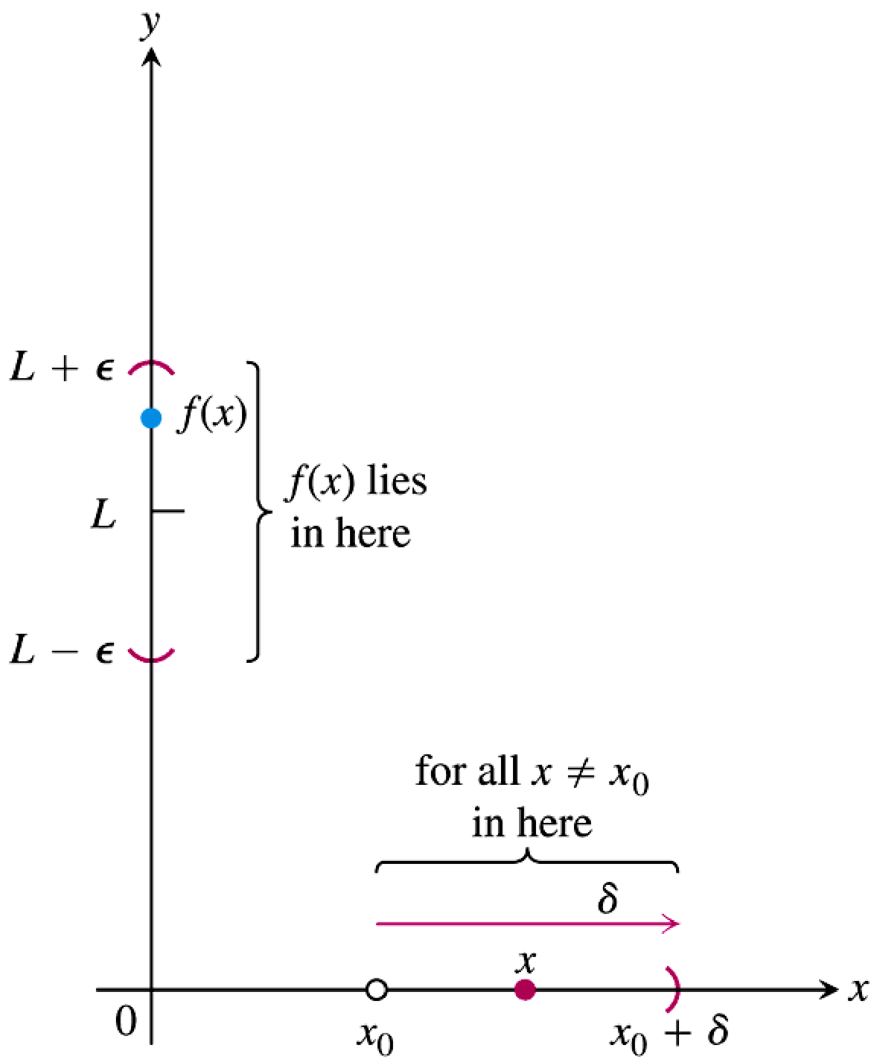


FIGURE 2.25 Intervals associated with the definition of right-hand limit.

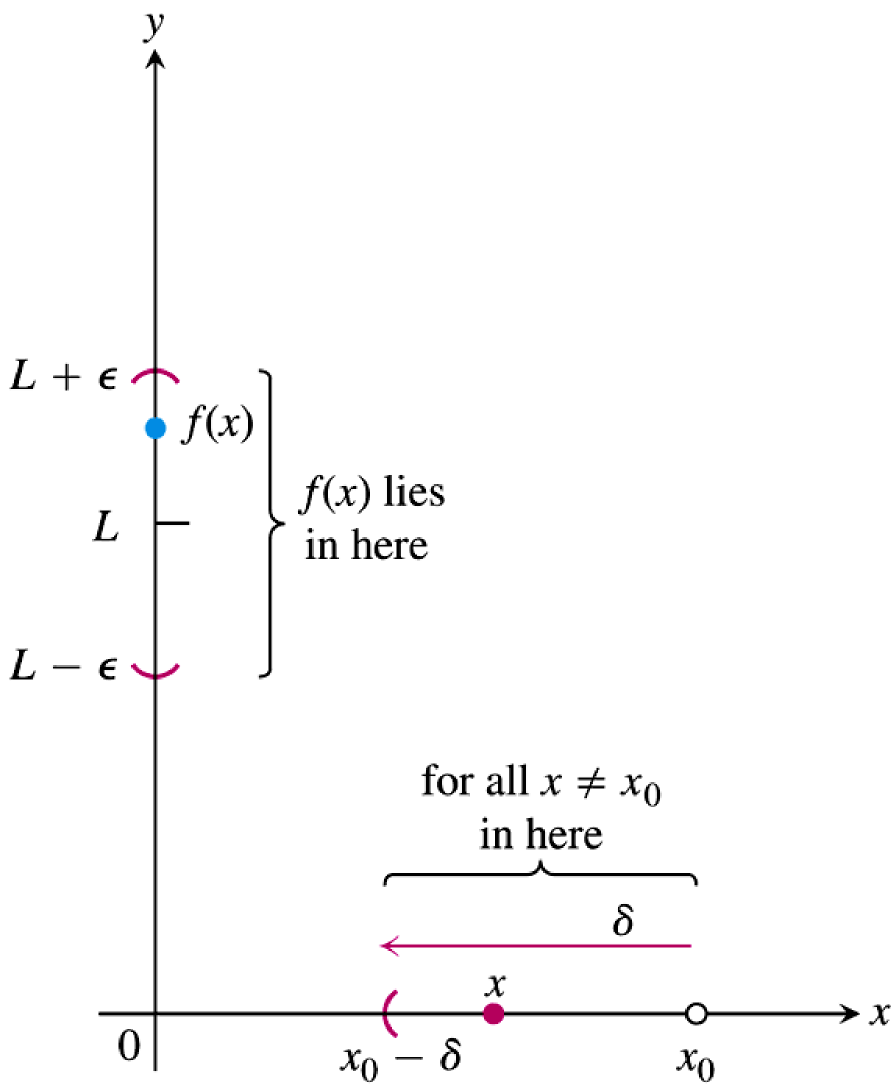


FIGURE 2.26 Intervals associated with the definition of left-hand limit.

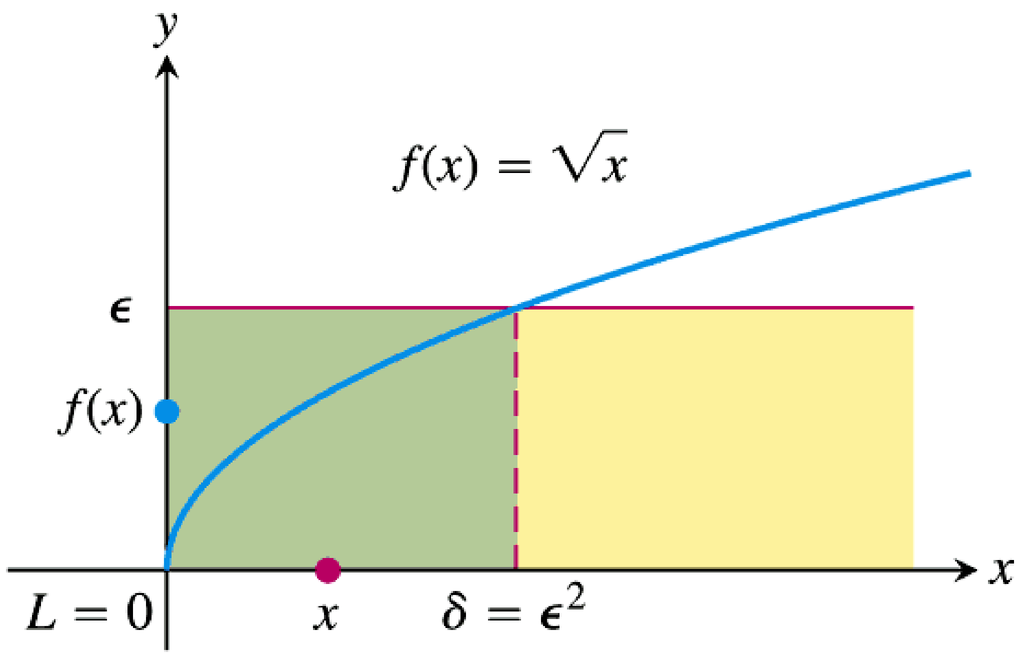


FIGURE 2.27 $\lim_{x \to 0^+} \sqrt{x} = 0$ in Example 3.

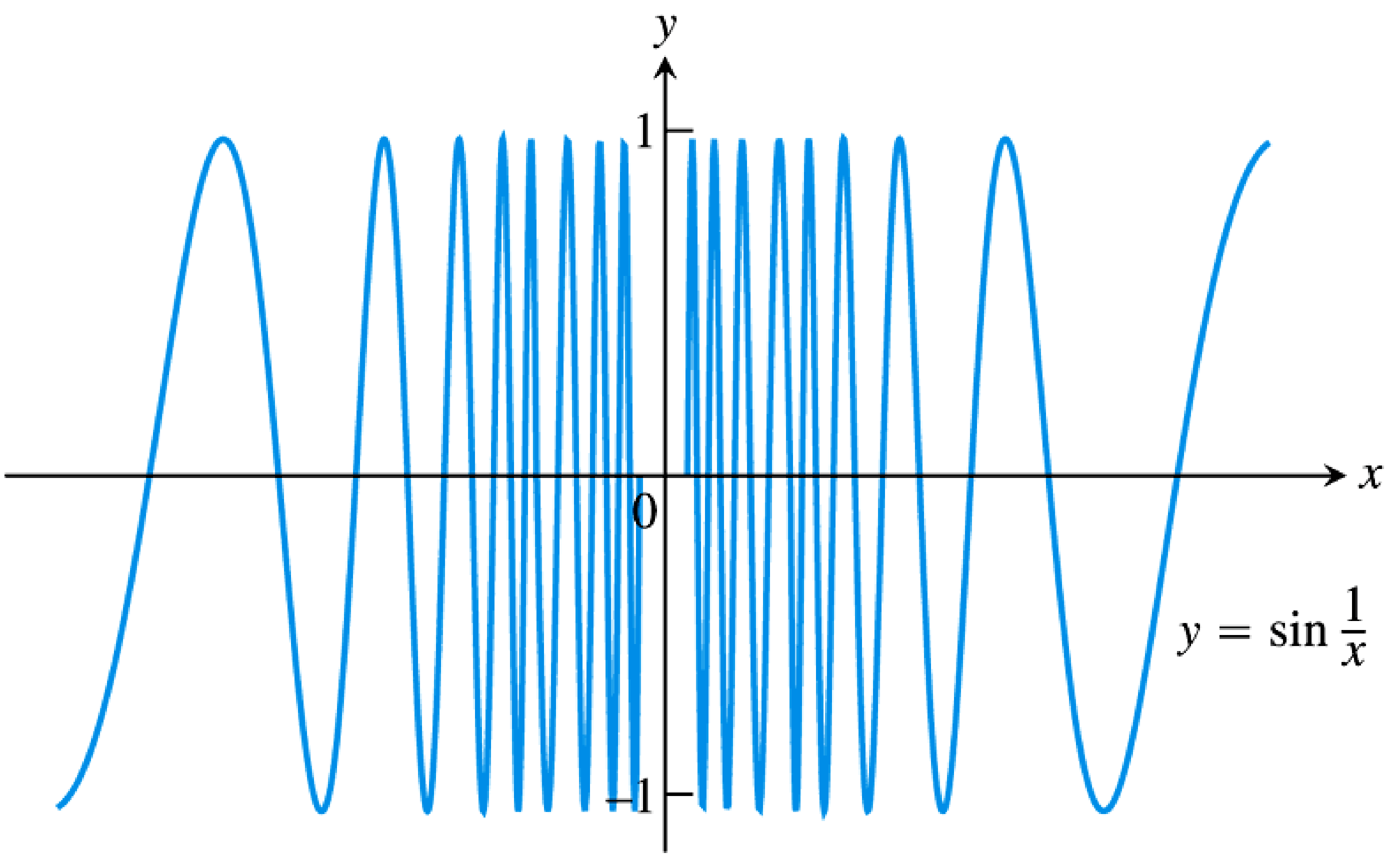


FIGURE 2.28 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4).

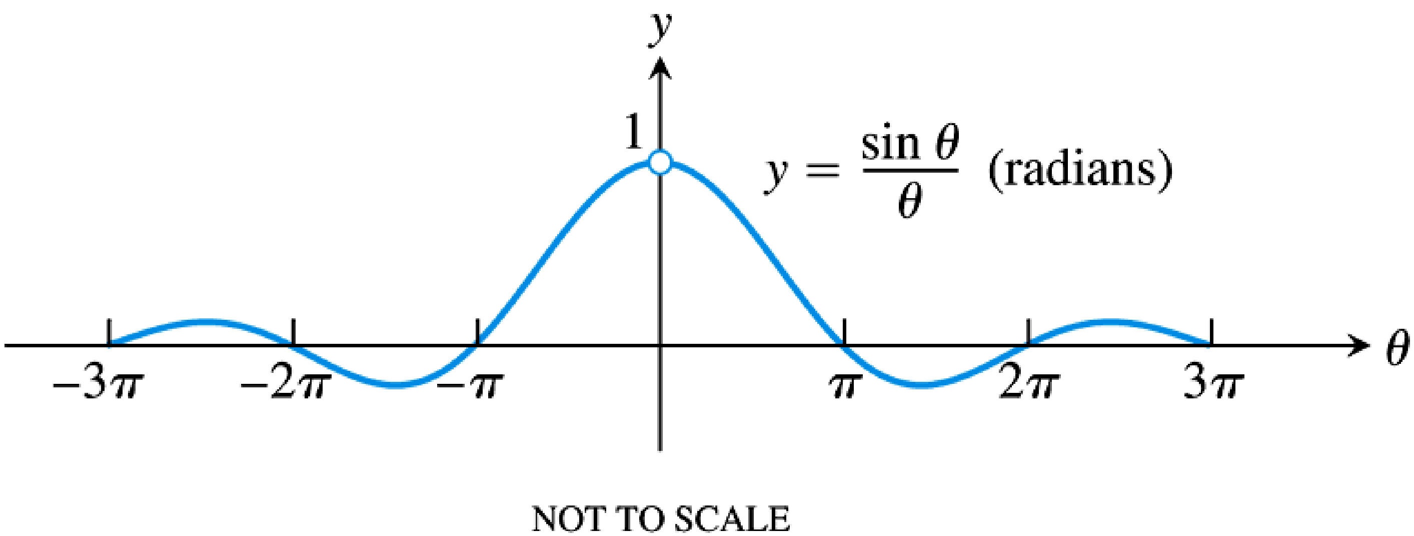


FIGURE 2.29 The graph of $f(\theta) = (\sin \theta)/\theta$.

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

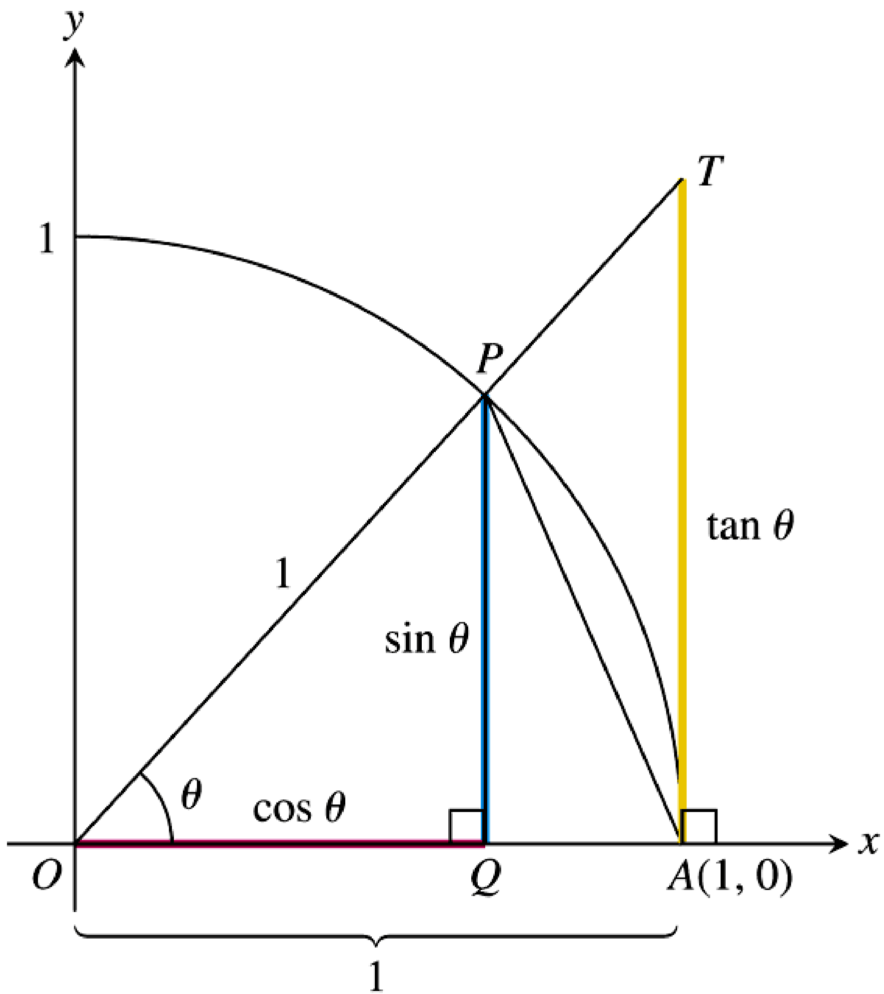


FIGURE 2.30 The figure for the proof of Theorem 7. $TA/OA = \tan\theta$, but $OA = 1$, so $TA = \tan\theta$.

DEFINITIONS **Limit as x approaches ∞ or $-\infty$**

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

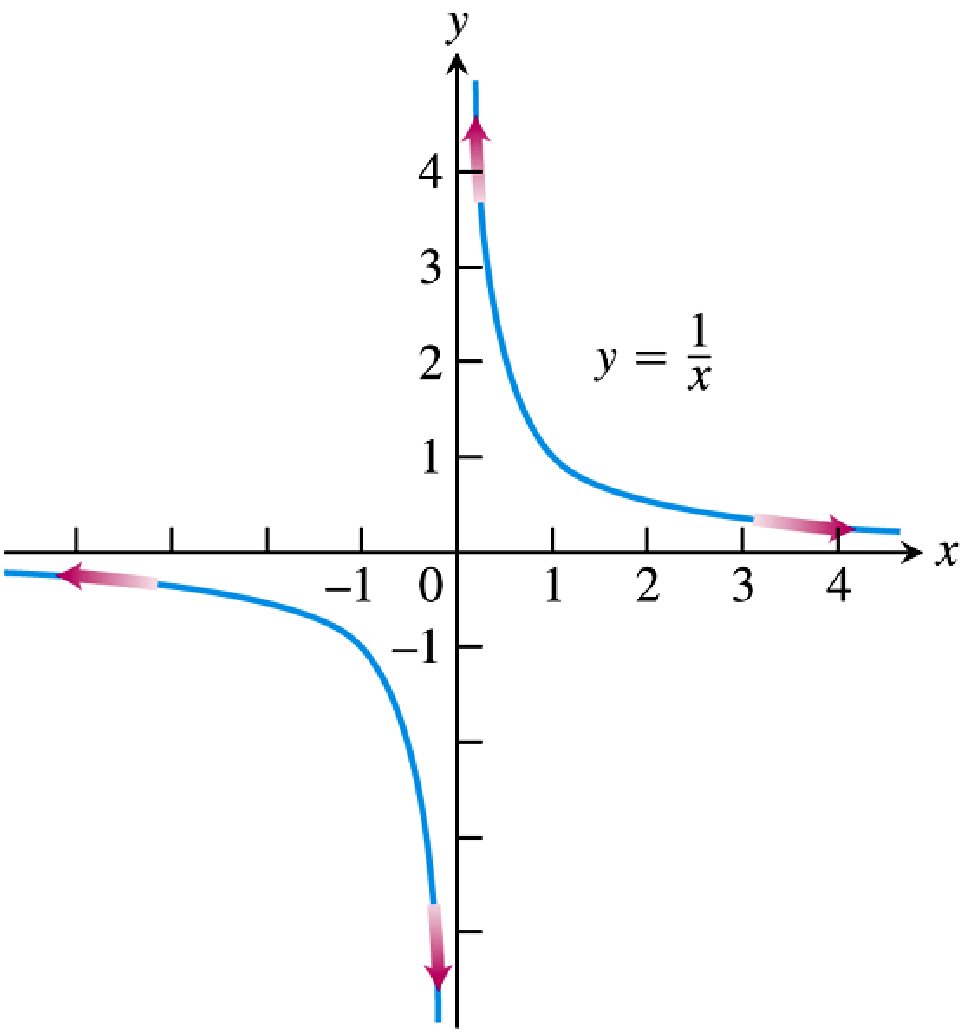


FIGURE 2.31 The graph of $y = 1/x$.

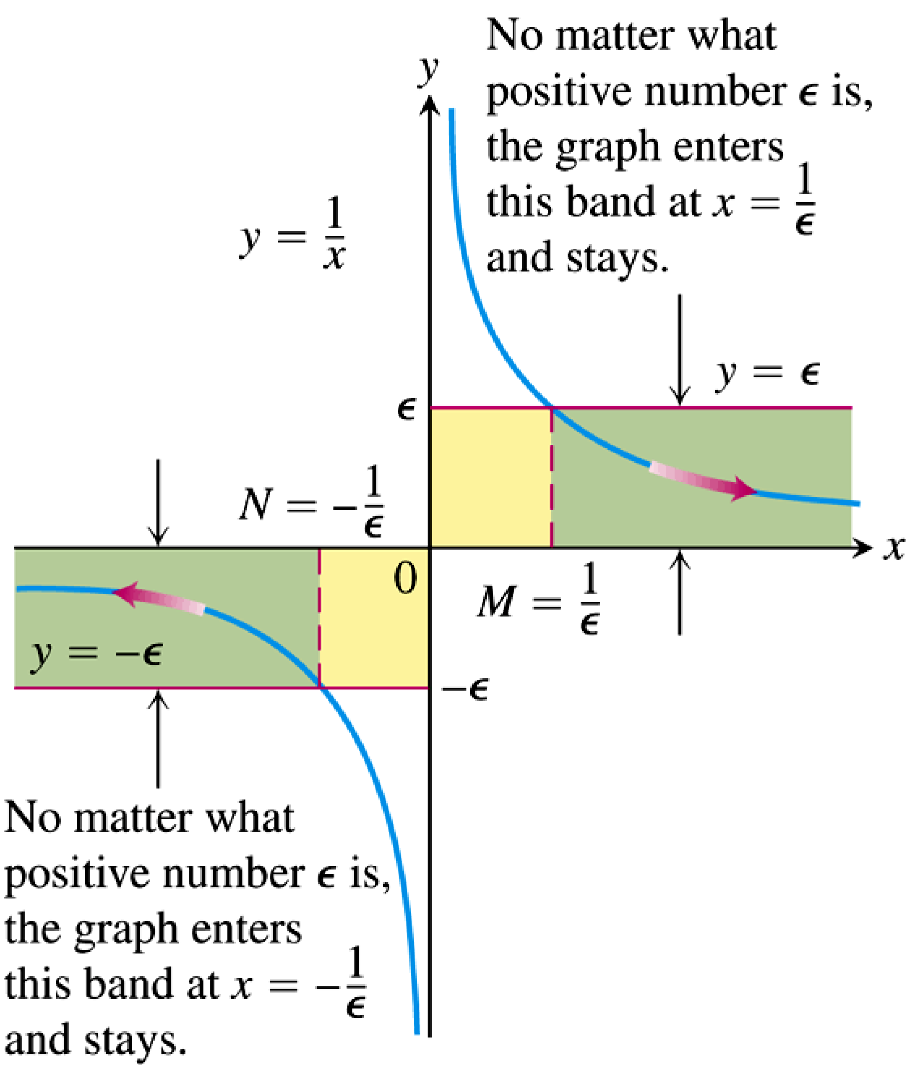


FIGURE 2.32 The geometry behind the argument in Example 6.

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $$\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$$
2. *Difference Rule:* $$\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$$
3. *Product Rule:* $$\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$$
4. *Constant Multiple Rule:* $$\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$$
5. *Quotient Rule:* $$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

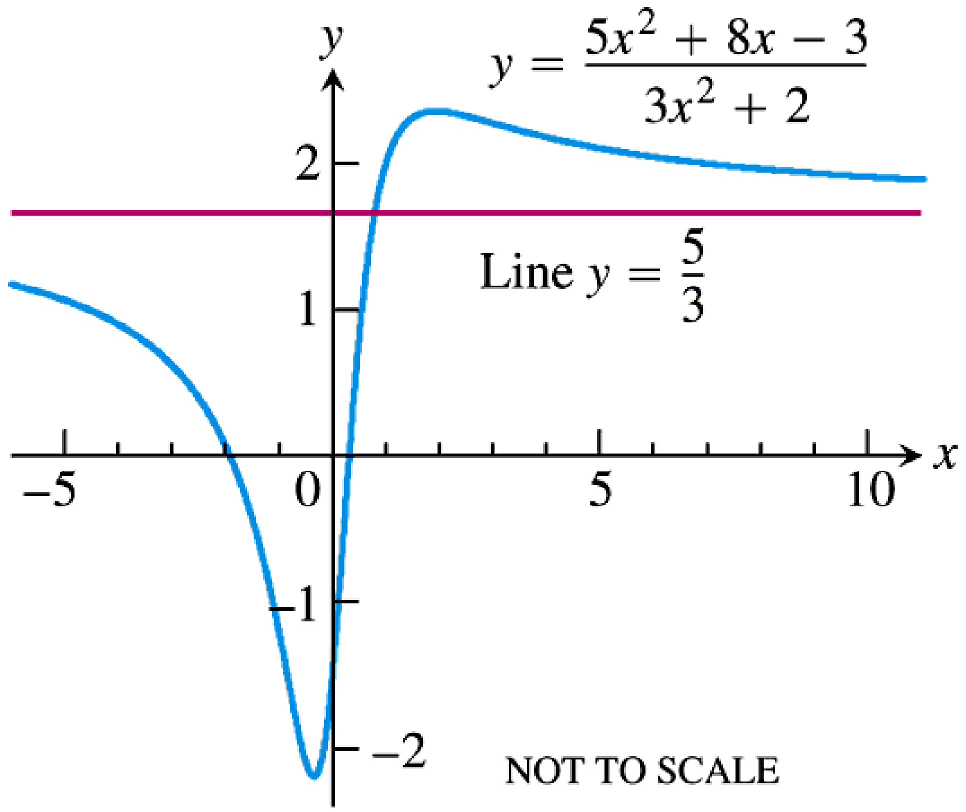


FIGURE 2.33 The graph of the function in Example 8. The graph approaches the line $y = 5/3$ as $|x|$ increases.

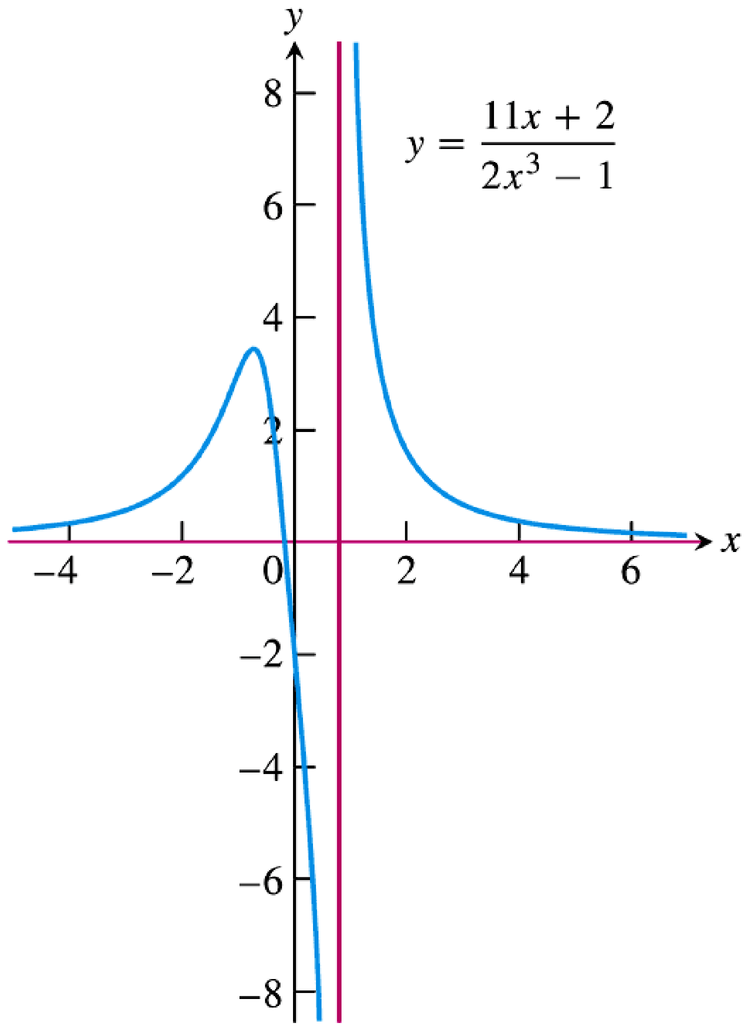


FIGURE 2.34 The graph of the function in Example 9. The graph approaches the x-axis as $|x|$ increases.

DEFINITION **Horizontal Asymptote**

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

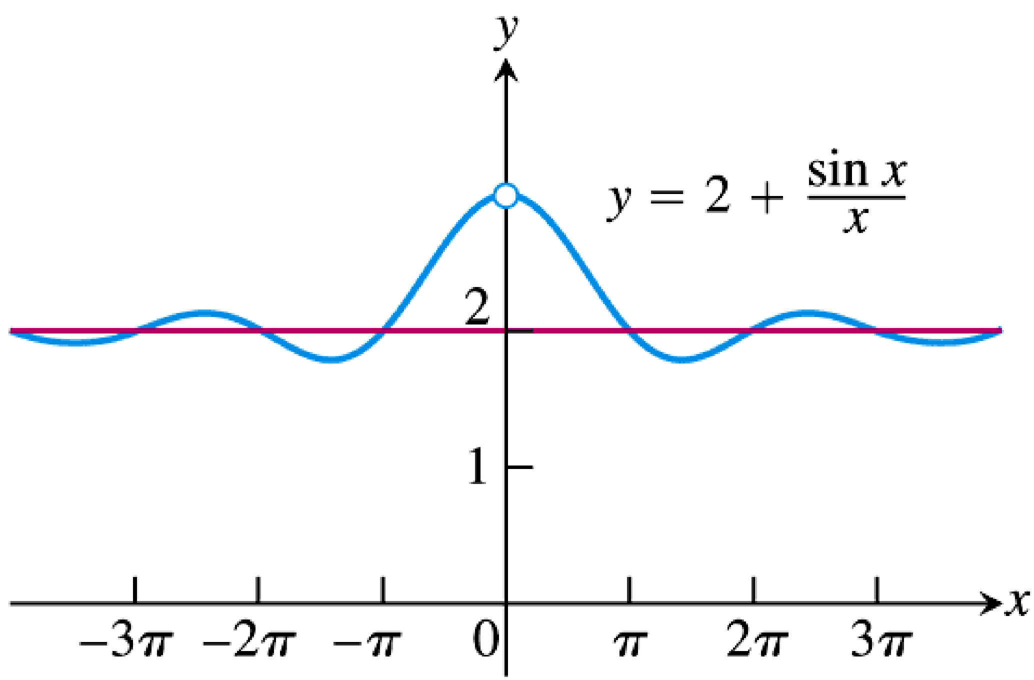


FIGURE 2.35 A curve may cross one of its asymptotes infinitely often (Example 11).

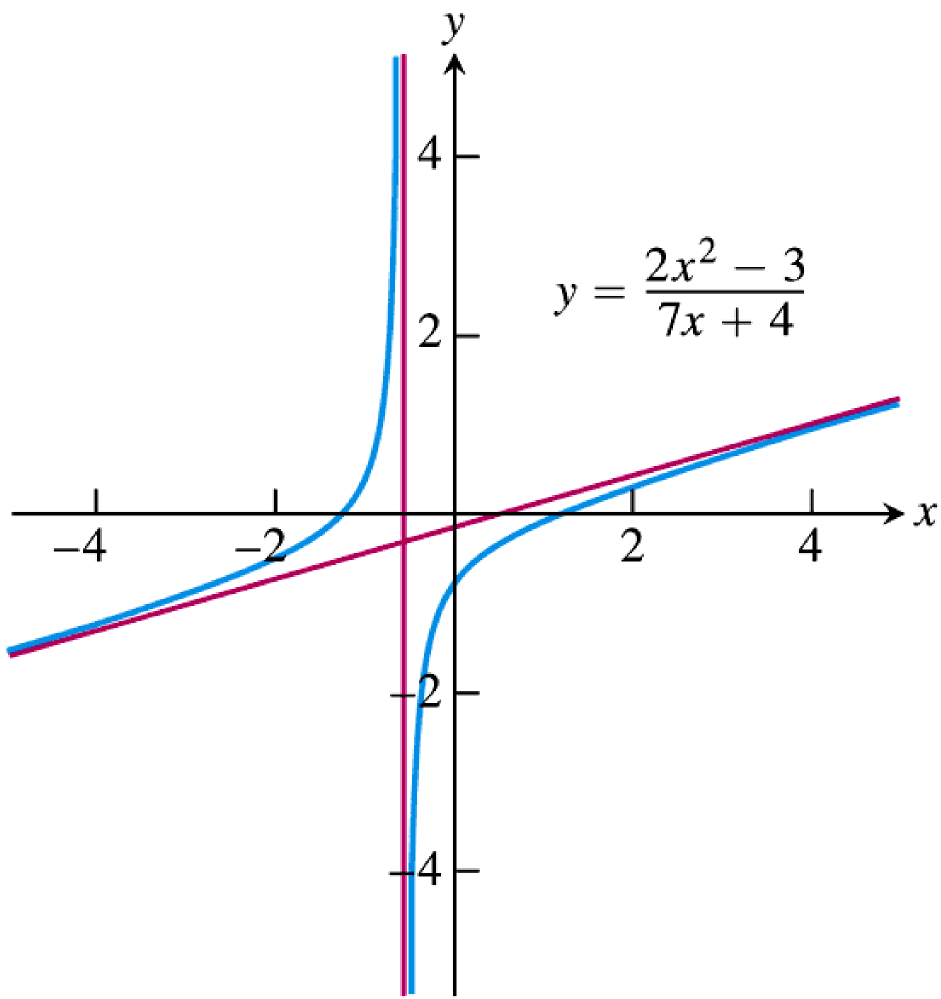


FIGURE 2.36 The function in Example 12 has an oblique asymptote.

2.5

Infinite Limits and Vertical Asymptotes

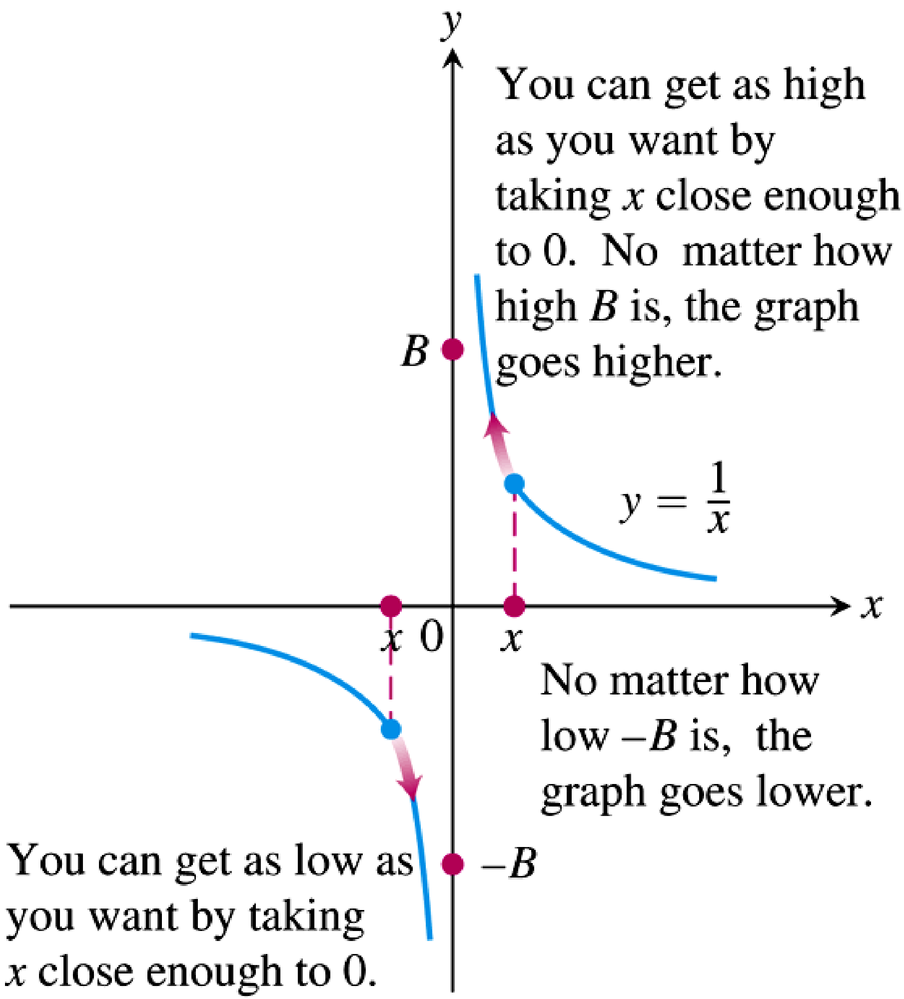


FIGURE 2.37 One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

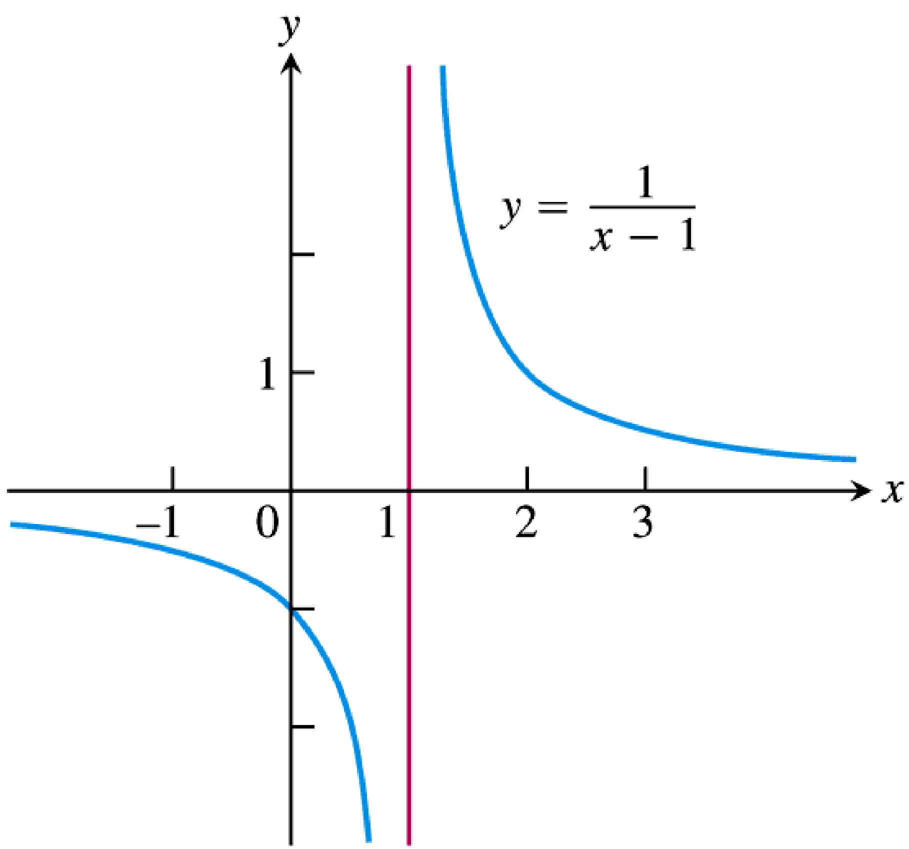


FIGURE 2.38 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 1).

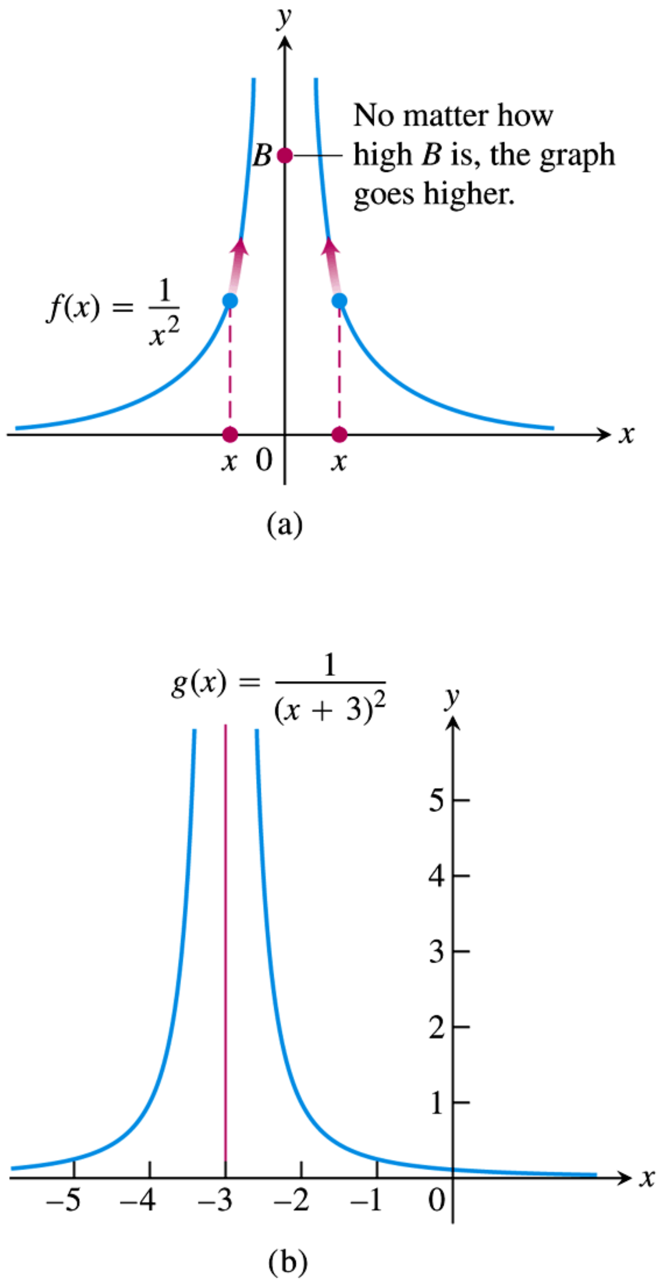


FIGURE 2.39 The graphs of the functions in Example 2. (a) $f(x)$ approaches infinity as $x \to 0$. (b) $g(x)$ approaches infinity as $x \to -3$.

DEFINITIONS Infinity, Negative Infinity as Limits

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) > B.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) < -B.$$

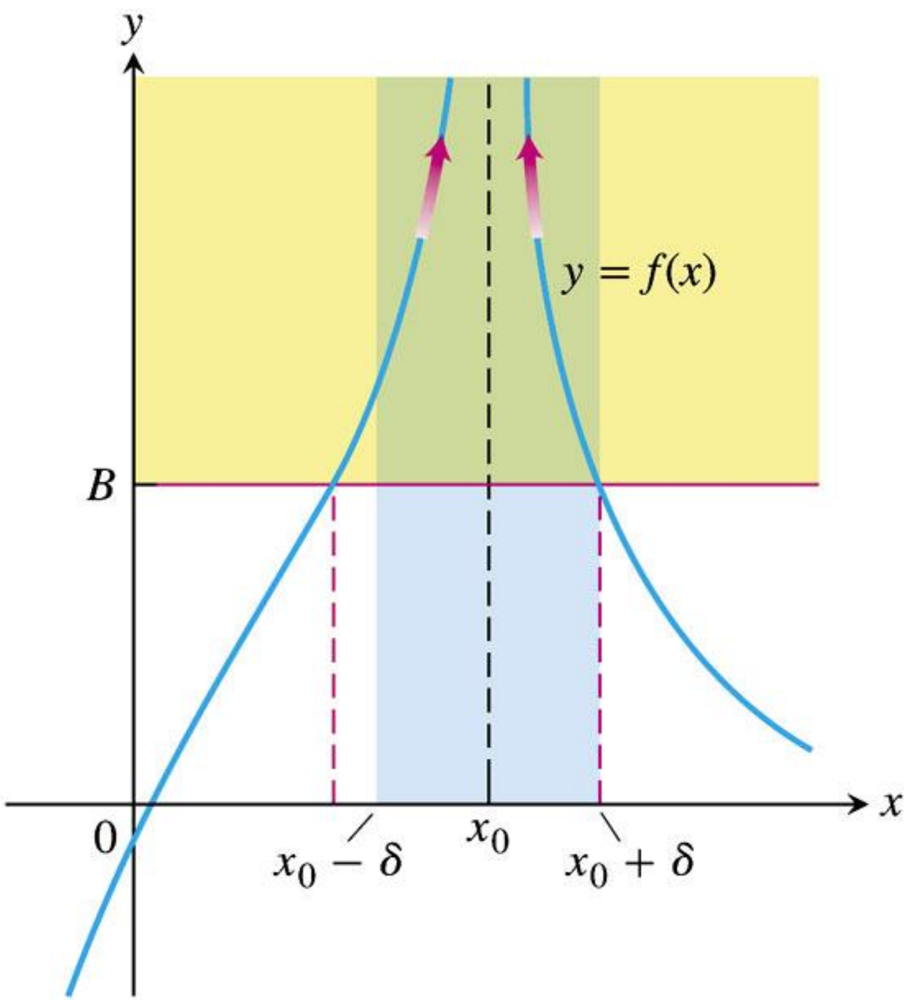


FIGURE 2.40 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies above the line $y = B$.

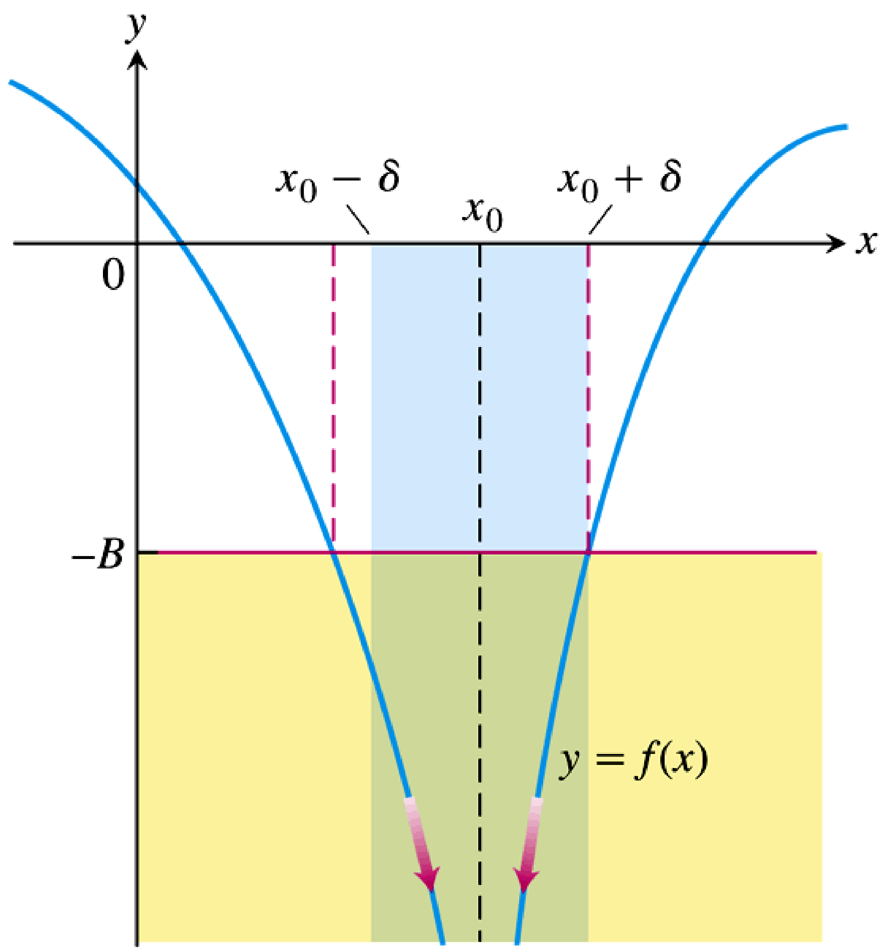


FIGURE 2.41 For $x_0 - \delta < x < x_0 + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITION Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

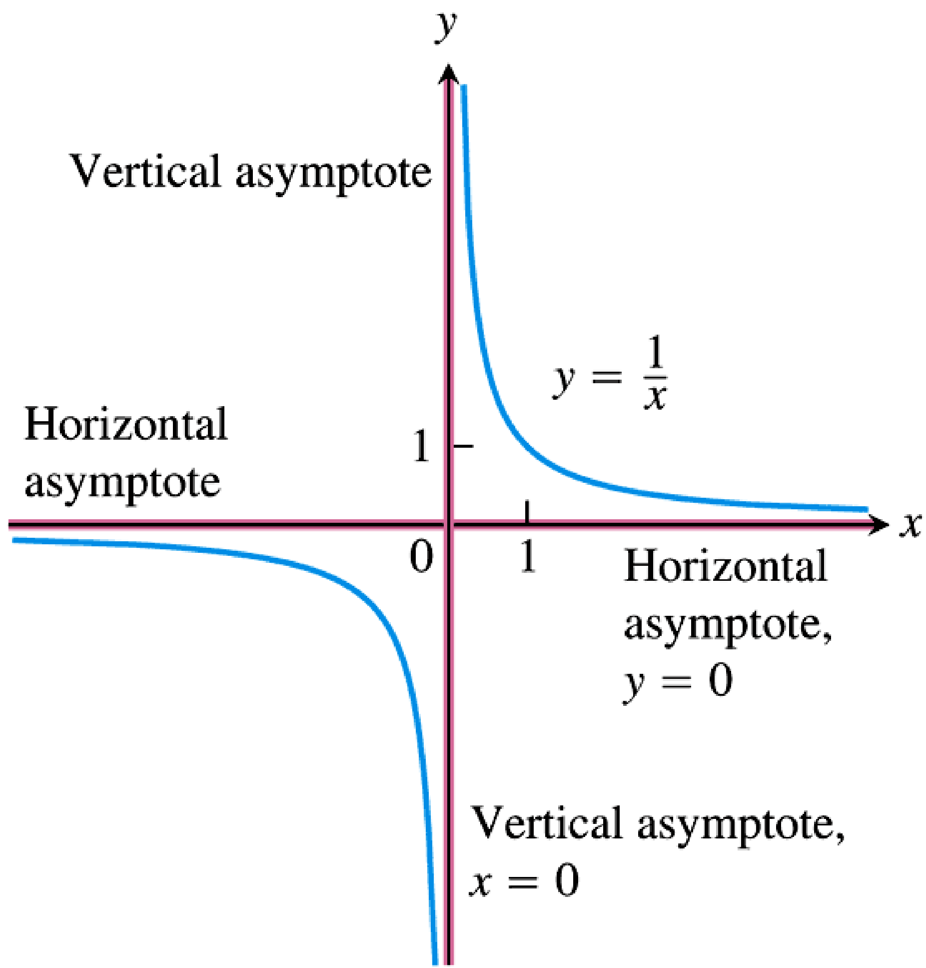


FIGURE 2.42 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

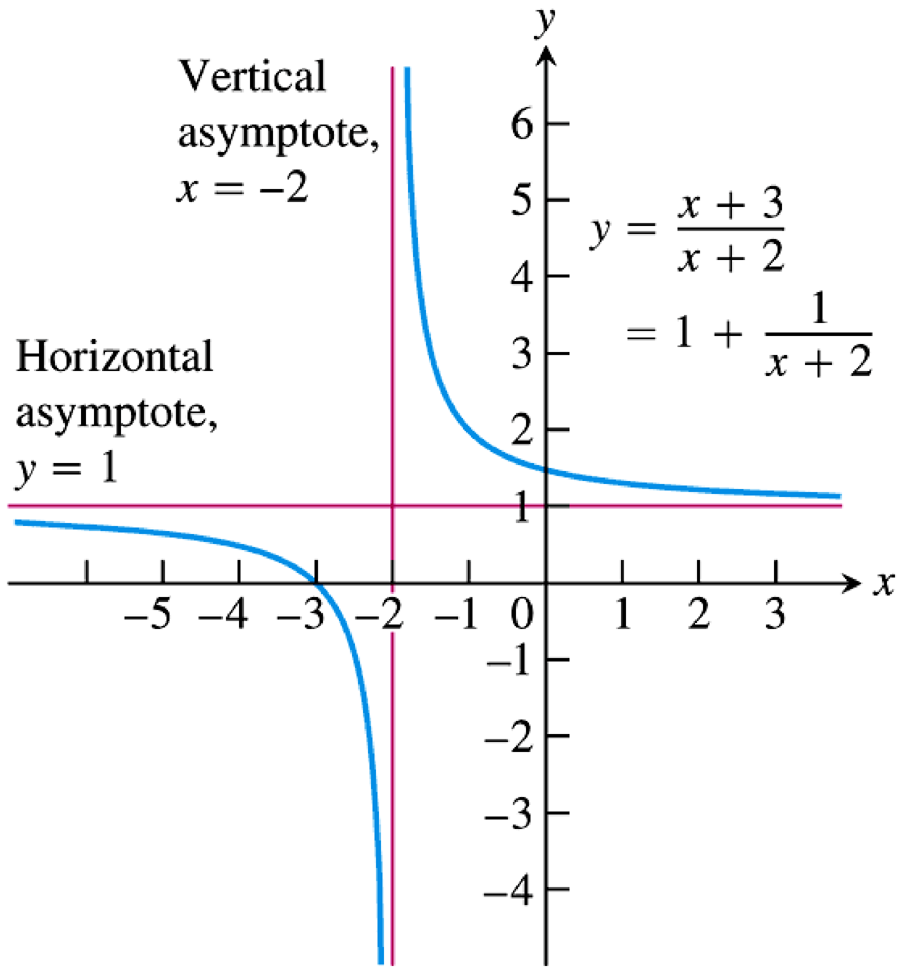


FIGURE 2.43 The lines $y = 1$ and $x = -2$ are asymptotes of the curve $y = (x + 3)/(x + 2)$ (Example 5).

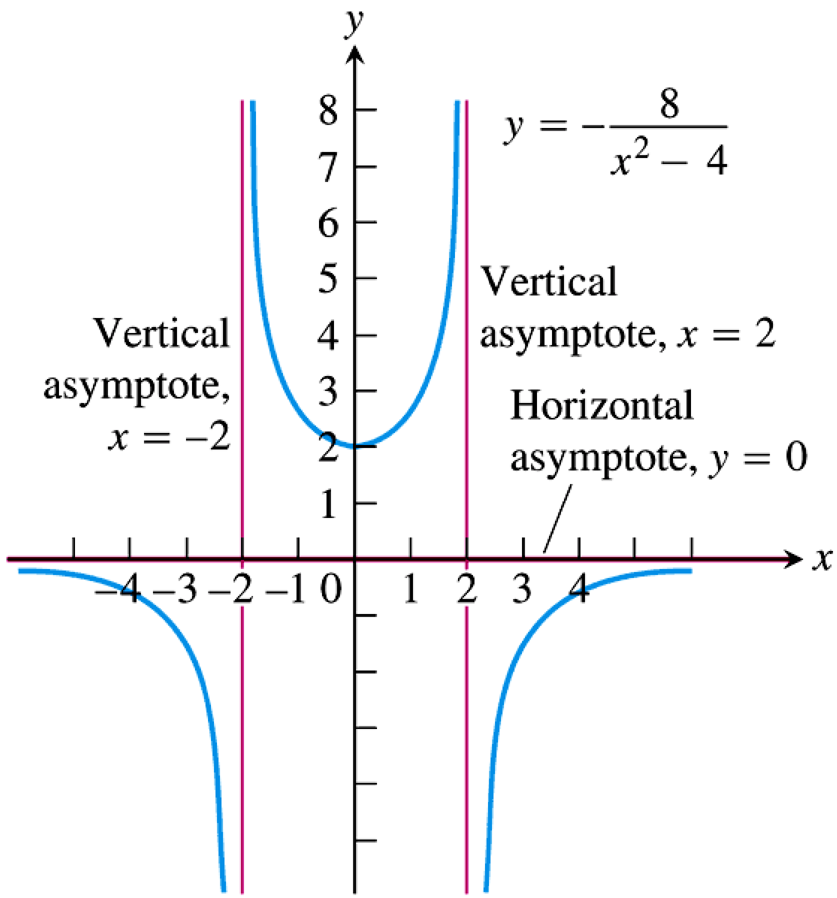


FIGURE 2.44 Graph of $y = -8/(x^2 - 4)$. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided (Example 6).

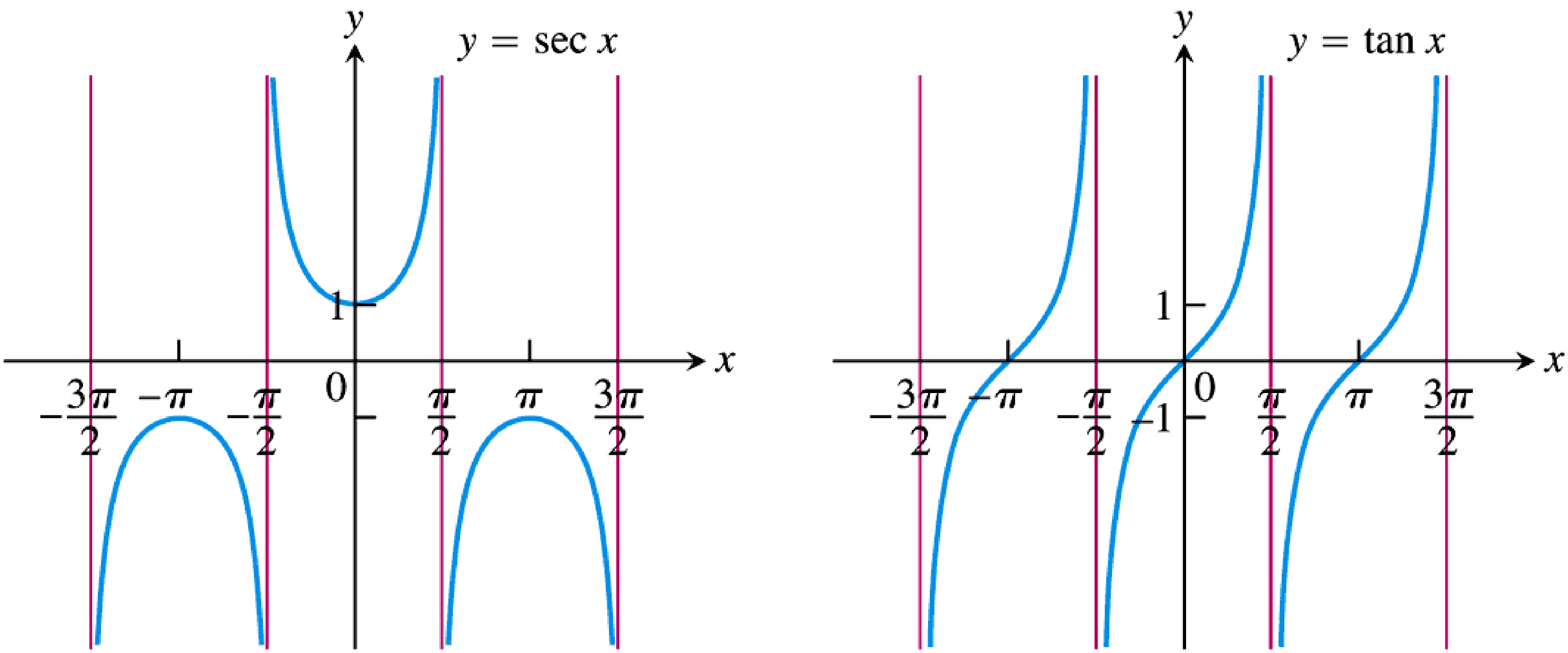


FIGURE 2.45 The graphs of sec x and tan x have infinitely many vertical asymptotes (Example 7).

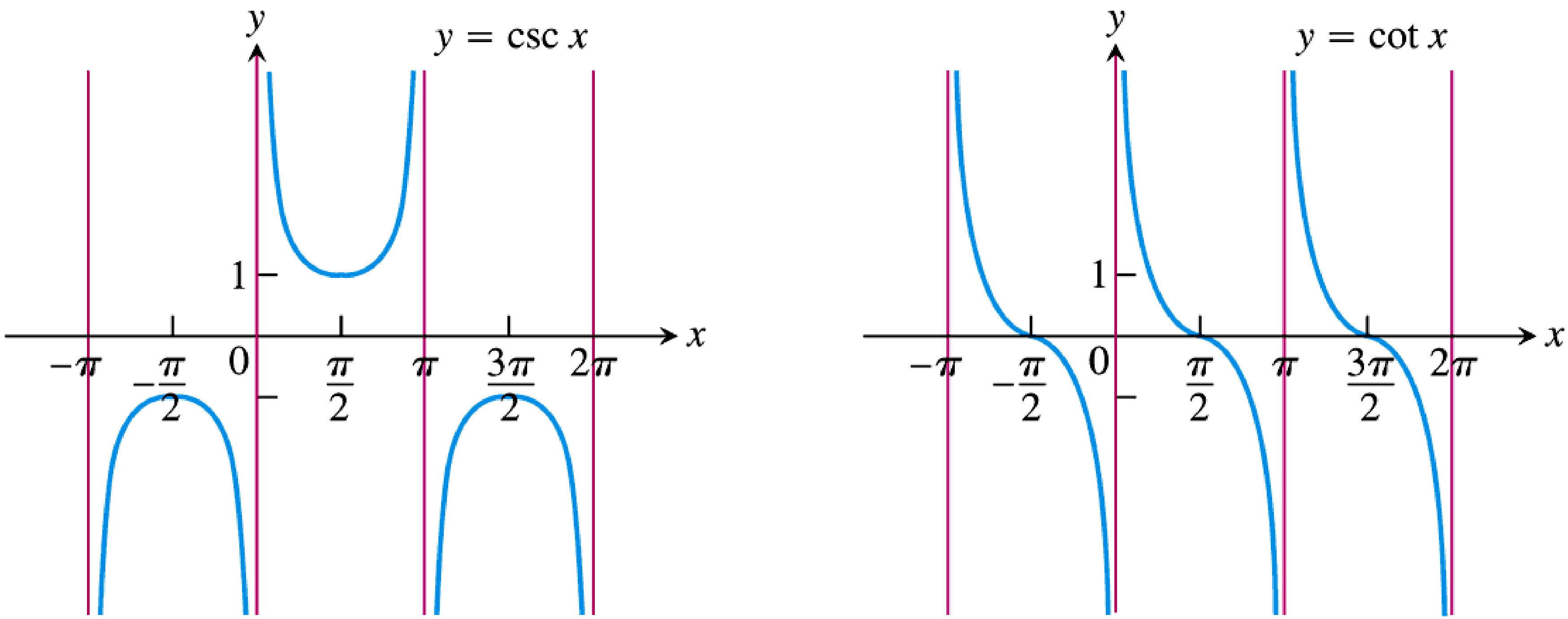


FIGURE 2.46 The graphs of csc x and cot x (Example 7).

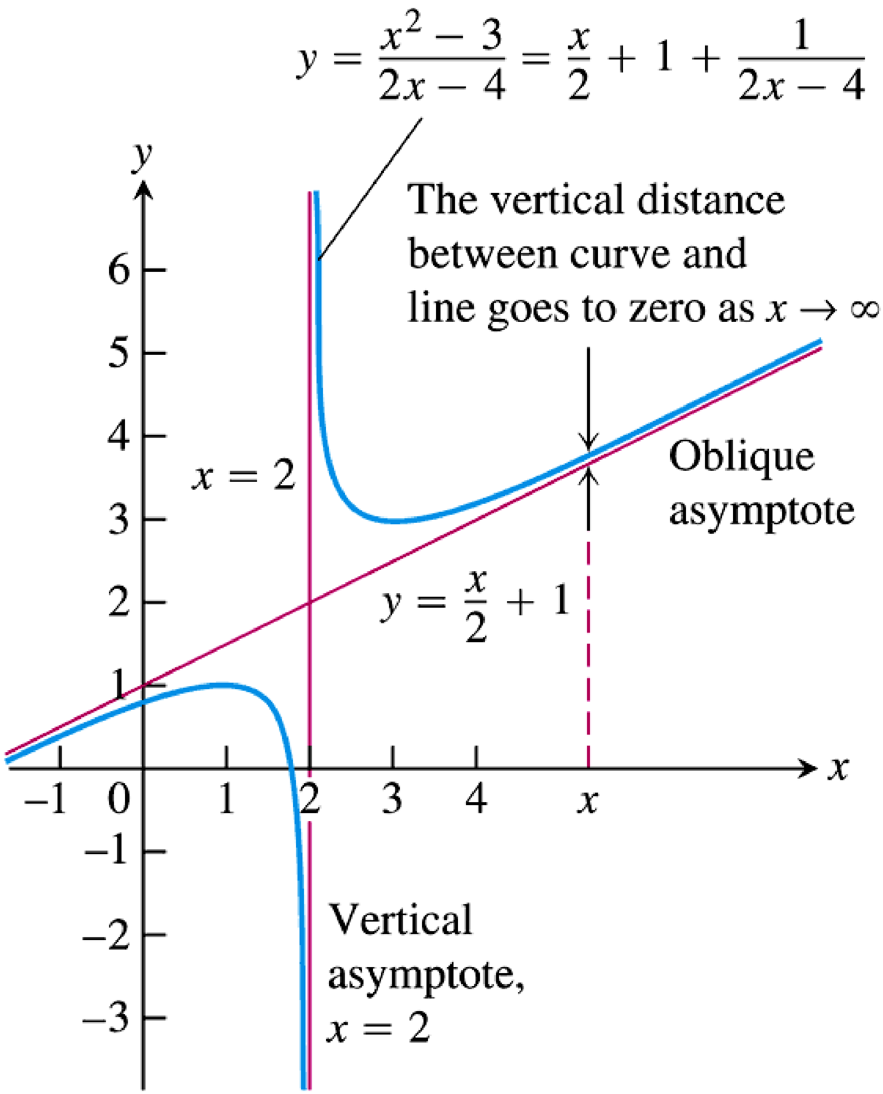


FIGURE 2.47 The graph of $f(x) = (x^2 - 3)/(2x - 4)$ has a vertical asymptote and an oblique asymptote (Example 8).

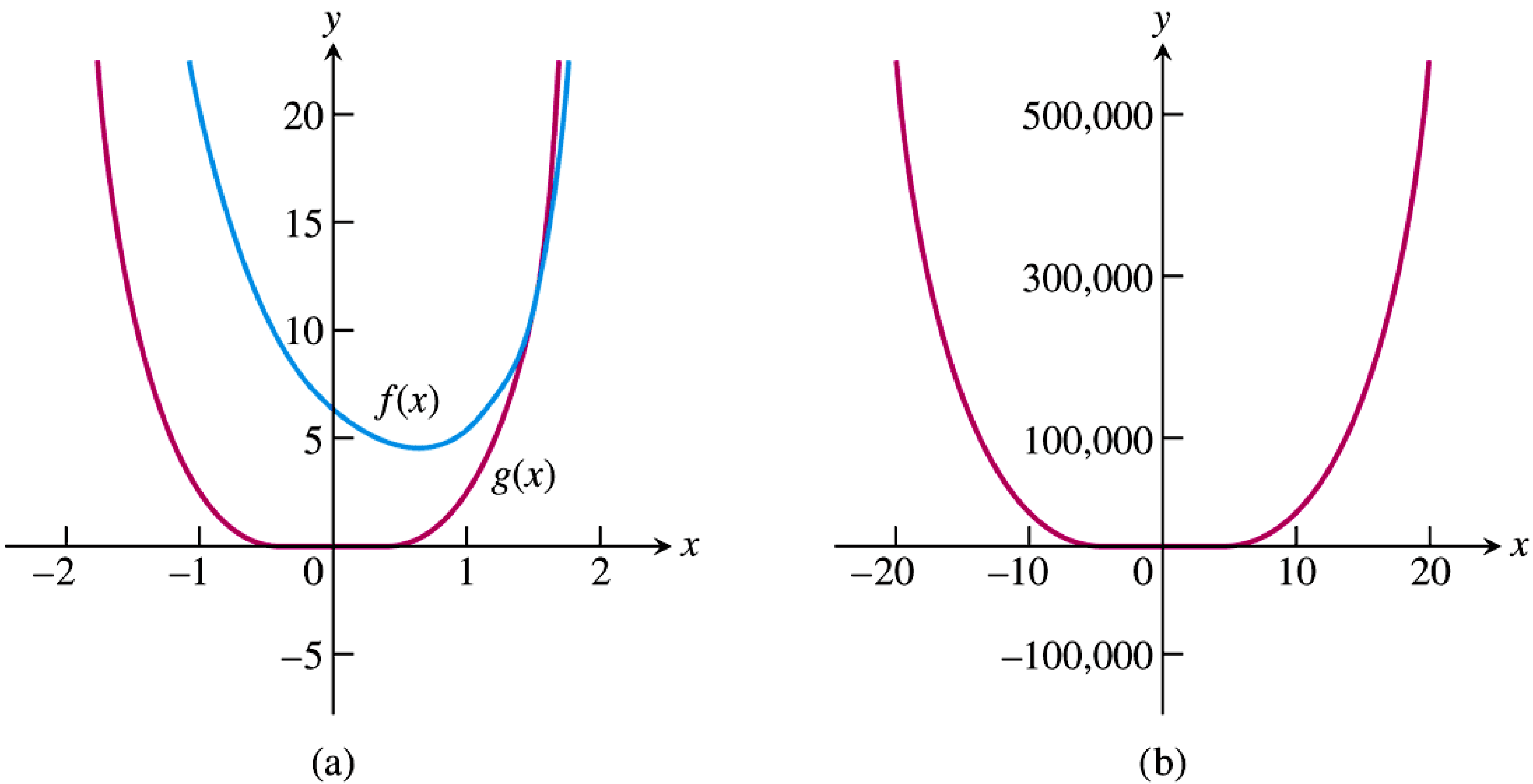


FIGURE 2.48 The graphs of f and g, (a) are distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 9).

2.6

Continuity

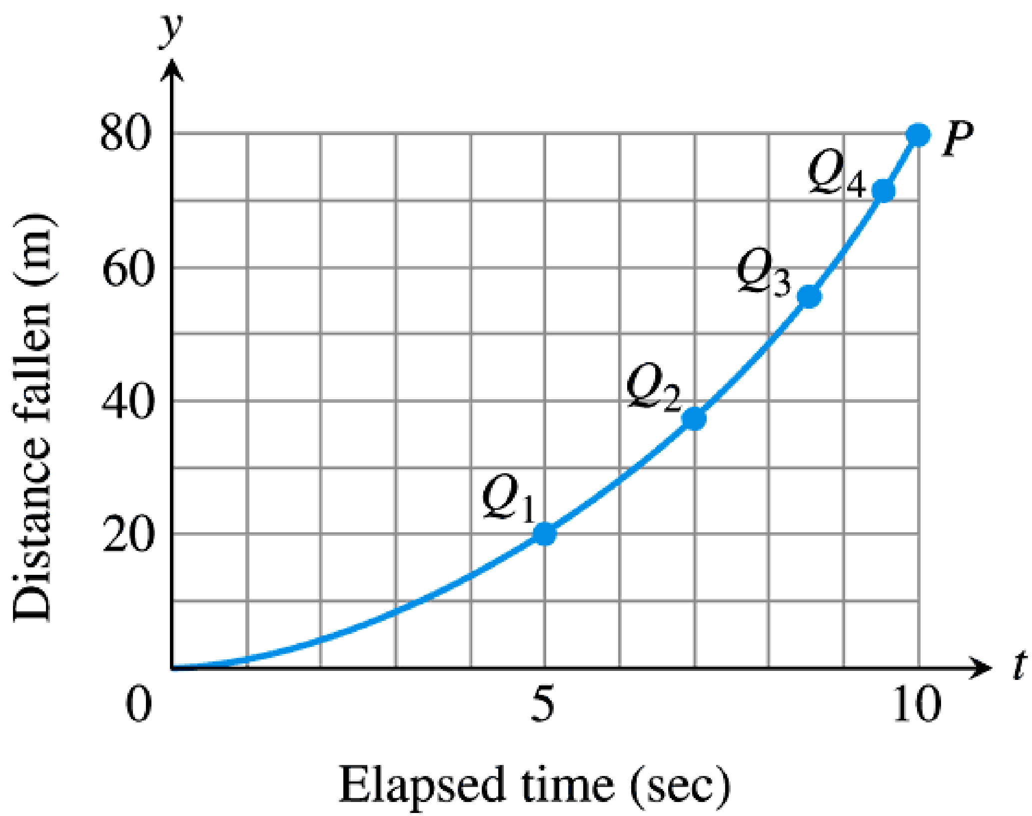


FIGURE 2.49 Connecting plotted points by an unbroken curve from experimental data $Q_1, Q_2, Q_3, \ldots$ for a falling object.

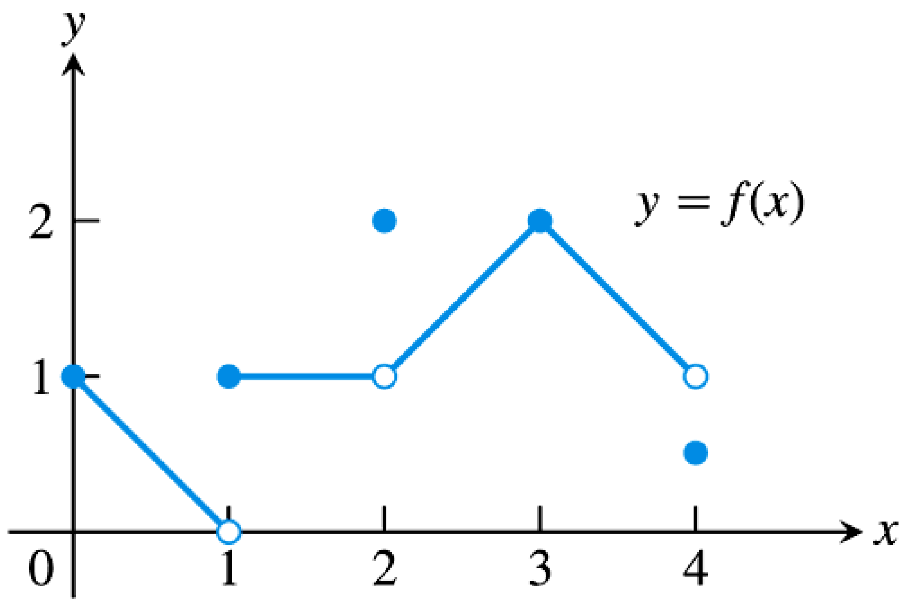


FIGURE 2.50 The function is continuous on [0, 4] except at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

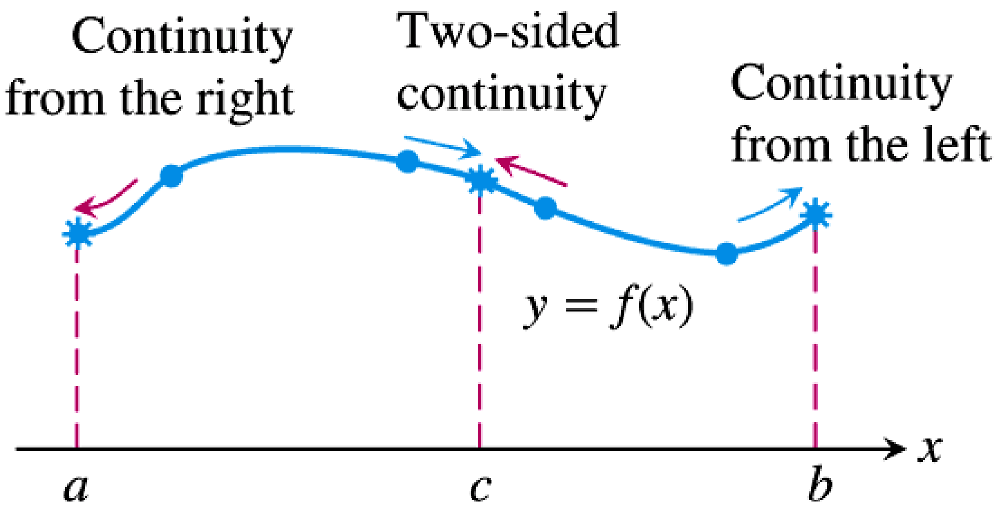


FIGURE 2.51 Continuity at points a, b, and c.

DEFINITION **Continuous at a Point**

Interior point: A function $y = f(x)$ is **continuous at an interior point** $\boldsymbol{c}$ of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** $\boldsymbol{a}$ or is **continuous at a right endpoint** $\boldsymbol{b}$ of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

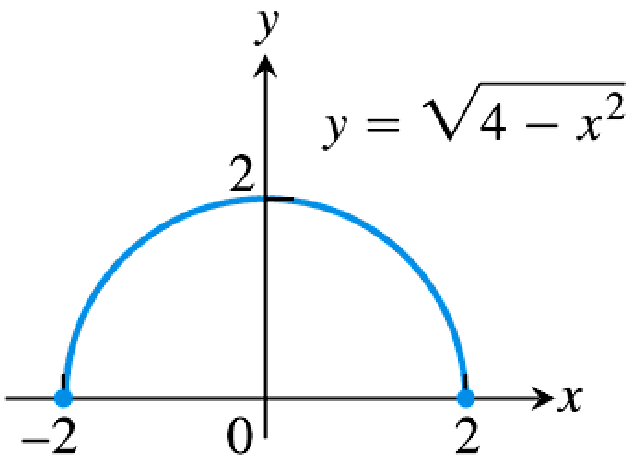


FIGURE 2.52 A function that is continuous at every domain point (Example 2).

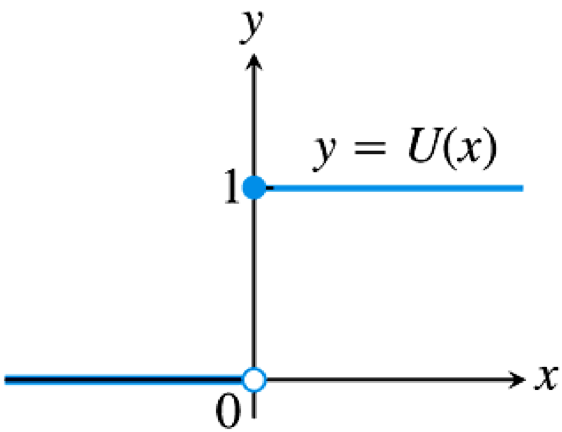


FIGURE 2.53 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there (Example 3).

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (*c* lies in the domain of *f*)
2. $\lim_{x \to c} f(x)$ exists (*f* has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

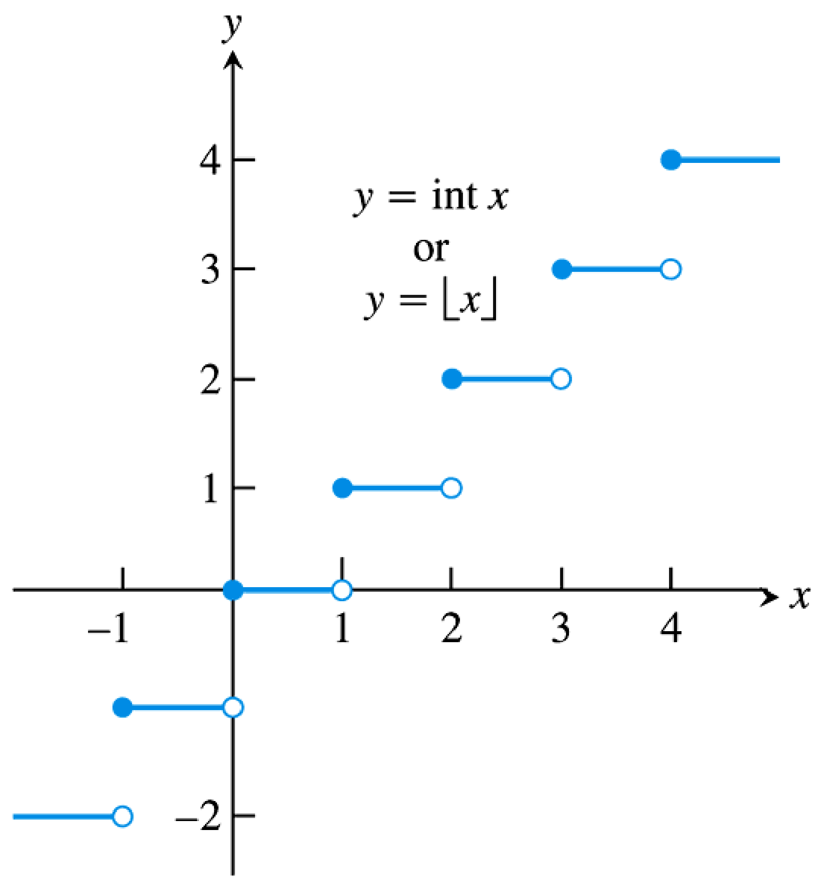


FIGURE 2.54 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

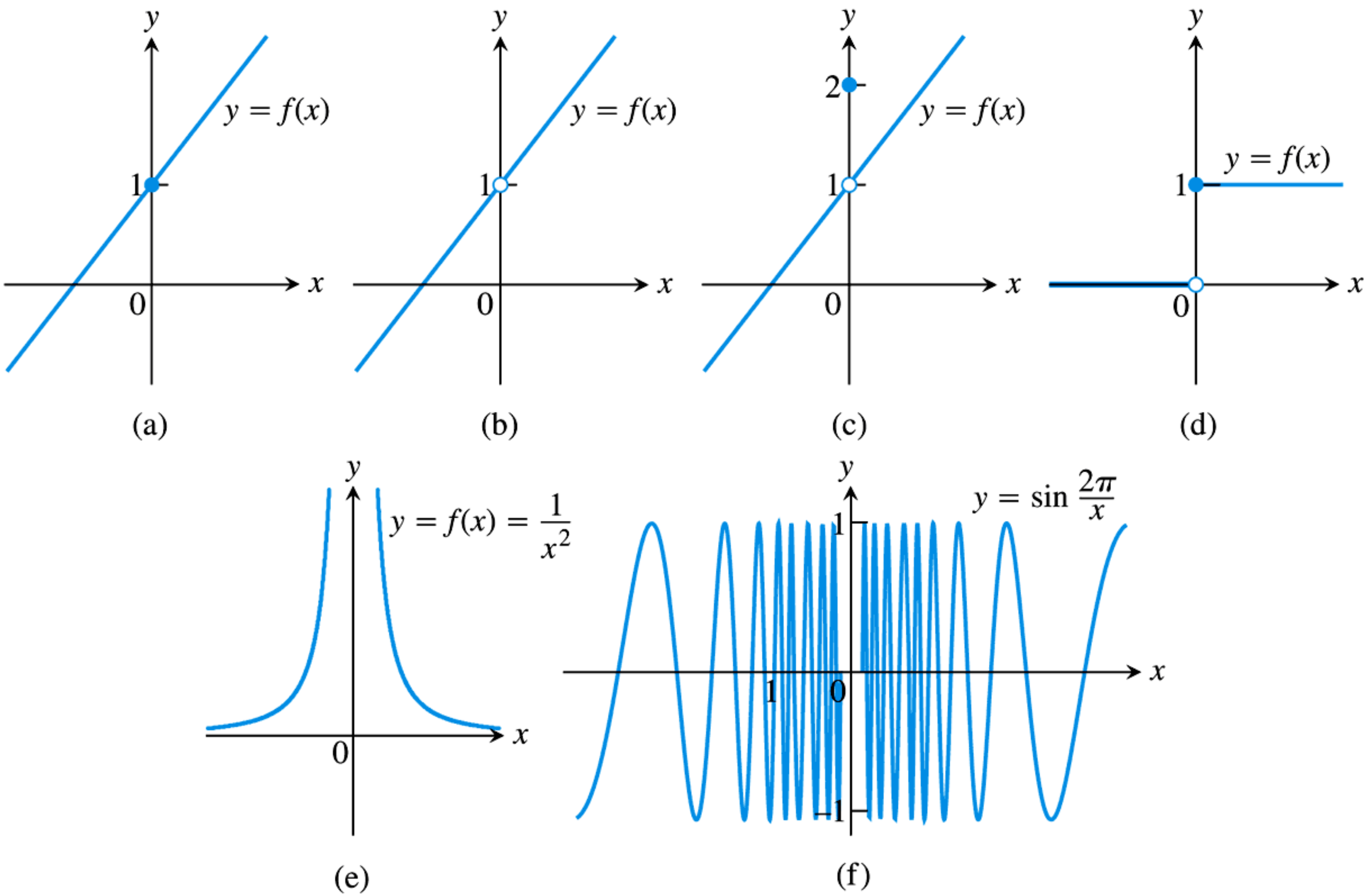


FIGURE 2.55 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

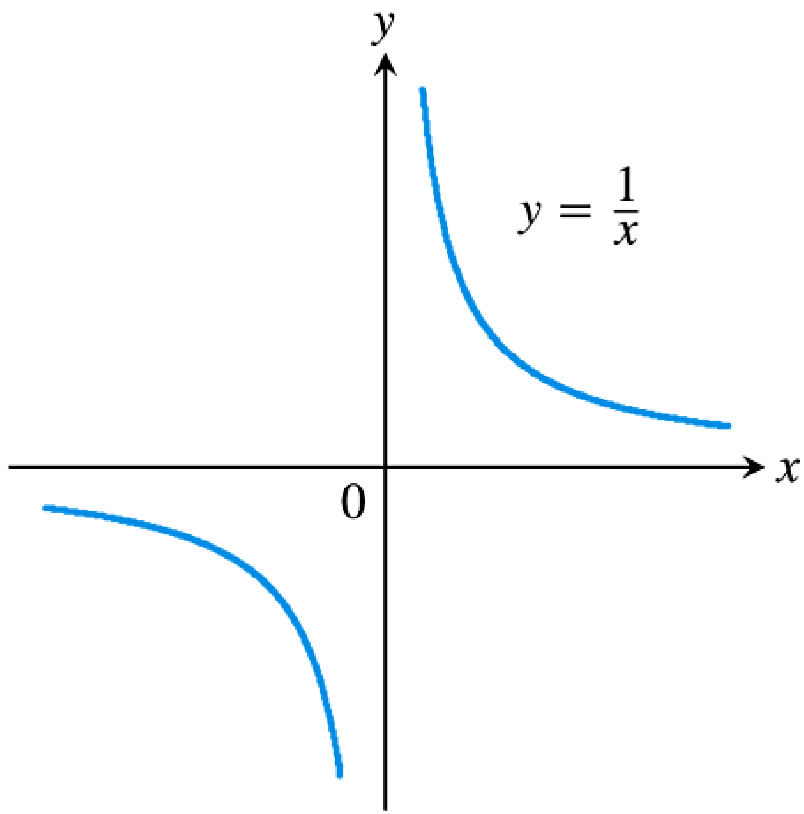


FIGURE 2.56 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$ (Example 5).

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

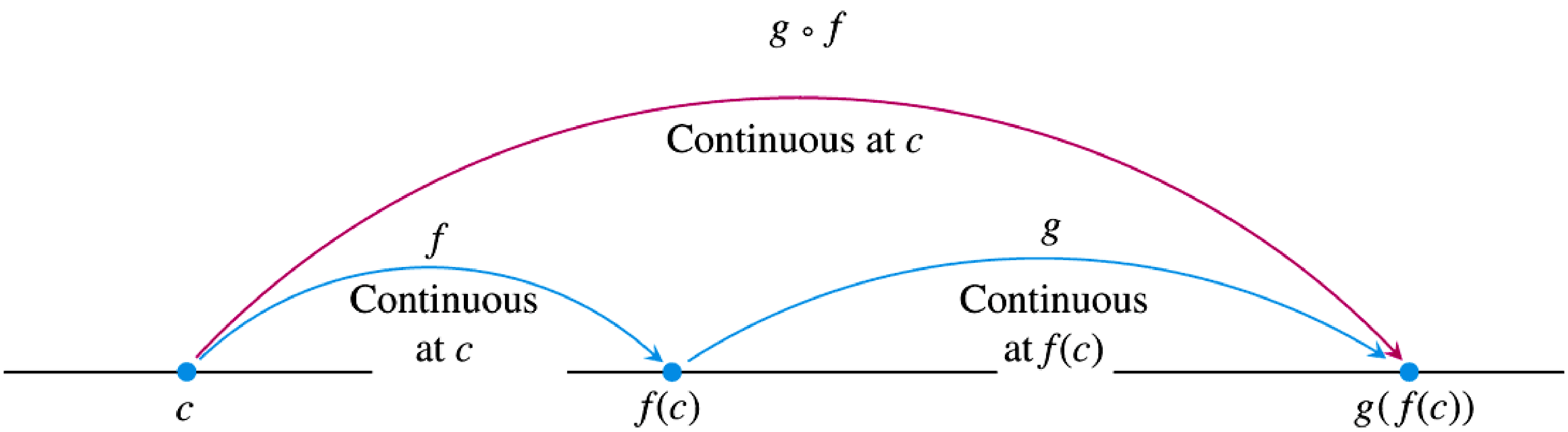


FIGURE 2.57 Composites of continuous functions are continuous.

THEOREM 10 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

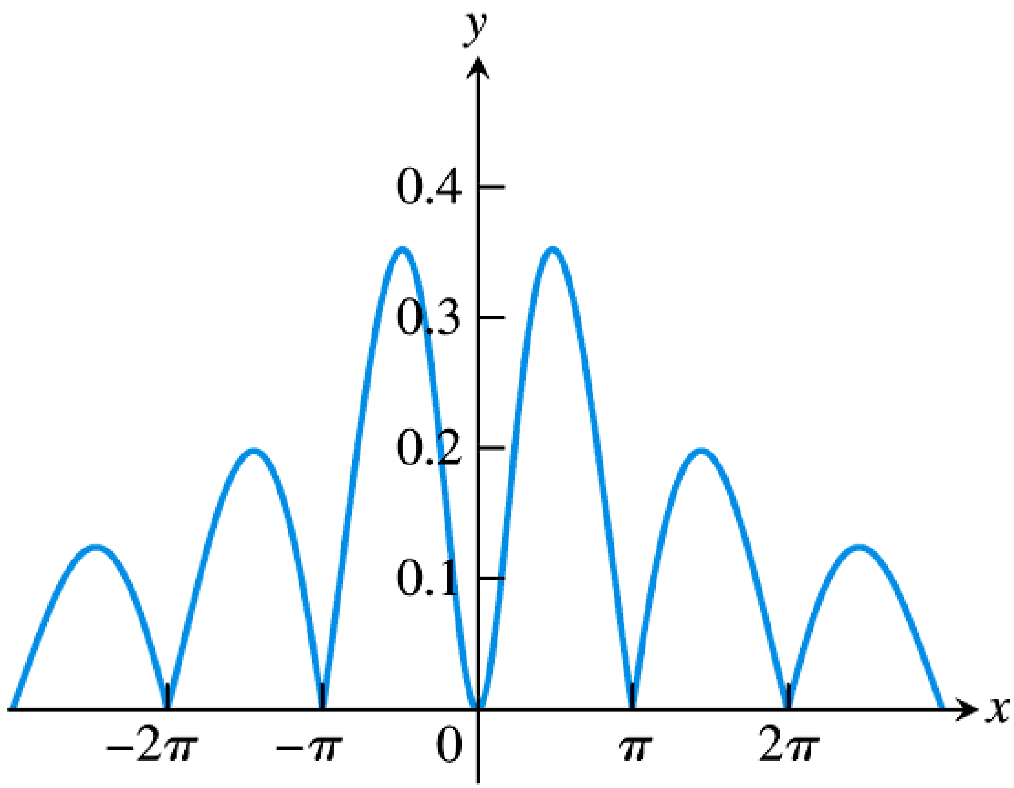


FIGURE 2.58 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

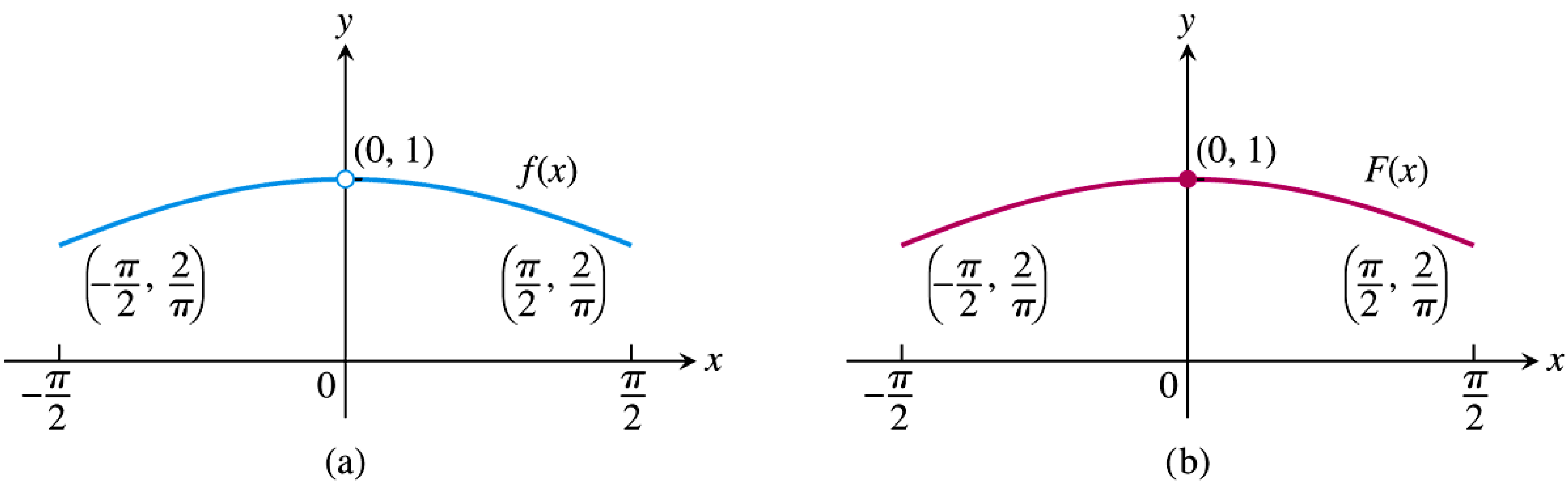
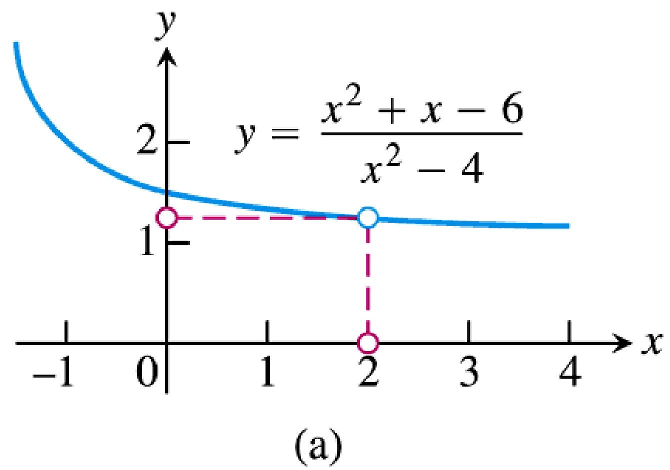


FIGURE 2.59 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \to 0} f(x)$.



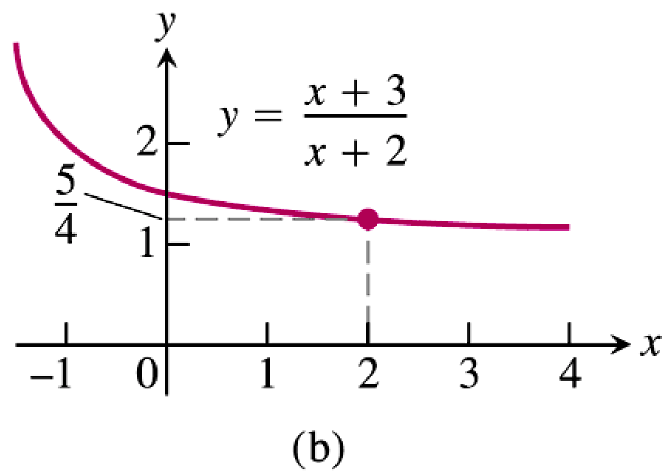
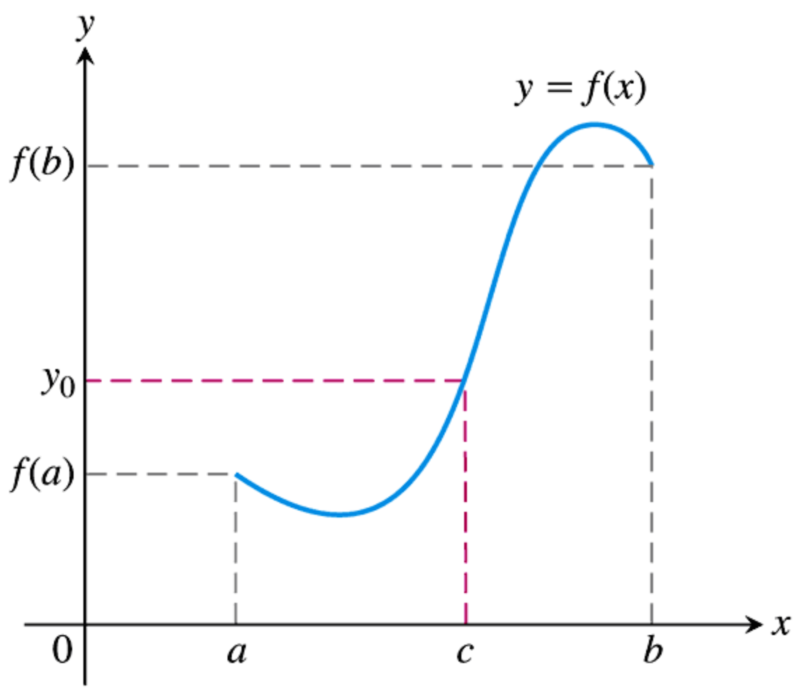


FIGURE 2.60 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 9).

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



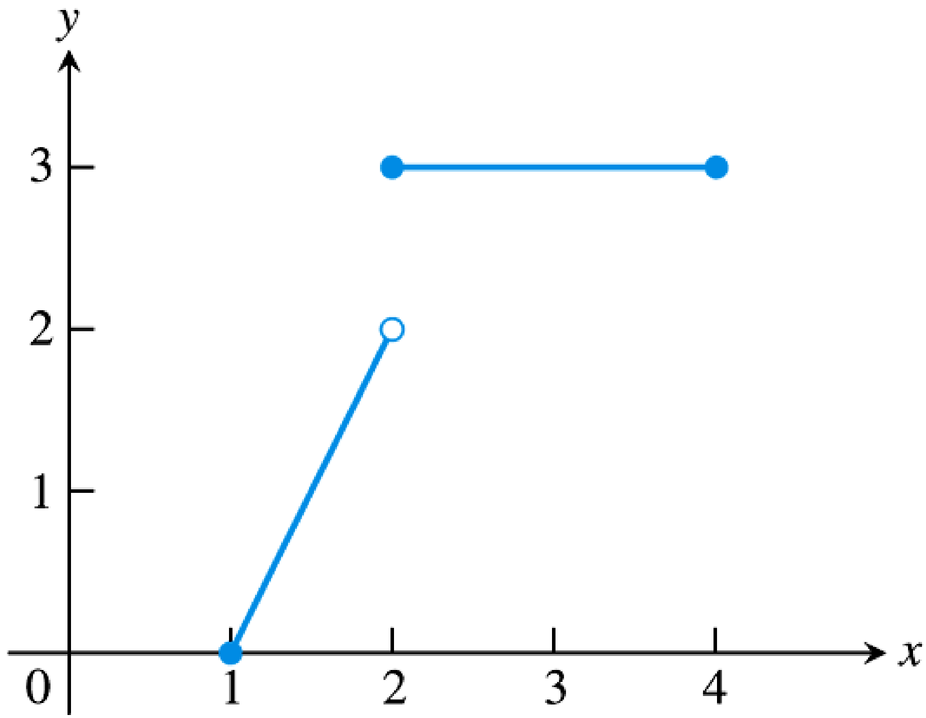


FIGURE 2.61 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

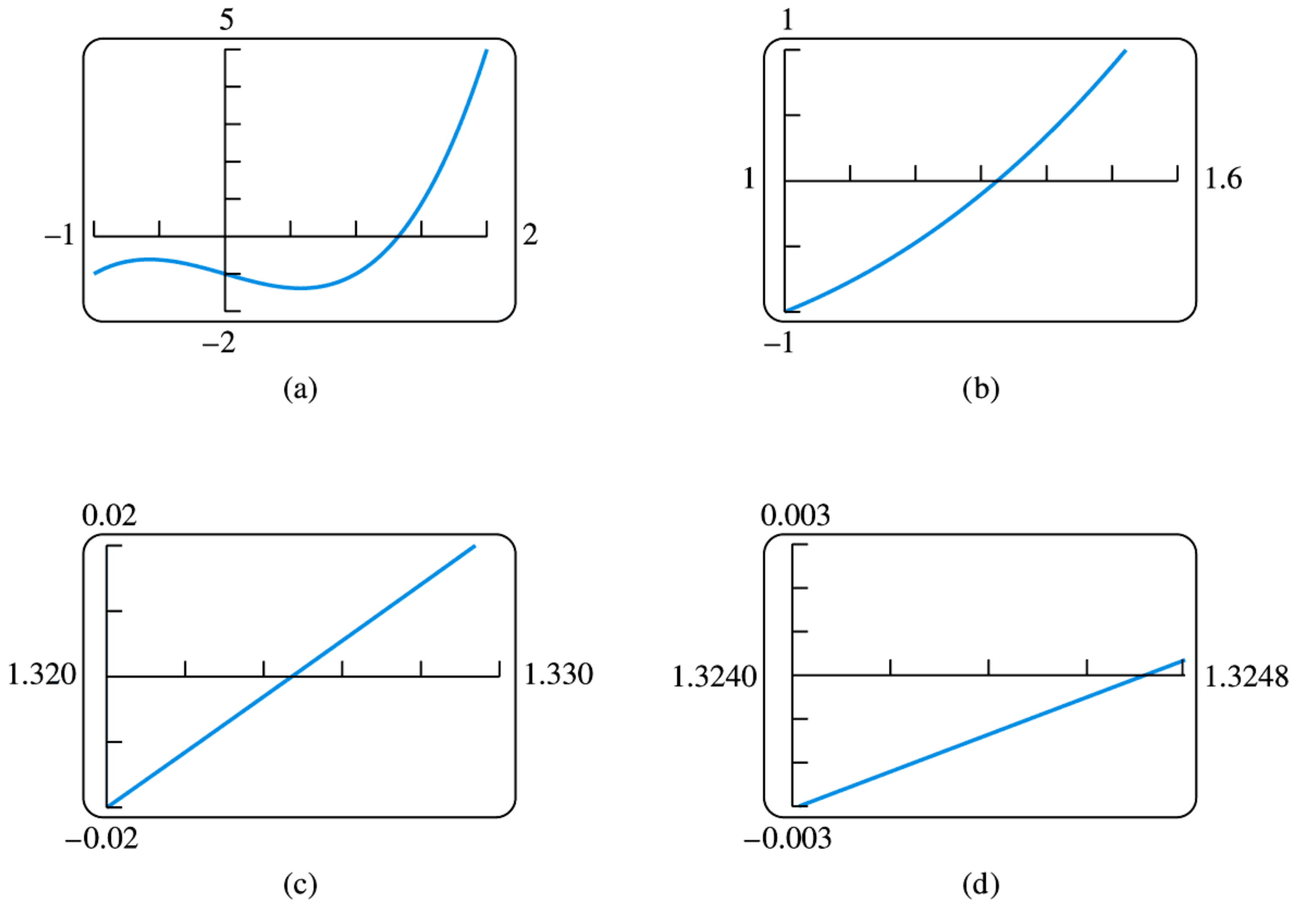


FIGURE 2.62 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$.

2.7

Tangents and Derivatives

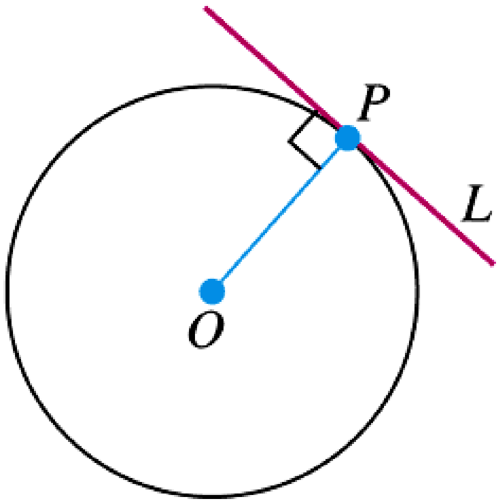
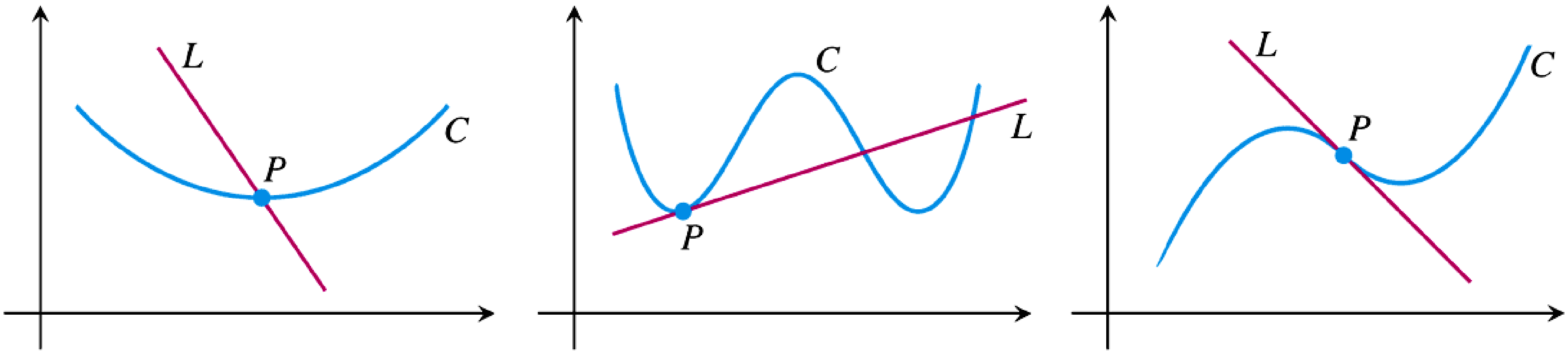


FIGURE 2.63 *L* is tangent to the circle at *P* if it passes through *P* perpendicular to radius *OP*.



L meets *C* only at *P* but is not tangent to *C*.

L is tangent to *C* at *P* but meets *C* at several points.

L is tangent to *C* at *P* but lies on two sides of *C*, crossing *C* at *P*.

FIGURE 2.64 Exploding myths about tangent lines.

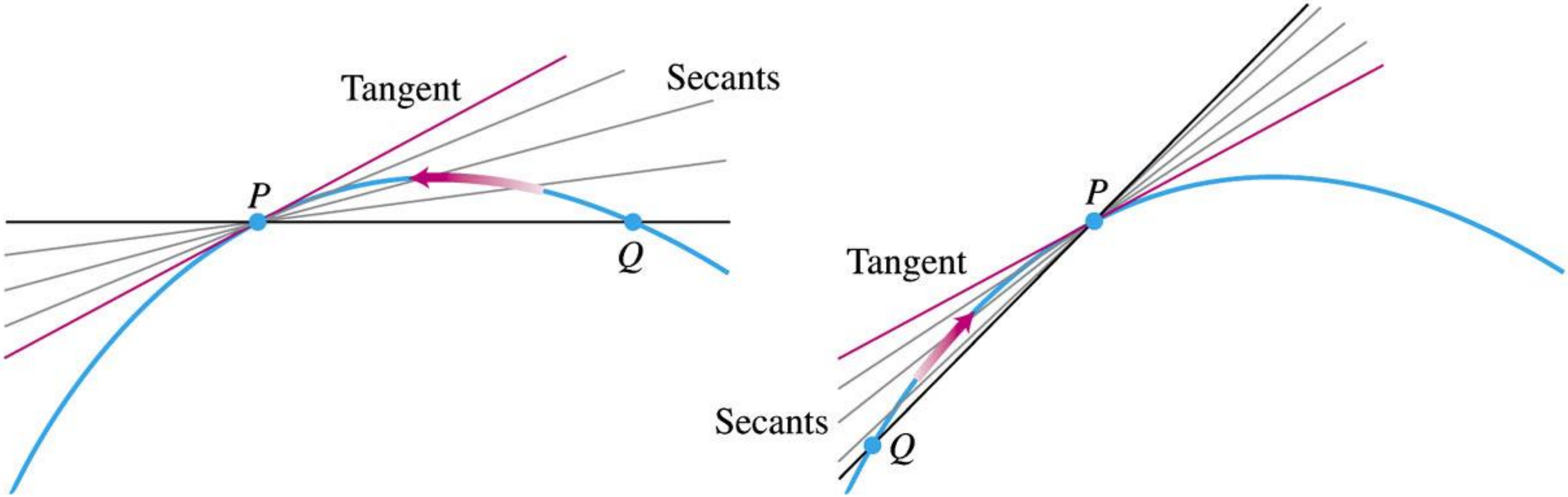


FIGURE 2.65 The dynamic approach to tangency. The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \to P$ from either side.

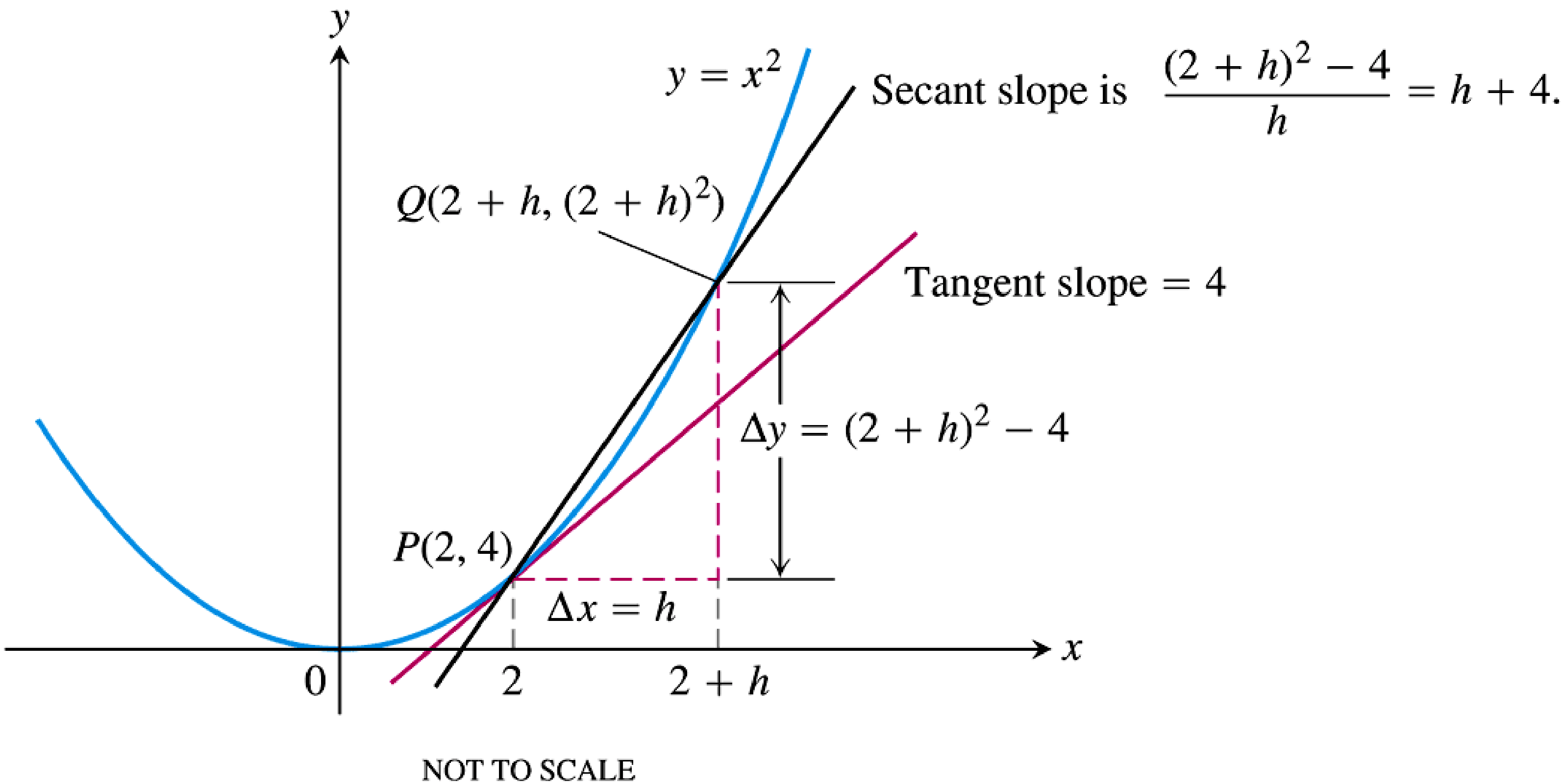


FIGURE 2.66 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ (Example 1).

DEFINITIONS Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

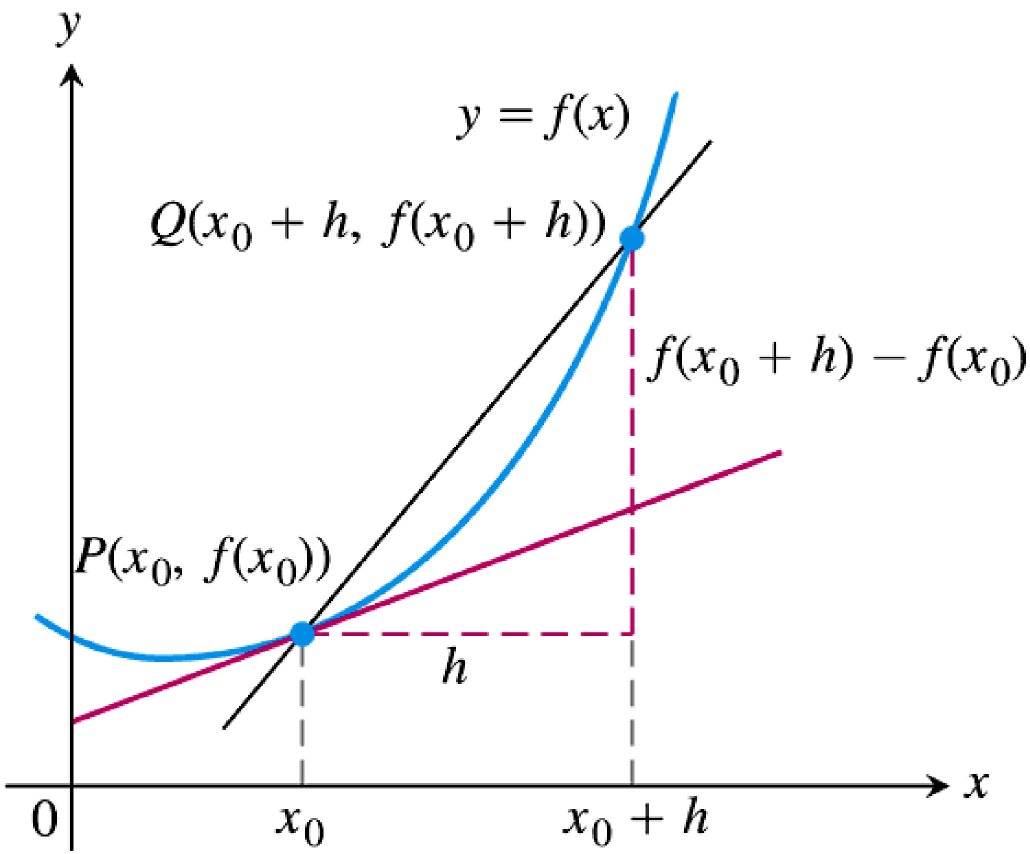


FIGURE 2.67 The slope of the tangent line at P is $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

3. If the limit exists, find the tangent line as

$$y = y_0 + m(x - x_0).$$

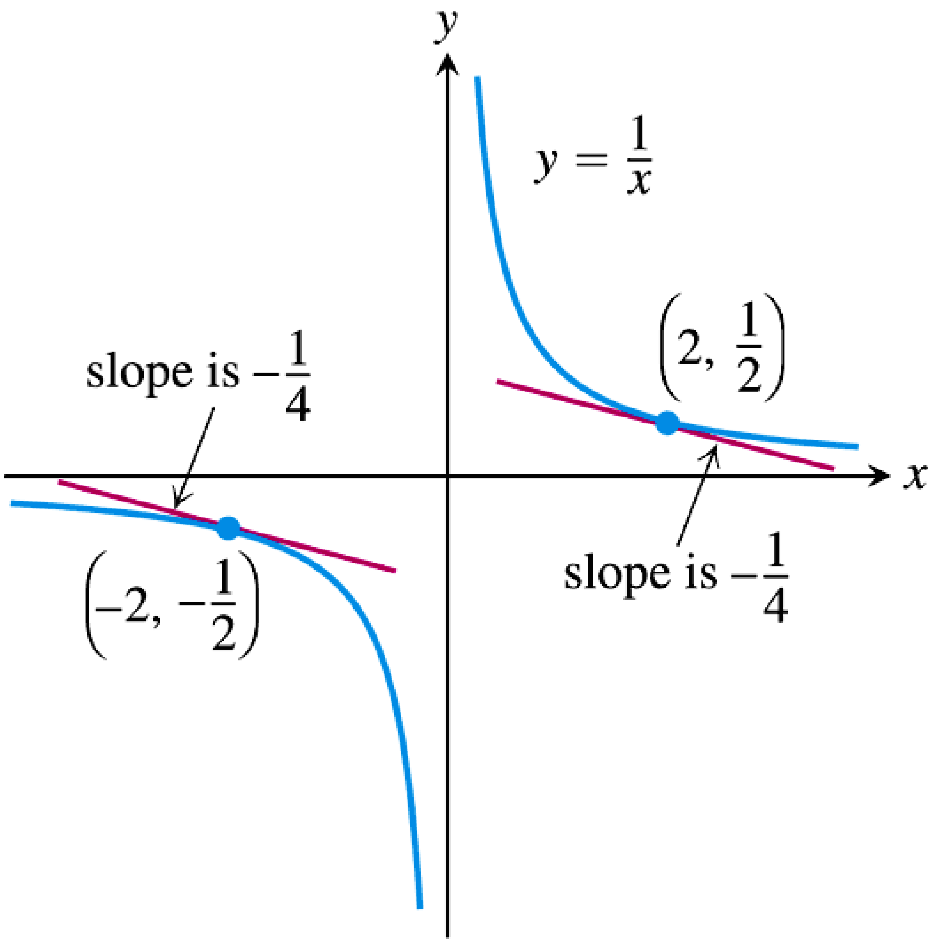


FIGURE 2.68 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 3).

1. The slope of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative of f at $x = x_0$
5. The limit of the difference quotient, $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$

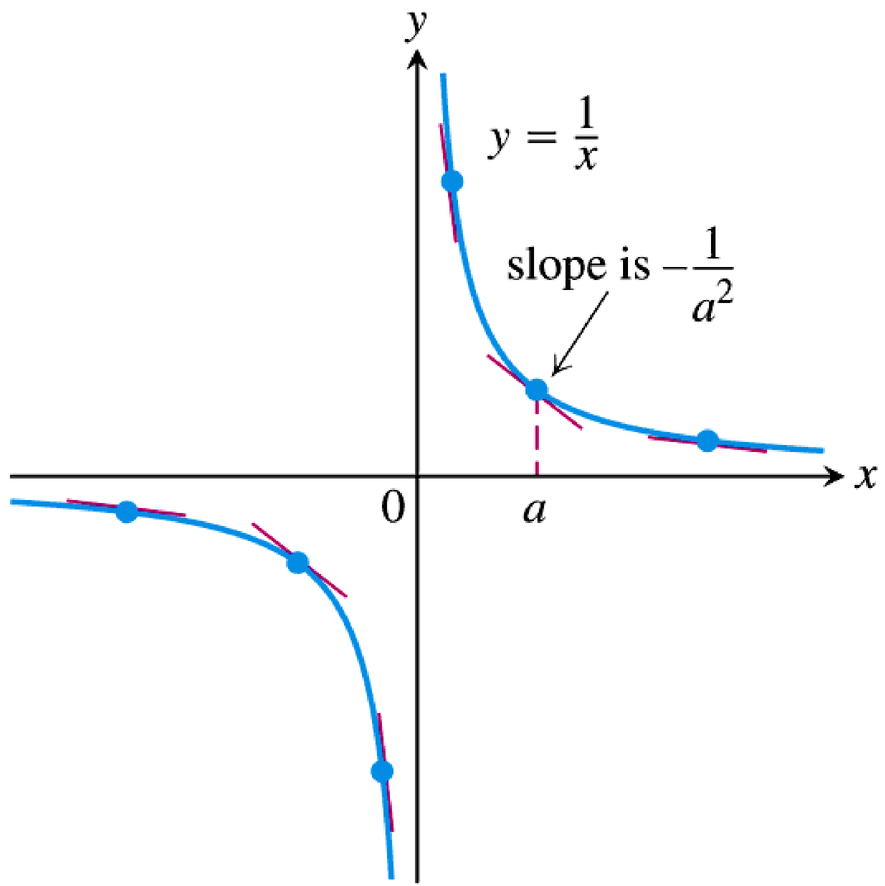


FIGURE 2.69 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away.

CH. 2

2.2

Notes

- التركيز على المفاهيم الاساسية.
- شرح أبواب المنهج حسب الخطة.
- أمثلة توضيحية وتدريبات.
- نماذج اختبارات.

السعدي

The limit of the Function

رياضيات ١١٠

Math. 110

جمال السعدي

استاذ الرياضيات والإحصاء للمرحلة الجامعية

0566664790

CH. 2

2.2

The limit of A function

If: $f(x) = x + 3$

what is the value of f(x) approaches it ? when x approaches 2 .

$\longrightarrow 2^-$ $\qquad\qquad$ $2^+ \longleftarrow$

X	1	1.8	1.9	2	2.1	2.2	3
F(x) = X + 3	4	4.8	4.9	5	5.1	5.2	6

النهاية اليسرى

$$\lim_{x \to 2^-} (x+3) = 5$$

النهاية اليمنى

$$\lim_{x \to 2^+} (x+3) = 5$$

$$\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 5$$

اليسرى $\qquad$ اليمنى

$$\therefore \lim_{x \to 2} f(x) = 5$$

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Definition :

If: f(x) approaches (تقترب) L when X approaches (تقترب) a

$$f(x) \longrightarrow L \quad \text{when} \quad x \longrightarrow a$$

- X close to a on either side of a but $\neq a$
- F(x) tends to get closer and closer to L .

$$\lim_{x \to a} f(x) = L$$

Note that :

① If: $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ (النهايه اليمنى) (النهايه اليسرى)

$$\Rightarrow \lim_{x \to a} f(x) = L$$

---

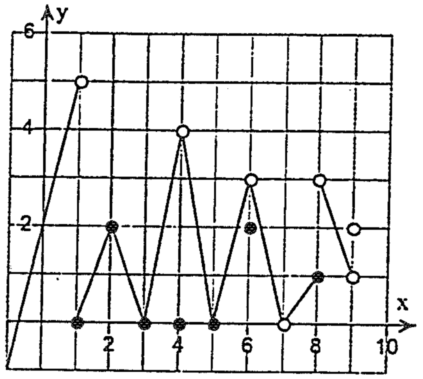
② If: $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

$$\Rightarrow \lim_{x \to a} f(x) \quad \text{does not exist.}$$

(غير موجوده)

عدم تساوي النهايه اليمنى واليسرى

يؤدي إلى النهايه غير موجوده (does not exist)



$* \lim_{x \to 1^-} f(x) = 5 \quad * \lim_{x \to 1^+} f(x) = 0$

اليسار ≠ اليمين

$\therefore \lim_{x \to 1} f(x)$ does not exist

$* \lim_{x \to 2^-} f(x) = 2 \quad * \lim_{x \to 2^+} f(x) = 2$

اليسار = اليمين

$\therefore \lim_{x \to 2} f(x) = 2$

$* \lim_{x \to 8^-} f(x) = 1 \quad * \lim_{x \to 8^+} f(x) = 3$

اليسار ≠ اليمين

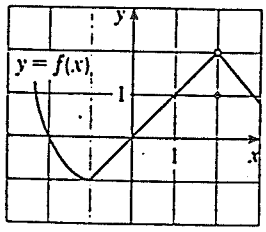
$\therefore \lim_{x \to 8} f(x)$ does not exist

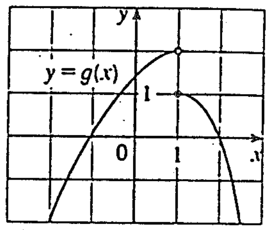
$* F(1) = 0 \quad * F(4) = 0$

$*$ F(7) does not exist

$* F(6) = 2 \quad * F(8) = 1$

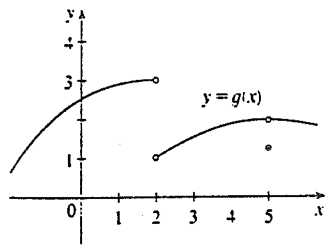
The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.





(a) $\lim_{x \to 2} [f(x) + g(x)]$

(b) $\lim_{x \to 1} [f(x) + g(x)]$



$* \lim_{x \to 2^-} g(x) = 3 \quad * \lim_{x \to 2^+} g(x) = 1$

اليسار ≠ اليمين

$\therefore \lim_{x \to 2} g(x)$ does not exist

$* \lim_{x \to 5^-} g(x) = 2 \quad * \lim_{x \to 5^+} g(x) = 2$

اليسار = اليمين

$\therefore \lim_{x \to 5} g(x) = 2 \quad * g(5) = 1$

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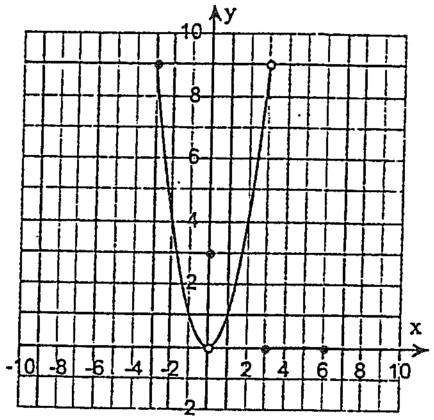
Which of the following statements about the function y = f(x) graphed here are true, and Use the graph below to determine whether the statements about the function y = f(x) are true or false.



* $\lim_{x \to -3^+} f(x) = 9$

* $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$

اليسرى = اليمنى

$\therefore \lim_{x \to 0} f(x) = 0$

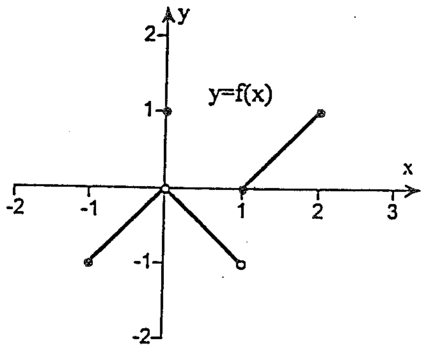
* $\lim_{x \to 3^-} f(x) = 9$ * $\lim_{x \to 3^+} f(x) = 0$

اليسرى ≠ اليمنى

$\therefore \lim_{x \to 3} f(x)$ does not exist

* $\lim_{x \to 6^-} f(x) = 0$

* $f(-2) = 9$ * $f(0) = 3$ * $f(3) = 0$



* $\lim_{x \to 0^-} f(x) = 0$ النهاية اليسرى

* $\lim_{x \to 0^+} f(x) = 0$ النهاية اليمنى

∴ اليمنى = اليسرى

$\therefore \lim_{x \to 0} f(x) = 0$ exist
موجودة

* $\lim_{x \to 1^-} f(x) = -1$ النهاية اليسرى

* $\lim_{x \to 1^+} f(x) = 0$ النهاية اليمنى

∴ اليمنى ≠ اليسرى

$\therefore \lim_{x \to 0} f(x)$ does not exist
غير موجودة

* $F(0) = 1$ * $F(1) = 0$ * $F(2) = 1$

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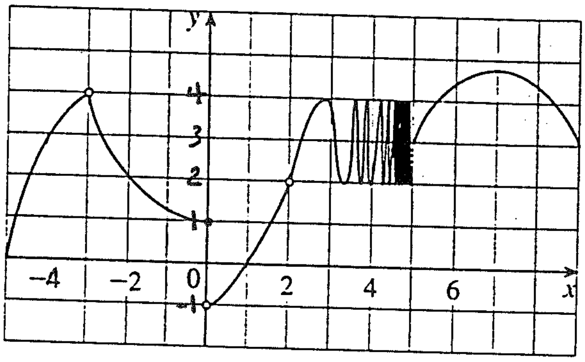
For the function $h(x)$ whose graph is given.

Find :



① * $\lim_{x \to -3^-} h(x) = 4$

* $\lim_{x \to -3^+} h(x) = 4$

$\because \lim_{x \to -3^-} h(x) = \lim_{x \to -3^+} h(x) = 4 \Rightarrow \lim_{x \to -3} h(x) = 4$

* $h(-3)$ does not exist.

---

② * $\lim_{x \to 0^-} h(x) = 1$ * $\lim_{x \to 0^+} h(x) = -1$

$\because \lim_{x \to 0^-} h(x) \neq \lim_{x \to 0^+} h(x)$ النهايه اليسرى $\neq$ النهايه اليمنى

* $\lim_{x \to 0} h(x)$ does not exist * $h(0) = 1$

---

③ * $\lim_{x \to 5^+} h(x) = 3$ * $\lim_{x \to 5^-} h(x) =$ does not exist

$\because$ النهايه اليمنى $\neq$ النهايه اليسرى

$\Rightarrow \lim_{x \to 5} h(x)$ does not exist.

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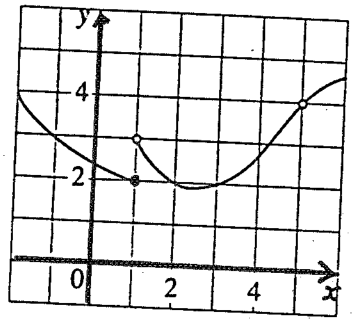
Use the given graph of F(x) to find:



① $\lim_{x \to 1} f(x)$ نوجد النهاية اليمنى واليسرى

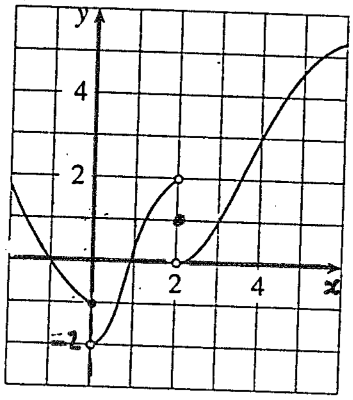
* $\lim_{x \to 1^-} f(x) = 2$ * $\lim_{x \to 1^+} f(x) = 3$

∵ النهاية اليمنى ≠ النهاية اليسرى

∴ $\lim_{x \to 1} f(x)$ does not exist. (غير موجوده)

② $\lim_{x \to 5} F(x) = 4$ ③ F(5) does not exist لوجود انفصال

Use the given graph of f(x) to find:



① * $\lim_{x \to 0^-} f(x) = -1$ * $\lim_{x \to 0^+} f(x) = -2$

∵ النهاية اليمنى ≠ النهاية اليسرى

∴ $\lim_{x \to 0} f(x)$ does not exist

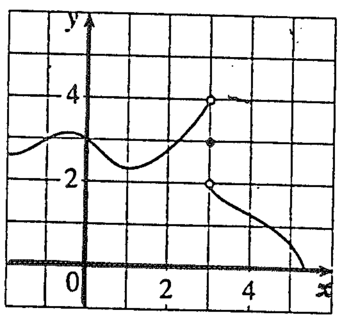
② * $\lim_{x \to 2^-} f(x) = 2$ * $\lim_{x \to 2^+} f(x) = 0$

النهاية اليسرى ≠ النهاية اليمنى

∴ $\lim_{x \to 0} f(x)$ does not exist

③ F(2) = 1 ④ $\lim_{x \to 4} F(x) = 3$

Use the given graph of f(x) to find:



① $\lim_{x \to 0} f(x) = 3$

② * $\lim_{x \to 3^-} f(x) = 4$ * $\lim_{x \to 3^+} f(x) = 2$

النهاية اليسرى ≠ النهاية اليمنى

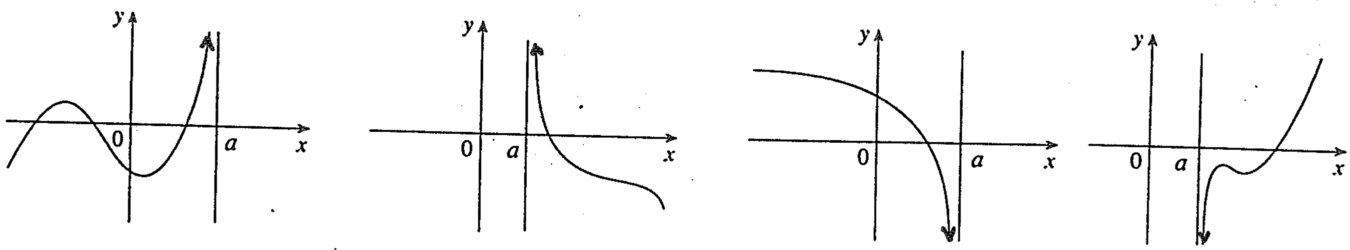
∴ $\lim_{x \to 3} f(x)$ does not exist

③ F(3) = 3

DEFINITION The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

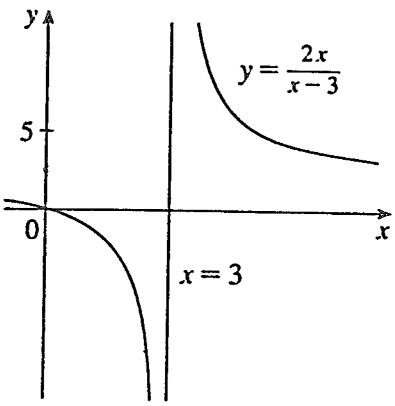
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

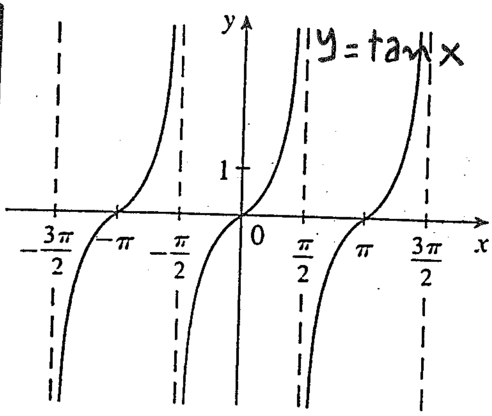


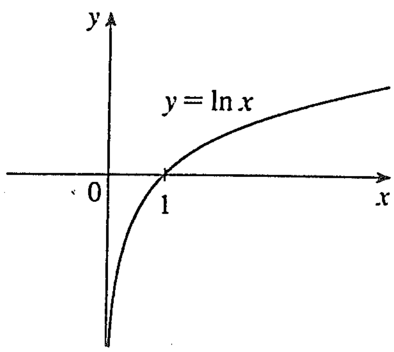
(a) $\lim_{x\to a^-} f(x) = \infty$ (b) $\lim_{x\to a^+} f(x) = \infty$ (c) $\lim_{x\to a^-} f(x) = -\infty$ (d) $\lim_{x\to a^+} f(x) = -\infty$

X = a is vertical asymptote.

Find the equation of the vertical asymptote:







V. asymptote:

$X = 3$

V. asymptote:

$X = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

. odd مضاعف π فردي

OR $X = \pm\frac{(2n+1)\pi}{2}$

where n is whole number عدد كلي

V. asymptote:

$X = 0$

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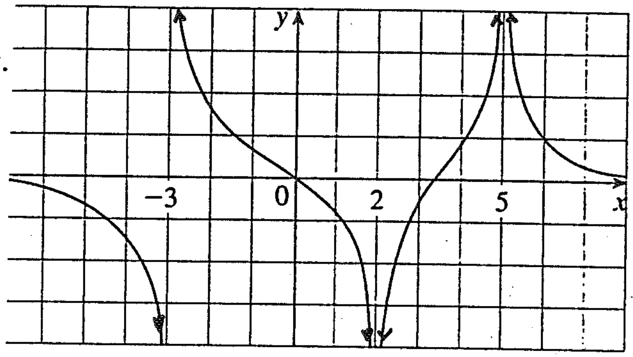
For the function R whose graph is shown, state the following.



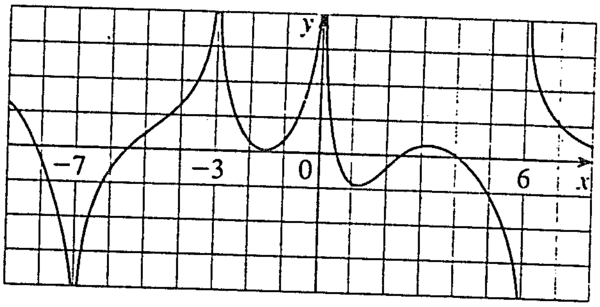
(a) $\lim_{x\to 2} R(x) = -\infty$ (b) $\lim_{x\to 5} R(x) = \infty$

(c) $\lim_{x\to -3^-} R(x) = -\infty$ (d) $\lim_{x\to -3^+} R(x) = \infty$

(e) The equations of the vertical asymptotes.

* $X = -3$ * $X = 2$ * $X = 5$

For the function f whose graph is shown, state the following.



(a) $\lim_{x\to -7} f(x) = -\infty$ (b) $\lim_{x\to -3} f(x) = \infty$ (c) $\lim_{x\to 0} f(x) = \infty$

(d) $\lim_{x\to 6^-} f(x) = -\infty$ (e) $\lim_{x\to 6^+} f(x) = \infty$

(f) The equations of the vertical asymptotes.

* $X = -7$ * $X = -3$ * $X = 0$ * $X = 6$

قاعده هامه

- في حالة النهاية اليمين أو اليسار لدالة كسرية
  - * تحديد اشارة المقام أولاً.
  - * تعويض مباشر ثانياً.
- في حالة النهاية لدالة كسرية بدون يمين أو يسار
  - * تعويض مباشر فقط.

Find the infinite limit

① $\lim_{x \to 5^+} \frac{6}{x-5} = \frac{+6}{+(5-5)} = \frac{6}{0} = \infty$

② $\lim_{x \to 5^-} \frac{6}{x-5} = \frac{+6}{-(5-5)} = -\frac{6}{0} = -\infty$

اذا عوضنا بعدد على يسار 5 في المقام يكون الناتج سالب

③ $\lim_{x \to 1} \frac{2-x}{(x-1)^2} = \frac{2-1}{(1-1)^2} = \frac{1}{0} = \infty$

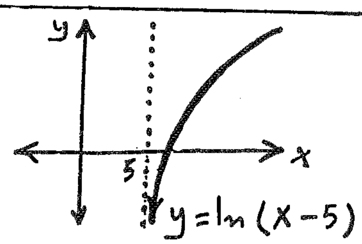
تعويض مباشر فقط لعدم وجود يمين أو يسار

④ $\lim_{x \to 5^-} \frac{e^x}{(x-5)^3} = \frac{+e^5}{-(5-5)^3} = -\frac{e^5}{0} = -\infty$

اذا عوضنا بعدد على يسار 5 في المقام يكون الناتج سالب

⑤ $\lim_{x \to 5^+} \ln(x-5) = -\infty$

عندما x تقترب من العدد 5 من اليمين فإن منحنى الدالة يقترب من $-\infty$



⑥ $\lim_{x \to -2^+} \frac{x-1}{x^2(x+2)} = \frac{-3}{+0} = -\infty$

x^2 موجب ، $(x+2)$ موجب

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(7) $\lim_{x \to -4^+} \frac{2x}{x+4} = \frac{-8}{+(-4+4)} = -\frac{8}{0} = -\infty$

(8) $\lim_{x \to -1^-} \frac{3x}{2x+2} = \frac{-3}{-(-2+2)} = +\frac{3}{0} = \infty$

(9) $\lim_{x \to 0^-} \frac{1}{x} = \frac{+1}{-0} = -\infty$

(10) $\lim_{x \to 0^-} \frac{1}{x^2} = \frac{+1}{+0} = \infty$

يسار zero عندما يُربع يتحول إلى موجب

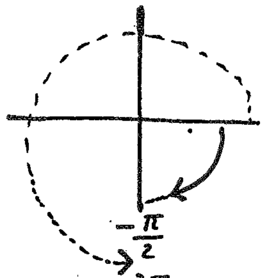
(11) $\lim_{x \to 0^-} \frac{2}{x^{2/3}} = \lim_{x \to 0^-} \frac{2}{\sqrt[3]{x^2}} = \frac{+2}{+0} = \infty$

موجب لوجود التربيع

(12) $\lim_{x \to 0^-} \frac{2}{x^{3/5}} = \lim_{x \to 0^-} \frac{2}{\sqrt[5]{x^3}} = \frac{+2}{-0} = -\infty$

تكعيب السالب يعطي سالب
الجذر الخامس لعدد سالب يعطي سالب

Note : $-\frac{\pi}{2}$ تناظر 270



* على يسارها :. في الربع الثالث
* ,, يمينها :. ,, ,, الرابع

(13) $\lim_{x \to (-\pi/2)^-} \sec x$

$= \lim_{x \to (-\frac{\pi}{2})^-} \frac{1}{\cos x}$

في الربع الثالث

cos x في الربع الثالث سالبه

$= \frac{+1}{-0}$

$= -\infty$

(14) $\lim_{x \to (\frac{\pi}{2})^-} \tan x = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sin x}{\cos x} = \frac{+1}{+0} = \infty$

في الربع الأول
كل الدوال المثلثيه موجبه

(15) $\lim_{x \to (-\frac{\pi}{2})^+} \tan x = \lim_{x \to (-\frac{\pi}{2})^+} \frac{\sin x}{\cos x} = \frac{-1}{+0} = -\infty$

في الربع الرابع

كل التمنيات بالنجاح والتوفيق

السعدي

2.3

Limits by using limits laws

9

Notes

- التركيز على المفاهيم الأساسية.
- شرح أبواب المنهج حسب الخطة.
- أمثلة توضيحية وتدريبات.
- نماذج اختبارات.

السعدي

رياضيات ١١٠

Math.110

جمال السعدي

استاذ الرياضيات والإحصاء للمرحلة الجامعية

0566664790

2.3

Calculating limits using the limits laws

In this section:

To calculate limits we use the following properties of limits called "The limits laws"

limit laws

suppose that: $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

and c is constant.

① $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

توزيع النهاية على الجمع والطرح .

② $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

توزيع النهاية على الضرب .

③ $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ (If: $\lim_{x \to a} g(x) \neq 0$)

توزيع النهاية على القسمة .

$= \frac{L}{M}$ (If: $M \neq 0$)

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④ $\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x) = c \cdot L$

⑤ $\lim_{x \to a} c = c$ نهاية الثابت نفس الثابت

where c is constant

⑥ $\lim_{x \to a} x = a$ التعويض عن x ب a

⑦ $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n = L^n$

⑧ $\lim_{x \to a} x^n = a^n$ التعويض عن x ب a

⑨ $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (If: n is even (زوجي)
a must be positive (موجب))

⑩ $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$

(If: n is even
L must be positive)

Example :

Evaluate the following limits

① $\lim_{x \to 5} (2x^2 - 3x + 4)$ [by direct substitution]
بالتعويض المباشر

$= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4$ ← * ممكن عدم كتابة هذه الخطوة

$= 2(5)^2 - 3(5) + 4$

$= 50 - 15 + 4$

$= 39$

---

② $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

$= \dfrac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x}$ ← * ممكن عدم كتابة هذه الخطوة

$= \dfrac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$

$= \dfrac{-8 + 8 - 1}{5 + 6} = \dfrac{-1}{11}$

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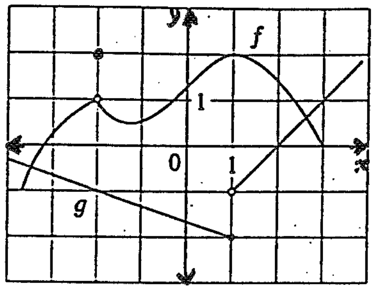
Example :

Use the limit laws and graphs of F and g in figure to evaluate the following limits (if they exist).



① $\lim_{x \to -2} [f(x) + 5g(x)]$

$= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x)$

$= 1 + 5(-1)$

$= 1 - 5 = \boxed{-4}$

من الرسم نجد أن :

* $\lim_{x \to -2} f(x) = 1$
* $\lim_{x \to -2} g(x) = -1$

---

② $\lim_{x \to 1} [f(x)\, g(x)]$

Does not exist

* $\lim_{x \to 1} f(x) = 2$
* $\lim_{x \to 1} g(x)$ does not exist

because :
the $\lim_{x \to 1} g(x)$ is not exist

where $\lim_{x \to 1^+} g(x) \neq \lim_{x \to 1^-} g(x)$

$\lim_{x \to 1^+} g(x) \to (-1)$, $\lim_{x \to 1^-} g(x) \to (-2)$

---

* $\lim_{x \to 2} f(x) \approx 1.4$
* $\lim_{x \to 2} g(x) = 0$

③ $\lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{\lim_{x \to 2} f(x)}{\lim_{x \to 2} g(x)} = \frac{1.4}{0} \Rightarrow$ Does not exist

لأن المقام zero

النهاية من حالة الدوال المعرفة بأكثر من قاعدة (فرعين أو أكثر)

• الحالة الأولى: وجود العلامات $>$ ، $<$

* نوجد النهاية اليمنى بالتعويض من الفرع الذي يحتوي على علامة أكبر من
* // // اليسرى // // // // // // // // أصغر من

□ إذا كانت: النهاية اليمنى = النهاية اليسرى تكون النهاية موجودة.

□ إذا كانت: النهاية اليمنى ≠ النهاية اليسرى تكون النهاية غير موجودة.
(Does not exist)

Example:

If: $$f(x) = \begin{cases} \sqrt{x-4} & ; x > 4 \\ 8-2x & ; x < 4 \end{cases}$$

Find the $\lim\limits_{x \to 4} f(x)$?

* نوجد النهاية اليمنى من الفرع الذي يحتوي على أكبر من

$$\lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = \sqrt{0} = 0$$

* نوجد النهاية اليسرى من الفرع الذي يحتوي على أصغر من

$$\lim_{x \to 4^-} (8-2x) = 8-2(4) = 8-8 = 0$$

∵ النهاية اليمنى = النهاية اليسرى = zero

∴ النهاية موجودة Exist وتساوي zero

$$\therefore \lim_{x \to 4} f(x) = 0$$

• الحالة الثانية: وجود علامة $\neq$ ، $=$

Example:

$$f(x) = \begin{cases} 2x+1 & ; x \neq 3 \\ x+5 & ; x = 3 \end{cases}$$

Find: $\lim\limits_{x \to 3} f(x)$?

نستخدم الفرع الذي يحتوي على ≠

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} (2x+1) = 2(3)+1 = \boxed{7}$$

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• في حاله وجود داله القيمه المطلقه $f(x) = |\quad|$

لابد من أعاده تعريف المطلقه

ثم ايجاد النهايه اليمنى من عند اكبر من

والنهايه اليسرى من عند اصغر من .

Example :

Find : $\lim_{x \to 0} \frac{|x|}{x}$?

لابد من اعاده تعريف المطلقه :

$|x|$

نضع مابداخل المطلقه ← $x = 0$

عكس		مثل
$-x$	0	$+x$

* $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} (1) = \boxed{1}$ النهايه اليمنى

* $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = \boxed{-1}$ النهايه اليسرى

$\because \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\therefore \lim_{x \to 0} f(x)$ Does not exist.

اى أن النهايه غير موجوده

لعدم تساوى النهايتين

اليمنى واليسرى .

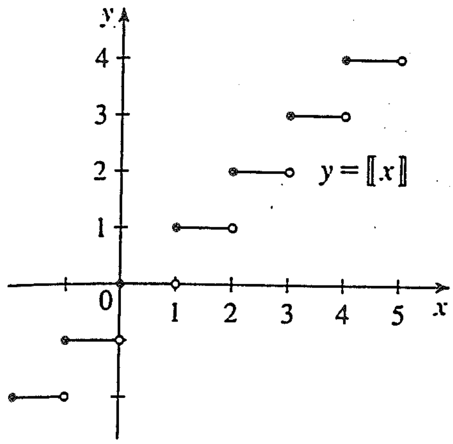
• The greatest integer function (داله الصحيح)

is defined by

$[\![x]\!]$ = the largest integer that is less than or equal to x

$$[\![x]\!] = a \quad \text{for } a \leq x < a+1$$

Other notations for $[\![x]\!]$ are $[x]$ and $\lfloor x \rfloor$. The greatest integer function is sometimes called the *floor function*.



Greatest integer function

Note:

$[\![$ عدد صحيح $]\!]$ = نفس العدد الصحيح

$[\![$ عدد غير صحيح $]\!]$ = العدد الصحيح الأقل من العدد الغير صحيح (الموجود على يساره)

Example: Find the value of :

$[\![2]\!] = 2$ ، $[\![-2]\!] = -2$ ، $[\![2.9]\!] = 2$ ، $[\![-2.9]\!] = -3$

$[\![\pi]\!] = 3$ (π ↳ 3.14) ، $[\![e]\!] = 2$ (e ↳ 2.7) ، $[\![\sqrt{3}]\!] = 1$ ($\sqrt{3}$ ↳ 1.7) ، $[\![-\sqrt{3}]\!] = -2$

Example: Find $\lim_{x \to 3} [\![x]\!]$

* $\lim_{x \to 3^+} [\![x]\!] = \boxed{3}$ كأنه تعويض مباشر

* $\lim_{x \to 3^-} [\![x]\!] = \boxed{2}$ كأنه تعويض مباشر -1 ⇒ اليمين ≠ اليسار

$\Rightarrow \lim_{x \to 3} [\![x]\!]$ Does not exist.

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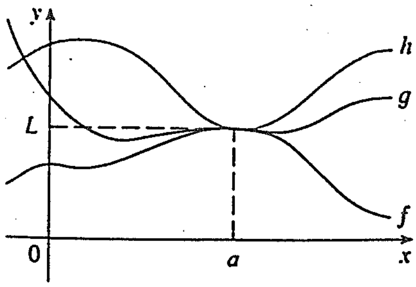
نظريات هامه

① If: $f(x) \leq g(x) \Rightarrow \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$

---

② (squeeze theorem)



If: $f(x) \leq g(x) \leq h(x)$?

and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

then $\lim_{x\to a} g(x) = L$

Example:

If: $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$

Find: $\lim_{x\to 4} F(x)$?

* $\lim_{x\to 4} (4x-9) = 16-9 = \boxed{7}$
* $\lim_{x\to 4} (x^2 - 4x + 7) = 16 - 16 + 7 = \boxed{7}$

$\lim_{x\to 4} f(x) = 7$

---

③ من حاله ايجاد نهايه حاصل ضرب دالتين احدهما نهايتها صفر والأخرى محدوده يكون الناتج zero

Example:

① $\lim_{x\to 0} \underbrace{x^2}_{\text{نهايتها صفر}} \cdot \underbrace{\sin \frac{1}{x}}_{\text{داله محدوده}} = 0$

② $\lim_{x\to 0} \underbrace{x^4}_{\text{نهايتها صفر}} \cdot \underbrace{\cos \frac{2}{x}}_{\text{داله محدوده}} = 0$

$-1 \leq \begin{matrix}\sin\\ \cos\end{matrix} \leq 1$

اى أن دوال sin & cos
دوال محدوده دائماً
بين 1- و 1

مهم جداً : عند إيجاد النهاية

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

↓

تعويض مباشر عن x ب a

الناتج: عدد أو ∞ أو $-\infty$

نتوقف
STOP

الناتج : $\frac{0}{0}$ أو $\frac{\infty}{\infty}$ أو $\infty - \infty$

* نحلل البسط والمقام
* نختصر المتشابه بين البسط والمقام
* نعوض بعد الاختصار عن x ب a
* في حالة وجود ($\sqrt{\ }$ - عدد) أو (عدد - $\sqrt{\ }$) أو ($\sqrt{\ } - \sqrt{\ }$)
نضرب في المرافق ← conjugate
* في حالة وجود كسور ← توحيد مقامات.

●● هناك تصرف أسرع وأسهل (بدلاً من التحليل ----)
وهو استخدام قاعدة لوبيتال ← L'Hopital Rule
بأن نشتق البسط والمقام كلاً على حده ثم التعويض بعد الاشتقاق عن x ب a
إذا كان الناتج بعد الاشتقاق :

(1) عدد أو ∞ أو $-\infty$ نتوقف STOP

(2) $\frac{0}{0}$ أو $\frac{\infty}{\infty}$ ---- نشتق مره أخرى ونعوض عن x ب a ------
حتى نحصل على الناتج عدد أو ∞ أو $-\infty$.

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Exercises :

① $\lim_{x \to 2} (2x+1) = 2(2)+1 = \boxed{5}$ تعويض مباشر

عدد ← stop.

② $\lim_{y \to 5} \frac{y^2}{5-y} = \frac{(5)^2}{5-5} = \frac{25}{0} = \boxed{\infty}$ تعويض مباشر

→ stop.

③ $\lim_{x \to 2} \frac{x^2+x-6}{x-2} = \frac{4+2-6}{2-2} = \frac{0}{0}$ (I.f.) حالة عدم تعيين

$\lim_{x \to 2} \frac{(x+3)(x-2)}{(x-2)} = \lim_{x \to 2}(x+3) = 2+3 = \boxed{5}$ ← * ممكن بالتحليل

* أو ممكن باستخدام قاعدة لوبيتال (by L.H.R)

بأن نشتق البسط والمقام كلاً على حده.

$\lim_{x \to 2} \frac{x^2+x-6}{x-2} = \lim_{x \to 2} \frac{2x+1}{1} = 2(2)+1 = \boxed{5}$

④ $\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4} = \frac{16-20+4}{16-12-4} = \frac{0}{0}$ (I.f.) حالة عدم تعيين

$\lim_{x \to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{x \to -4} \frac{x+1}{x-1} = \frac{-3}{-5} = \boxed{\frac{3}{5}}$ ← * ممكن بالتحليل

* ممكن باستخدام قاعدة لوبيتال (by L.H.R) ← أسهل وأسرع

$\lim_{x \to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{x \to -4} \frac{2x+5}{2x+3} = \frac{-8+5}{-8+3} = \frac{-3}{-5} = \boxed{\frac{3}{5}}$

⑤ $\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} = \frac{16-16}{16-12-4} = \frac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x\to 4} \frac{2x-4}{2x-3} = \frac{8-4}{8-3} = \boxed{\frac{4}{5}}$

---

⑥ $\lim_{t\to -3} \frac{t^2-9}{2t^2+7t+3} = \frac{9-9}{18-21+3} = \frac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{t\to -3} \frac{2t}{4t+7} = \frac{-6}{-12+7} = \frac{\not{-}6}{\not{-}5} = \boxed{\frac{6}{5}}$

---

⑦ $\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right) = \frac{1}{0} - \frac{1}{0} = \infty - \infty$ (I.f.)

* في حالة الكسور توحيد مقامات أولاً

$= \lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t(t+1)}\right) = \lim_{t\to 0} \left(\frac{t+1}{t(t+1)} - \frac{1}{t(t+1)}\right)$

$= \lim_{t\to 0} \left(\frac{t+\not{1}-\not{1}}{t(t+1)}\right) = \lim_{t\to 0} \frac{t}{t(t+1)}$

$= \lim_{t\to 0} \frac{1}{t+1} = \frac{1}{0+1}$

$= \frac{1}{1} = \boxed{1}$

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⑧ $\lim_{h \to 0} \dfrac{(4+h)^2 - 16}{h} = \dfrac{(4+0)^2 - 16}{0} = \dfrac{16-16}{0} = \dfrac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{h \to 0} \dfrac{2(4+h)' \cdot 1}{1} = \lim_{h \to 0} 2(4+h) = 2(4+0) = \boxed{8}$

⑨ $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1} = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x \to 1} \dfrac{3x^2}{2x} = \lim_{x \to 1} \dfrac{3x}{2} = \dfrac{3(1)}{2} = \boxed{\dfrac{3}{2}}$

⑩ $\lim_{x \to -2} \dfrac{x+2}{x^3+8} = \dfrac{-2+2}{-8+8} = \dfrac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x \to -2} \dfrac{1}{3x^2} = \dfrac{1}{3(-2)^2} = \dfrac{1}{3(4)} = \boxed{\dfrac{1}{12}}$

⑪ $\lim_{t \to 9} \dfrac{9-t}{3-\sqrt{t}} = \dfrac{9-9}{3-3} = \dfrac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{t \to 9} \dfrac{-1}{-\frac{1}{2\sqrt{t}}} = \lim_{t \to 9} 1 \cdot \dfrac{2\sqrt{t}}{1} = 2\sqrt{9} = 2(3) = \boxed{6}$

(12) $\lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7} = \frac{\sqrt{9} - 3}{7 - 7} = \frac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x \to 7} \frac{\frac{1}{2\sqrt{x+2}}}{1} = \lim_{x \to 7} \frac{1}{2\sqrt{x+2}} = \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}$

---

(13) $\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} = \frac{\sqrt{1+0} - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{h \to 0} \frac{\frac{1}{2\sqrt{1+h}}}{1} = \lim_{h \to 0} \frac{1}{2\sqrt{1+h}} = \frac{1}{2\sqrt{1+0}} = \boxed{\frac{1}{2}}$

---

(14) $\lim_{x \to 2} \frac{x^4 - 16}{x - 2} = \frac{16 - 16}{2 - 2} = \frac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x \to 2} \frac{4x^3}{1} = 4(2)^3 = 4(8) = \boxed{32}$

---

(15) $\lim_{x \to -4} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} = \frac{\frac{1}{4} + \frac{1}{-4}}{4 + (-4)} = \frac{\frac{1}{4} - \frac{1}{4}}{4 - 4} = \frac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x \to -4} \frac{0 + (\frac{-1}{x^2})}{0 + 1} = \lim_{x \to -4} -\frac{1}{x^2} = -\frac{1}{(-4)^2} = \boxed{-\frac{1}{16}}$

مشتقة $\frac{1}{x}$ هي $-\frac{1}{x^2}$

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(16) $\lim_{x \to 9} \dfrac{x^2 - 81}{\sqrt{x} - 3} = \dfrac{81 - 81}{3 - 3} = \dfrac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x \to 9} \dfrac{2x}{\frac{1}{2\sqrt{x}}} = \lim_{x \to 9} 2x \cdot 2\sqrt{x}$

$= \lim_{x \to 9} 4x\sqrt{x} = 4(9)\sqrt{9} = 36(3) = \boxed{108}$

---

(17) $\lim_{h \to 0} \dfrac{(3+h)^{-1} - 3^{-1}}{h} = \dfrac{3^{-1} - 3^{-1}}{0} = \dfrac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{h \to 0} \dfrac{-1(3+h)^{-2} \cdot 1}{1} = \lim_{h \to 0} \dfrac{-1}{(3+h)^2}$

$= \dfrac{-1}{(3+0)^2} = \dfrac{-1}{(3)^2} = \boxed{\dfrac{-1}{9}}$

---

(18) $\lim_{x \to -4} \dfrac{\sqrt{x^2+9} - 5}{x + 4} = \dfrac{\sqrt{16+9} - 5}{-4 + 4} = \dfrac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x \to -4} \dfrac{\frac{2x}{2\sqrt{x^2+9}}}{1} = \lim_{x \to -4} \dfrac{x}{\sqrt{x^2+9}}$

$= \dfrac{-4}{\sqrt{16+9}} = \dfrac{-4}{\sqrt{25}} = \boxed{\dfrac{-4}{5}}$

(19) $\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\frac{1}{0}-\frac{1}{0}=\infty-\infty$ (I.f.)

توحيد مقامات .

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{\sqrt{1+t}}{t\sqrt{1+t}}\right)=\lim_{t\to 0}\left(\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\right)$$

$$=\frac{0}{0}\ (\text{I.f.})$$

بالضرب في مرافقه البسط
(بسطاً ومقاماً)

$$\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\cdot\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}$$

$$=\lim_{t\to 0}\frac{1-(1+t)}{t\sqrt{1+t}\,(1+\sqrt{1+t})}=\lim_{t\to 0}\frac{\cancel{1}-\cancel{1}-t}{t\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$$=\lim_{t\to 0}\frac{-\cancel{t}}{\cancel{t}\sqrt{1+t}\,(1+\sqrt{1+t})}=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$$=\frac{-1}{\sqrt{1+0}\,(1+\sqrt{1+0})}=\frac{-1}{1\,(1+1)}=\boxed{\frac{-1}{2}}$$

---

(20) $\lim_{x\to 0}\sqrt{x^3+x^2}\ \sin\frac{\pi}{x}=\boxed{0}$ نظرية

$\sqrt{x^3+x^2}$ ↓ داله نهايتها zero

$\sin\frac{\pi}{x}$ ↓ داله محدوده بين ١- ، ١

↓ (3) page 8

(21) $\lim_{x \to 3} (2x + |x-3|)$

اعادة تعريف المطلقة

$x - 3 = 0$

$x = 3$

مثل | عكس

$-(x-3)$ | 3 | $(x-3)$

$= -x + 3$

النهاية اليمنى

$* \lim_{x \to 3^+} (2x + x - 3)$

$= \lim_{x \to 3^+} (3x - 3) = 9 - 3 = \boxed{6}$

النهاية اليسرى

$* \lim_{x \to 3^-} (2x - x + 3)$

$= \lim_{x \to 3^-} (x + 3) = 3 + 3 = \boxed{6}$

النهاية اليمنى = النهاية اليسرى

$\because \lim_{x \to 3^+} = \lim_{x \to 3^-} = 6 \Rightarrow \lim_{x \to 3} (2x + |x-3|) = \boxed{6}$

---

(22) $\lim_{x \to -6} \frac{2x+12}{|x+6|}$

اعادة تعريف المطلقة

$x + 6 = 0$

$x = -6$

مثل | عكس

$-(x+6)$ | -6 | $(x+6)$

$* \lim_{x \to -6^+} \frac{2(x+6)}{(x+6)} = \boxed{2}$

$* \lim_{x \to -6^-} \frac{2(x+6)}{-(x+6)} = \frac{2}{-1} = \boxed{-2}$

$\therefore \lim_{x \to -6^+} \neq \lim_{x \to -6^-}$

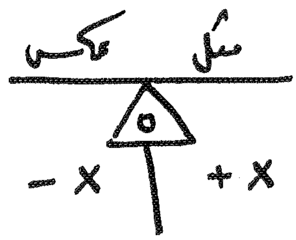
النهاية اليمنى ≠ النهاية اليسرى

$\therefore \lim_{x \to -6} \frac{2x+12}{|x+6|}$ does not exist

(23) $\lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

اعادة تعريف المطلقة

$x = 0$



$= \lim_{x \to 0^-} \left(\frac{1}{x} - \frac{1}{-x} \right)$

$= \lim_{x \to 0^-} \left(\frac{1}{x} + \frac{1}{x} \right)$

$= \lim_{x \to 0^-} \left(\frac{2}{x} \right) = \frac{2}{0} = \boxed{\infty}$

---

(24) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

$= \lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{x} \right)$

$= \lim_{x \to 0^+} (0) = \boxed{0}$

---

(25) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{|x|} \right)$ does not exist.

لأن النهاية اليمنى $\neq$ النهاية اليسرى .

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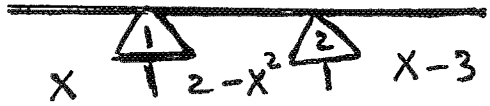
(26)

let : $g(x) = \begin{cases} x & ; \ x < 1 \\ 3 & ; \ x = 1 \\ 2 - x^2 & ; \ 1 < x \leq 2 \\ x - 3 & ; \ x > 2 \end{cases}$

Evaluat each of the following limits if it exists :

(1) $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^-} (x) = \boxed{1}$



(2) $\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} (2 - x^2) = 2 - 1 = \boxed{1}$

(3) $\lim\limits_{x \to 1} g(x) = 1$ (لأن النهاية اليمنى = النهاية اليسرى = 1)

(4) $g(1) = 3$

---

(5) $\lim\limits_{x \to 2^-} g(x) = \lim\limits_{x \to 2^-} (2 - x^2) = 2 - 4 = \boxed{-2}$

(6) $\lim\limits_{x \to 2^+} g(x) = \lim\limits_{x \to 2^+} (x - 3) = 2 - 3 = \boxed{-1}$

(7) $\lim\limits_{x \to 2} g(x)$ does not exist

لأن النهاية اليمنى $\neq$ النهاية اليسرى .

(27) If n is integer عدد صحيح

Find

① $\lim_{x \to n^-} [\![x]\!] = n - 1$ في النهاية اليسرى تعويض مباشر -1

② $\lim_{x \to n^+} [\![x]\!] = n$ في النهاية اليمنى تعويض مباشر فقط

③ $\lim_{x \to n} [\![x]\!]$ does not exist
لأن النهاية اليمنى ≠ النهاية اليسرى

(28) If : $f(x) = [\![x]\!] + [\![-x]\!]$

Find: $\lim_{x \to 2} f(x)$? and $f(2)$?

* $\lim_{x \to 2^+} ([\![x]\!] + [\![-x]\!]) = (2) + (-3) = 2-3 = -1$ (2.1 , -2.1)

* $\lim_{x \to 2^-} ([\![x]\!] + [\![-x]\!]) = (1) + (-2) = 1-2 = -1$ (1.9 , -1.9)

∴ النهاية اليمنى = النهاية اليسرى

∴ $\lim_{x \to 2} f(x) = -1$

* $F(2) = [\![2]\!] + [\![-2]\!]$
$= 2 + (-2) = 2-2 = 0$

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(29) If: $\lim_{x \to 1} \frac{f(x) - 8}{x - 1} = 10$

Find $\lim_{x \to 1} f(x)$?

$\because \lim_{x \to 1} \frac{f(x) - 8}{x - 1} = 10$

$\Rightarrow \frac{\lim_{x \to 1} f(x) - \lim_{x \to 1} 8}{\lim_{x \to 1} x - \lim_{x \to 1} 1} = 10$

$\Rightarrow \lim_{x \to 1} f(x) - \lim_{x \to 1} 8 = 10 \left(\lim_{x \to 1} x - \lim_{x \to 1} 1 \right)$

$\Rightarrow \lim_{x \to 1} f(x) - 8 = 10 (1 - 1)$

$\Rightarrow \lim_{x \to 1} f(x) = 10 (0) + 8$

0

$\therefore \lim_{x \to 1} f(x) = 8$

(30) If:

$$\lim_{x \to 0} \frac{f(x)}{x^2} = 5$$

Find:

(a) $\lim_{x \to 0} f(x)$

$\because \lim_{x \to 0} \frac{f(x)}{x^2} = 5$

$$\frac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} x^2} = 5$$

$\Rightarrow \lim_{x \to 0} f(x) = 5 \cdot \lim_{x \to 0} x^2$

$\lim_{x \to 0} f(x) = 5 \cdot (0)^2$

$\therefore \lim_{x \to 0} f(x) = \boxed{0}$

(b) $\lim_{x \to 0} \frac{f(x)}{x}$

$\because \lim_{x \to 0} \frac{f(x)}{x^2} = 5$

$$\lim_{x \to 0} \left(\frac{F(x)}{x} \cdot \frac{1}{x} \right) = 5$$

$$\lim_{x \to 0} \frac{f(x)}{x} \cdot \lim_{x \to 0} \frac{1}{x} = 5$$

$$\lim_{x \to 0} \frac{f(x)}{x} = \frac{5}{\lim_{x \to 0} \frac{1}{x}}$$

$$= \frac{5}{\frac{1}{0}}$$

$$= 5 \cdot \frac{0}{1}$$

$\therefore \lim_{x \to 0} \frac{F(x)}{x} = 5 \cdot (0) = \boxed{0}$

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(31) $\lim_{x \to 2} \dfrac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} = \dfrac{\sqrt{6-2} - 2}{\sqrt{3-2} - 1} = \dfrac{\sqrt{4}-2}{\sqrt{1}-1}$

$= \dfrac{2-2}{1-1} = \dfrac{0}{0} \; (I.f.)$

* ممكن الضرب من المرافقه مره للبسط ومره للمقام
لكن الأسهل والأسرع استخدام لوبيتال

$$= \lim_{x \to 2} \dfrac{\dfrac{-1}{2\sqrt{6-x}}}{\dfrac{-1}{2\sqrt{3-x}}}$$

$$= \lim_{x \to 2} \dfrac{-1}{2\sqrt{6-x}} \cdot \dfrac{2\sqrt{3-x}}{-1}$$

$$= \lim_{x \to 2} \dfrac{\sqrt{3-x}}{\sqrt{6-x}} = \dfrac{\sqrt{3-2}}{\sqrt{6-2}} = \dfrac{\sqrt{1}}{\sqrt{4}} = \boxed{\dfrac{1}{2}}$$

(32) If there a number a such that :

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$ exist موجوده

(1) Find the value of a.

(2) Find the value of the limit .

① ∵ $\lim_{x \to -2} \frac{3(-2)^2 + a(-2) + a + 3}{(-2)^2 + (-2) - 2}$

$$= \frac{12 - 2a + a + 3}{4 - 2 - 2} = \frac{15 - a}{0}$$

∵ النهايه exist موجوده

∴ لابد أن البسط = 0

$15 - a = 0 \Rightarrow -a = -15 \Rightarrow \boxed{a = 15}$

② $\lim_{x \to -2} \frac{3x^2 + 15x + 15 + 3}{x^2 + x - 2}$

$$= \lim_{x \to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \frac{12 - 30 + 18}{4 - 2 - 2} = \frac{0}{0}$$ (I.F.)

(by L.H.R) $= \lim_{x \to -2} \frac{6x + 15}{2x + 1} = \frac{-12 + 15}{-4 + 1} = \frac{3}{-3} = \boxed{-1}$

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(33) Find: $\lim_{x \to 0} \sqrt{x}\; e^{\sin(\frac{\pi}{x})}$

$\sqrt{x}$ ← دالة نهايتها zero

$e^{\sin(\frac{\pi}{x})}$ ← دالة محدودة حيث أن

$$-1 \leq \sin(\tfrac{\pi}{x}) \leq 1$$

$$e^{-1} \leq e^{\sin(\frac{\pi}{x})} \leq e^{1}$$

∴ $e^{\sin(\frac{\pi}{x})}$ دالة محدودة بين e^{-1} ، e

∴ $\lim_{x \to 0} \sqrt{x}\; e^{\sin(\frac{\pi}{x})} = 0$ (نظرية)

$\sqrt{x}$ ← الدالة الأولى نهايتها صفر

$e^{\sin(\frac{\pi}{x})}$ ← دالة محدودة بين e^{-1} ، e

(نظرية) ← (3) page 8

مع كل التمنيات بالنجاح والتوفيق

السعدي

2.5 Continuity
الاتصال

10

Notes

- التركيز على المفاهيم الاساسية.
- شرح أبواب المنهج حسب الخطة.
- أمثلة توضيحية وتدريبات.
- نماذج اختبارات.

السعدي

رياضيات ١١٠
Math.110

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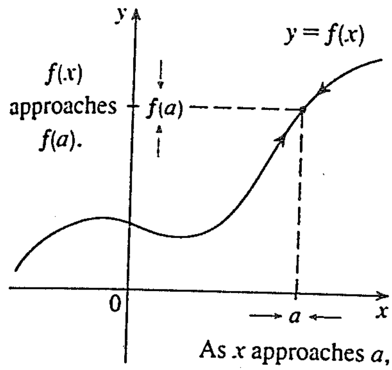
2.5

Continuity
الأتصال

- Continuous at the number $X = a$ ← الأتصال عند عدد

- requires three things
  يتطلب ثلاثة اشياء



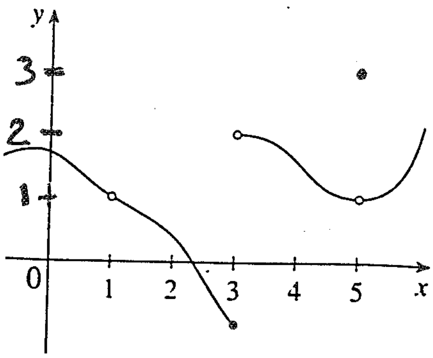
① $f(a)$ is defined (الداله معرفه عند a)

② $\lim_{x \to a} f(x)$ exist (النهايه موجوده)

③ $\lim_{x \to a} f(x) = F(a)$ (النهايه = قيمه الداله)

* اذا لم تتحقق الشروط الثلاثه السابقه معاً
تكون الداله غير متصله عند $x = a$ ⟵ (discontinuous at $\underline{a}$)

Example: From the figure



** at $X = 1$

$F(1)$ is not defined

$\therefore f(x)$ is discontinuous at $X=1$

** at $X = 3$

$\lim_{x \to 3} f(x)$ does not exist

$\therefore f(x)$ is discontinuous at $X=3$

** at $X = 5$

$(F(5) = 3)$ $(\lim_{x \to 5} f(x) = 1)$

$\therefore f(x)$ is discontinuous at $X=5$

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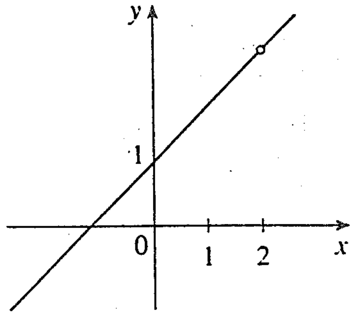
Example:

Where are each of the following functions discontinuous?

① $f(x) = \dfrac{x^2 - x - 2}{x - 2}$



$f(x) = \dfrac{x^2 - x - 2}{x - 2}$

$f(2)$ is not defined

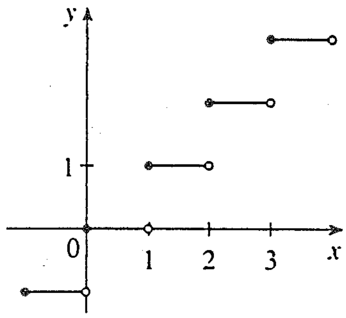
So $f(x)$ is discontinuous at $x = 2$

or $f(x)$ is continuous on $R - \{2\}$

بشكل آخر $f(x)$ 〃 〃 on $(-\infty, 2) \cup (2, \infty)$

---

② $f(x) = [\![x]\!]$

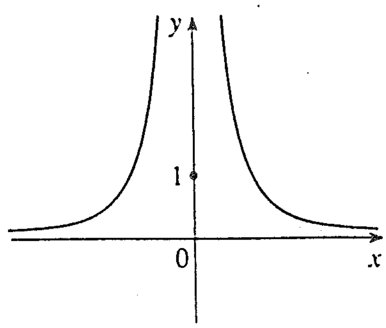


$f(x) = [\![x]\!]$

$f(x)$ is discontinuous at all of the integers where the $\lim\limits_{x \to n} [\![x]\!]$ does not exist (where n is integer)

---

③ $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{1}{x^2}$ does not exist

So $f(x)$ is discontinuous at $x = 0$

Definition :

If: $\lim_{x \to a^+} f(x) = f(a) \Rightarrow f$ is continuous from the right a.
متصله على يمينا a

If: $\lim_{x \to a^-} f(x) = f(a) \Rightarrow f$ is continuous from the left a.
متصله على يسار a

Example :

If: $f(x) = [\![x]\!]$

at each integer a

* $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} [\![x]\!] = a$ * $f(a) = a$

$\therefore$ $f(x)$ is continuous from the right a

متصله على يمين العدد a

* $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} [\![x]\!] = a - 1$ * $f(a) = a$

$\therefore$ $f(x)$ is discontinuous from the left a

غير متصله على يسار العدد a

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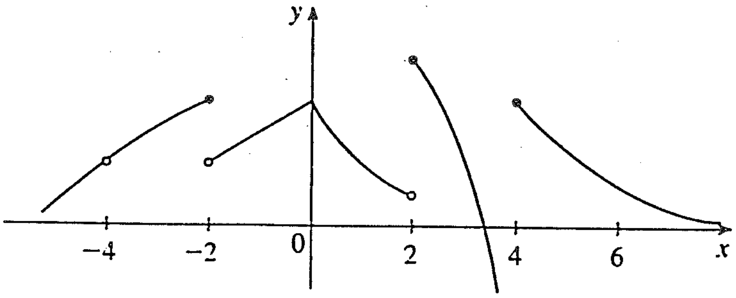
3. (a) From the graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.



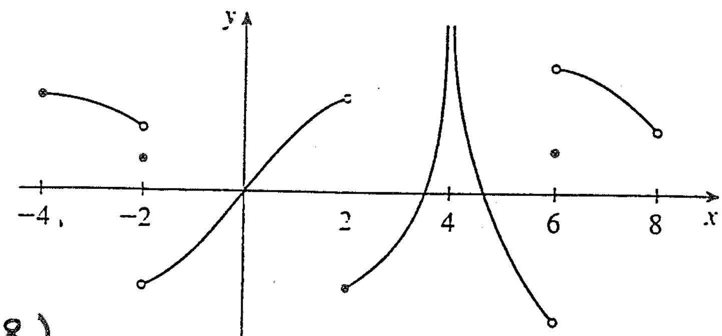
* at $x = -4$ F(x) is discontinuous where f(-4) undefined
F(x) <u>neither</u> continuous from right <u>nor</u> from left.

* at $x = -2$ f(x) is discontinuous (Jump) → قفزة
F(x) is continuous from the left. ($\lim_{x \to -2^-} f(x) = f(-2)$)

* at $x = 2$ f(x) is discontinuous (Jump)
F(x) is continuous from the right. ($\lim_{x \to 2^+} f(x) = f(2)$)

* at $x = 4$ f(x) is discontinuous (Jump)
F(x) is continuous from the right. ($\lim_{x \to 4^+} f(x) = f(4)$)

4. From the graph of g, state the intervals on which g is continuous.



g(x) is continuous on:
* [-4, -2) * (-2, 2)
* [2, 4) * (4, 6) * (6, 8)

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● Continuous on the interval — الاتصال على فترة

① اذا كانت $f(x)$ كثيرة حدود (Polynomial)
تكون متصلة على $R = (-\infty, \infty)$

② اذا كانت $f(x)$ كسرية (rational)
تكون متصلة على $R - \{$اصفار المقام$\}$

③ اذا كانت $f(x)$ جذرية (root function)

$$F(x) = \sqrt[n]{\ }$$

n is odd — دليل الجذر فردي

- الجذر في المقام: تكون الدالة متصلة على $R - \{$اصفار المقام$\}$
  - $F(x) = \dfrac{1}{\sqrt[3]{x-2}}$ متصلة على $R - \{2\}$
- الجذر في البسط: تكون الدالة متصلة على $R = (-\infty, \infty)$
  - $F(x) = \sqrt[3]{x-2}$ متصلة على $(-\infty, \infty)$

n is even — دليل الجذر زوجي

- الجذر في المقام: تكون الدالة متصلة على الفترات الموجبة مفتوحة من عند العدد
  - $F(x) = \dfrac{1}{\sqrt{x-2}}$ متصلة على $(2, \infty)$
- الجذر في البسط: تكون الدالة متصلة على الفترات الموجبة مغلقة من عند العدد
  - $F(x) = \sqrt{x-2}$ متصلة على $[2, \infty)$

Note that :

The following types of function

are continuous (متصلة) on their domain (مجالها).

* trigonometric function. الدوال المثلثية
* Inverse trigonometric function. الدوال المثلثية العكسية
* exponential function. الدوال الأسية
* logarithmic function. الدوال اللوغاريتمية

Exmple :

① $F(x) = \ln(x-2)$ دالة ln متصلة على الفترات الموجبة مفتوحة من عند العدد 2

$\therefore f(x)$ is continuous on $(2, \infty)$

---

② $f(x) = \tan^{-1}x$ دالة $\tan^{-1}x$ متصلة على مجالها

$\therefore f(x)$ is continuous on $(-\infty, \infty)$

---

③ $f(x) = \ln(x-2) + \tan^{-1}x$ متصلة على المجال المشترك

$\therefore f(x)$ is continuous

on $(2, \infty) \cap (-\infty, \infty) = (2, \infty)$

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Example :

Where is the function $f(x)$ continuous?

① $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$

* الدالة $f(x)$ متصلة على المجال المشترك لدالة $\ln x$ ، $\tan^{-1} x$
باستبعاد اصفار المقام

$\tan^{-1} x \rightarrow (-\infty, \infty)$

$\ln x \rightarrow (0, \infty)$

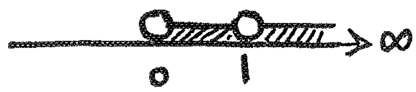
$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

$\therefore f(x)$ is continuous on $(-\infty, \infty) \cap (0, \infty) - \{-1, 1\}$

$= (0, \infty) - \{-1, 1\}$



$= (0, 1) \cup (1, \infty)$

---

② $f(x) = 2x^3 - x^2 + 1 \longrightarrow$ Polynomial كثيرة حدود

$f(x)$ is continuous on $(-\infty, \infty) = \mathbb{R}$

---

③ * $f(x) = 2$ * $f(x) = \sqrt{5}$ * $f(x) = -\frac{2}{3}$ * $f(x) = 0$

are continuous on $(-\infty, \infty) = \mathbb{R}$

④ $f(x) = |x-3|$ continuous on $(-\infty,\infty)$

⑤ $f(x) = \dfrac{1}{|x-3|}$ متصلة على $R -$ {اصفار المقام}

$\therefore f(x)$ continuous on $R-\{3\} = (-\infty,3)\cup(3,\infty)$

⑥ $f(x) = \dfrac{1}{|x|-3}$ اصفار المقام $|x|-3 = 0$

$|x| = 3 \Rightarrow x = \pm 3$

$\therefore f(x)$ continuous on $R-\{-3,3\} = (-\infty,-3)\cup(-3,3)\cup(3,\infty)$

⑦ $f(x) = \dfrac{2x-1}{|x|+3}$ المقام ليس له أصفار

لأن $|x|+3 = 0$

$|x| = -3$ مرفوض (discard)

$\therefore f(x)$ continuous on R.

⑧ $f(x) = \dfrac{3x}{x^2-9} \Rightarrow$ اصفار المقام $x^2-9 = 0$

$x^2 = 9 \Rightarrow x = \pm 3$

$\therefore f(x)$ continuous on $R-\{-3,3\} = (-\infty,-3)\cup(-3,3)\cup(3,\infty)$

⑨ $f(x) = \dfrac{3x}{x^2+9}$ المقام ليس له أصفار

لأن x^2+9

لا يمكن أن تساوي zero

$\therefore f(x)$ continuous on R.

⑩ $f(x) = \sqrt[3]{x^2-4}$ جذر تكعيبي في البسط

$\therefore f(x)$ continuous on $R = (-\infty,\infty)$

⑪ $f(x) = \dfrac{2x-1}{\sqrt[3]{x^2-4}}$ جذر تكعيبي في المقام

$\therefore$ متصلة على $R -$ {اصفار المقام}

$f(x)$ continuous on $R-\{-2,2\}$

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⑫ $f(x) = \sqrt[5]{x^2-x}$ الجذر الخامس (دليل الجذر n فردي)
والجذر من البسط
∴ f(x) متصلة على R .

---

⑬ $F(x) = \frac{2}{x}$ دالة كسرية
متصلة على
R − { اصفار المقام }

متصلة
∴ F(x) continuous on $R - \{0\}$

غير متصلة
OR f(x) is discontinuous at $x = 0$

---

⑭ $f(x) = \frac{2x-1}{x^2-5x+6} - 5x$ دالة كسرية
اصفار المقام
$x^2 - 5x + 6 = 0$
$(x-3)(x-2) = 0$
$x = 3$ ، $x = 2$

f(x) is continuous on $R - \{2, 3\}$

---

⑮ $F(x) = \frac{2x-1}{x^2-5x+6} + \frac{2x^2}{3}$

discontinuous at x = اصفار المقام
$x = 2, 3$

(16) Find the interval

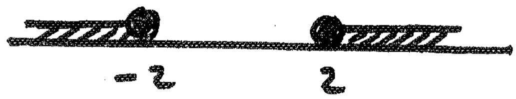
on which (1) $f(x) = \sqrt{|x|-2}$ is continuous.

* جذر تربيعي في البسط
∴ الداله متصله على الفترات الموجبه مُغلقه من عند العدد

$\Longrightarrow \quad |x| - 2 \geq 0$

$|x| \geq 2$

$x \geq 2$ or $x \leq -2$ نظريه



∴ $f(x)$ is continuous on $(-\infty, -2] \cup [2, \infty)$

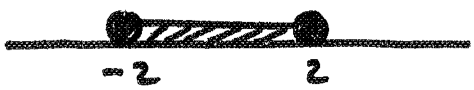
---

(2) $f(x) = \sqrt{2-|x|}$

$\Longrightarrow \quad 2 - |x| \geq 0$

$\Rightarrow -|x| \geq -2 \Longrightarrow |x| \leq 2$

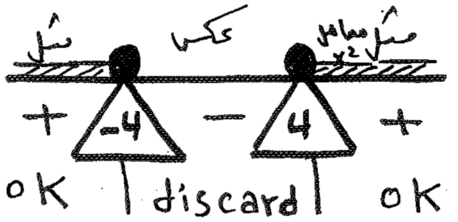
$-2 \leq x \leq 2$ نظريه



∴ $f(x)$ is continuous on $[-2, 2]$

(17) $f(x) = \sqrt{x^2-16}$

* جذر تربيعي في البسط
∴ متصله على الفترات الموجبه مغلقه من عند العدد



$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

$f(x)$ continuous on $(-\infty, -4] \cup [4, \infty)$

** $f(x)$ discontinuous on $(-4, 4)$

---

(18) $f(x) = \dfrac{2x}{\sqrt{x^2-16}}$

* جذر تربيعي في المقام
∴ متصله على الفترات الموجبه مفتوحه من عند العدد

نفس المثال السابق

$f(x)$ continuous on $(-\infty, -4) \cup (4, \infty)$

Note that :

الداله المعرفه بأكثر من قاعده في حاله وجود أكبر من و أقل من لكي تكون الداله متصله عند $x = a$

$$f(x) = \begin{cases} g(x) & ; x \geq a \\ h(x) & ; x < a \end{cases}$$

لابد أنه :

$$\lim_{x \to a^+} g(x) = \lim_{x \to a^-} h(x) = g(a)$$

قيمة الداله = النهايه اليسرى = النهايه اليمنى أولاً

قيمة الداله: التعويض في الطرف الموجود به علامة المساواه.

النهايه اليسرى: التعويض من طرف أصغر من.

النهايه اليمنى: التعويض من طرف أكبر من.

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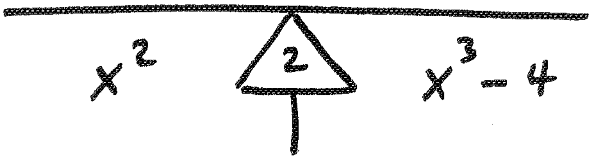
$$f(x) = \begin{cases} x^3 - 4 & ; \quad x \geq 2 \\ x^2 & ; \quad x < 2 \end{cases}$$



* $f(x)$ is continuous on $(-\infty, 2)$ and $(2, \infty)$
because : it is polynomail كثيرة حدود

* at $x = 2$ نقطة فاصلة
∴ لابد من إيجاد النهاية اليمنى ، اليسرى ، قيمة الدالة

• $\lim_{x \to 2^+} (x^3 - 4) = 8 - 4 = \underline{\underline{4}}$ * النهاية اليمنى
التعويض في طرف أكبر من

• $\lim_{x \to 2^-} x^2 = \underline{\underline{4}}$ * النهاية اليسرى
التعويض في طرف أصغر من

• $f(2) = (2^3 - 4) = 8 - 4 = \underline{\underline{4}}$ * قيمة الدالة
التعويض في الطرف الذي به علامة المساواة

$$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2)$$

∴ $f(x)$ is continuous at $x = 2$

** $f(x)$ is continuous on $(-\infty, \infty)$

Example :

Find the value of c

which makes

$$F(x) = \begin{cases} cx + 5 & ; x < 2 \\ cx^2 + 1 & ; x \geq 2 \end{cases}$$

is continuous at $x = 2$

$$\lim_{x \to 2^+} (cx^2+1) = \lim_{x \to 2^-} (cx+5)$$

$$c(2^2) + 1 = c(2) + 5$$

$$4c + 1 = 2c + 5$$

$$4c - 2c = 5 - 1$$

$$2c = 4 \Longrightarrow c = 2$$

Note that :

في حالة الدالة المعرفة بقاعدتين
احدهما تحتوي على $(x \neq a)$
والأخرى تحتوي على $(x = a)$
* نوجد النهاية من عند $\neq$ ، قيمة الدالة من عند $=$
اذا تساويا الناتجين تكون الدالة متصلة عند $x = a$
وإلا تكون الدالة غير متصلة عند $x = a$

Example :

$$\text{If}: \quad f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & ; \; x \neq 4 \\ 7 & ; \; x = 4 \end{cases}$$

Is $f(x)$ continuous at $x = 4$?

* نوجد النهاية من عند $x \neq 4$

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \frac{16 - 16}{4 - 4} = \frac{0}{0} \quad (\text{I.f.})$$

by L.H.R

$$\lim_{x \to 4} \frac{2x}{1} = \boxed{8}$$

* نوجد قيمة الدالة من عند $x = 4$

$$f(4) = \boxed{7}$$

$\because \lim_{x \to 4} f(x) \neq f(4)$ نهاية الدالة $\neq$ قيمة الدالة

$\therefore f(x)$ is discontinuous at $x = 4$

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Intermediate value theorem
الوسطى القيمة نظرية

* اذا كانت الداله $f(x)$ متصله على $[a, b]$
فإنه يوجد عدد واحد على الأقل c

where $a \leq c \leq b$

$$\Longrightarrow f(a) \leq f(c) \leq f(b)$$

Example :

For the function $f(x) = x^3 - x^2 + x$
there is a number $c \in [1, 3]$
Such that $f(c) = \ldots\ldots$

[A] 0 [B] 10 [C] -1 [D] 40

Solution

$$\because \quad c \in [1, 3]$$

$$1 \leq c \leq 3$$

$$f(1) \leq f(c) \leq f(3)$$

عوض بـ 1 في الداله $f(x)$ ، عوض بـ 3 في الداله $f(x)$

$$1 \leq f(c) \leq 21$$

الأختيار المناسب هو [B] 10 لأنه الوحيد المحصور بين 1 ، 21 .

• Removable discontinuity → ازاله عدم الأتصال

* اعاده تعريف الداله الغير متصله بحيث تصبح متصله *

اذا كانت f(x) داله غير متصله عند a

فإنه يمكن اعاده تعريف الداله f(x) بشكل آخر (داله أخرى g(x))

تكون متصله عند a كما يلي :

$$g(x) = \begin{cases} \underset{\text{قاعده الداله } f}{f(x)} & ; x \neq a \\ \underset{\text{نهايه الداله } f}{\lim_{x \to a} f(x)} & ; x = a \end{cases}$$

* مع العلم أن الداله f(x) : يمكن اعاده تعريفها اذا كانت نهايتها $\boxed{\frac{0}{0}}$

: لا يمكن اعاده تعريفها اذا كانت نهايتها $\boxed{\frac{\text{عدد}}{0}}$

Example:

which of the following functions f has removable discontinuity ?

① $f(x) = \dfrac{x^4+1}{x-1}$ discontinuous at $x=1$

$$\lim_{x \to 1} \frac{x^4+1}{x-1} = \frac{1+1}{1-1} = \frac{2}{0} = \boxed{\frac{\text{عدد}}{0}} = \infty$$

∴ discontinuity is not removable لا يمكن ازاله عدم الأتصال

(Infinite discontinuous)

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② $f(x) = \frac{x^4 - 1}{x - 1}$

دالة كسرية

∴ غير متصلة عند اصفار المقام

F(x) is discontinuous at $x = 1$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \frac{1-1}{1-1} = \boxed{\frac{0}{0}} \ (I.f.)$$

⇒ we can removable discontinuity

يمكننا ازالة عدم الاتصال كما يلي

$$* \lim_{x \to 1} \frac{4x^3}{1} = 4(1^3) = 4$$

$$\Rightarrow \quad g(x) = \begin{cases} \frac{x^4 - 1}{x - 1} & ; \ x \neq 1 \\ 4 & ; \ x = 1 \end{cases}$$

(قاعدة الدالة f ؛ نهاية الدالة f)

---

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If: f and g are continuous functions with $f(3) = 5$ and $\lim_{x \to 3} [2f(x) - g(x)] = 4$

Find: $g(3)$?

$$\lim_{x \to 3} [2f(x) - g(x)] = 4$$

$$2f(3) - g(3) = 4$$

$$2(5) - g(3) = 4 \quad \Rightarrow \quad g(3) = 10 - 4 = \boxed{6}$$

Example: page 125

Evaluate

$$\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$$

$$= \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}\right)$$

تعويض مباشر $\frac{0}{0}$

لوبيتال by L.H.R

$$= \arcsin\left(\lim_{x \to 1} \frac{-\frac{1}{2\sqrt{x}}}{-1}\right)$$

$$= \arcsin\left(\lim_{x \to 1} \frac{1}{2\sqrt{x}}\right)$$

$$= \arcsin\left(\frac{1}{2\sqrt{1}}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) = 30 = \frac{\pi}{6}$$

كل التمنيات بالنجاح والتوفيق

السعدي

2.6

- limits at infinity.
- Horizontal asymptotes.

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Notes

- التركيز على المفاهيم الاساسية.
- شرح أبواب المنهج حسب الخطة.
- أمثلة توضيحية وتدريبات.
- نماذج اختبارات.

السعدي

رياضيات ١١٠
Math. 110

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2.6

- Limits at infinity
  النهاية عند اللانهاية
- Horizontal asymptotes
  خطوط التقارب الأفقية

Note that :

* $(\infty)^n = \infty$ حيث n موجب

* $(-\infty)^n = \begin{cases} \infty \rightarrow & \text{حيث n زوجي} \\ -\infty \rightarrow & \text{حيث n فردي} \end{cases}$

* $(\pm\infty)^n = \text{zero}$ حيث n سالب

* $\dfrac{\text{عدد}}{\pm\infty} = 0$

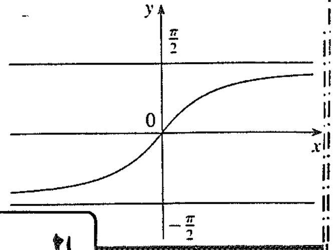
* $\dfrac{\pm\infty}{\text{عدد}} = \pm\infty$

* $\left(\dfrac{a}{b}\right)^{\infty} = 0$ اذا كانت a أصغر من b $\Rightarrow \left(\dfrac{2}{3}\right)^{\infty} = 0$

* $\left(\dfrac{a}{b}\right)^{\infty} = \infty$ اذا كانت a أكبر من b $\Rightarrow \left(\dfrac{3}{2}\right)^{\infty} = \infty$

* $e^{\infty} = \infty$ * $e^{-\infty} = 0$

* $\tan^{-1}\infty = \dfrac{\pi}{2}$ * $\tan^{-1}-\infty = \dfrac{-\pi}{2}$ → $\tan^{-1}x$



* أسهل طريقة لإيجاد النهاية

$$\lim_{x \to \pm\infty} \frac{\text{بسط}}{\text{مقام}}$$

(1) $\lim_{x \to \pm\infty} \frac{2x^2 - x}{3x^2 + 1} = \frac{2}{3}$

اذا كانت درجة البسط = درجة المقام)
يكون الناتج $\frac{\text{معامل أكبر أس لـ } x \text{ في البسط}}{\text{معامل أكبر أس لـ } x \text{ في المقام}}$

(2) $\lim_{x \to \pm\infty} \frac{2x + 1}{x^2 - x} = 0$

اذا كانت درجة البسط أصغر من درجة المقام)
يكون الناتج Zero

(3)* $\lim_{x \to \infty} \frac{x^3 - 2x^2}{x + 1}$

$= \frac{+}{+}\infty = \infty$

اذا كانت درجة البسط أكبر من درجة المقام)
الناتج ∞ ولتحديد اشارته
نعوض في الحد الذي يحتوي على أكبر أس في البسط
والحد الذي يحتوي على أكبر أس في المقام
عن x

* $\lim_{x \to -\infty} \frac{2x^3 - x - 2}{2x + x^2} = \frac{-}{+}\infty = -\infty$

* $\lim_{x \to -\infty} \frac{2x^3 - x - 2}{2x - x^2} = \frac{-}{-}\infty = \infty$

Find the limits :

(1) $\lim_{x \to \infty} \left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^\infty = 0$

البسط اصغر من المقام
الناتج ٥

(2) $\lim_{x \to \infty} \left(\frac{3}{2}\right)^x = \left(\frac{3}{2}\right)^\infty = \infty$

البسط أكبر من المقام
الناتج ∞

(3) $\lim_{x \to -\infty} \left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-\infty} = \left(\frac{3}{2}\right)^\infty = \infty$

(4) $\lim_{x \to -\infty} \left(\frac{\pi}{e}\right)^x = \left(\frac{\pi}{e}\right)^{-\infty} = \left(\frac{e}{\pi}\right)^\infty = 0$

$e \approx 2.7$ أصغر من $\pi \approx 3.14$

(5) $\lim_{x \to \pm\infty} \frac{1}{2 + \frac{1}{x}} = \frac{1}{2 + \frac{1}{\pm\infty}} = \frac{1}{2 + 0} = \frac{1}{2}$

(6) $\lim_{x \to \infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

درجة البسط ← -1
أكبر من
درجة المقام ← -2

$= \frac{+}{+}\infty = \infty$

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(7) $\lim_{x \to \infty} \left(\sqrt{x+2} - \sqrt{x} \right) = \infty - \infty \quad (I.F.)$

حالة عدم تعيين

الضرب في المرافق

ضرب ضرب

$$= \lim_{x \to \infty} \sqrt{x+2} - \sqrt{x} \cdot \frac{\sqrt{x+2} + \sqrt{x}}{\sqrt{x+2} + \sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{x + 2 - x}{\sqrt{x+2} + \sqrt{x}} = \lim_{x \to \infty} \frac{2}{\sqrt{x+2} + \sqrt{x}} = \frac{2}{\infty} = 0$$

(8) $\lim_{x \to \infty} \frac{\sqrt{1 + 4x^2}}{4 + x} = \frac{+\sqrt{4}}{+1} = 2$

x^2 تحت الجذر تعني x

اي أن البسط درجة أولى

والمقام درجة أولى

(9) $\lim_{x \to -\infty} \frac{\sqrt{1 + 4x^2}}{4 + x} = \frac{-\sqrt{4}}{1} = -2$

(10) $\lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)}{1 + \frac{1}{x}} = \frac{\cos\left(\frac{1}{\infty}\right)}{1 + \frac{1}{\infty}}$

$$= \frac{\cos 0}{1 + 0} = \frac{1}{1} = 1$$

نظرية هامة جداً :

$$\lim_{x \to a} f(x) \cdot g(x) = 0$$

نهاية حاصل ضرب دالتين احدهما نهايتها ٥ والأخرى محدوده

يكون الناتج ٥ .

* ملحوظة : داله sin ، cos دوال محدوده دائماً

$$-1 \leqslant \begin{matrix} \sin \\ \cos \end{matrix} \leqslant 1$$

Example :

Find the limits

① $\lim_{x \to \infty} \frac{1}{x} \cos x = 0$

$\frac{1}{x} \to \frac{1}{\infty} = 0$ ، $\cos x \to -1 \leqslant \cos x \leqslant 1$ محدوده

حاصل ضرب دالتين

* الأولى نهايتها ٥

$$\lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

* الثانية $\cos x$

محدوده بين ١- ، ١

② $\lim_{x \to \infty} \frac{\sin 2x}{x}$

$$= \lim_{x \to \infty} \frac{1}{x} \cdot \sin 2x = 0$$

$\frac{1}{x} \to \frac{1}{\infty} = 0$ ، $\sin 2x \to -1 \leqslant \sin 2x \leqslant 1$ محدوده

Find the limit :

① $\lim_{x \to \infty} \sqrt{x^2+x} - \sqrt{x^2-x} = \infty - \infty$ (I.F.)

حالة عدم تعيين

بالضرب في المرافق

ضرب ضرب

$$= \lim_{x \to \infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) \cdot \frac{\sqrt{x^2+x} + \sqrt{x^2-x}}{\sqrt{x^2+x} + \sqrt{x^2-x}}$$

$$= \lim_{x \to \infty} \frac{(x^2+x) - (x^2-x)}{\sqrt{x^2+x} + \sqrt{x^2-x}} = \lim_{x \to \infty} \frac{\cancel{x^2} + x - \cancel{x^2} + x}{\sqrt{x^2+x} + \sqrt{x^2-x}}$$

$$= \lim_{x \to \infty} \frac{2x}{\sqrt{x^2+x} + \sqrt{x^2-x}}$$

x^2 تحت $\sqrt{\ }$ تعني x عندما $x \to \infty$
وتعني $-x$ عندما $x \to -\infty$
∴ درجة البسط = درجة المقام

$$= \frac{2}{\sqrt{1} + \sqrt{1}} = \frac{2}{1+1} = \frac{2}{2} = \boxed{1}$$

---

② $\lim_{x \to \infty} \left(\frac{3}{x^2} - \cos \frac{1}{x} \right) \left(1 + \sin \frac{1}{x} \right)$ تعويض مباشر

$\frac{\text{عدد}}{\infty} = 0$

$$= \left(\frac{3}{\infty} - \cos \frac{1}{\infty} \right) \left(1 + \sin \frac{1}{\infty} \right)$$

$$= (0 - \cos 0)(1 + \sin 0)$$

$$= (0 - 1)(1 + 0) = (-1)(1) = \boxed{-1}$$

③ $\lim_{x \to \infty} \frac{\cos(\frac{1}{x})}{1 + \frac{1}{x}}$ تعويض مباشر

$$= \frac{\cos(\frac{1}{\infty})}{1 + \frac{1}{\infty}} = \frac{\cos 0}{1+0} = \frac{1}{1} = \boxed{1}$$

قاعده هامه

* $\lim_{x \to \pm\infty} \frac{\sin ax}{bx} = 0 \Rightarrow \lim_{x \to \pm\infty} \frac{\sin x}{3x} = 0$

* $\lim_{x \to \pm\infty} \frac{\cos ax}{bx} = 0 \Rightarrow \lim_{x \to \pm\infty} \frac{\cos 3x}{5x} =$

Example:

Find: $\lim_{x \to -\infty} \frac{2 - x + \sin x}{x + \cos x}$

للوصول لشكل القاعده السابقه نقسم بسطاً ومقاماً على X

$$= \lim_{x \to -\infty} \frac{\frac{2}{x} - 1 + \frac{\sin x}{x}}{1 + \frac{\cos x}{x}} = \frac{0 - 1 + 0}{1 + 0} = \frac{-1}{1}$$

$$= \boxed{-1}$$

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(11) Guess the value of the limit:

$$\lim_{x\to\infty} \frac{x^2}{2^x}$$

* عندما تزداد x حتى تقترب من ∞ فإن المقام يصل أسرع إلى ∞ قبل البسط الى أنه:

$$\frac{\text{عدد}}{\infty} = 0$$

x:	1	2	----	10	---> ∞
x^2:	1	4	-----	100	---> عدد
2^x:	2	4	-----	1024	---> ∞

$$\therefore \lim_{x\to\infty} \frac{x^2}{2^x} = \frac{\text{عدد}}{\infty} = 0$$

للوضوح

$$\lim_{x\to\infty} \frac{2^x}{x^2} = \frac{\infty}{\text{عدد}} = \infty$$

---

(28) $\lim_{x\to\infty} \cos x$

* عند ∞ تكون $\cos x$ اما [1] أو [-1]

⇐ لها نهايتان مختلفتان

$\therefore$ Does Not Exist.

* * $\lim_{x\to\infty} |\cos x| = 1$

في حالة 1 ، -1 المطلق يعطي 1.

(35) $\lim_{x\to\infty} \left(e^{-2x} \cos x \right) = 0$

حاصل ضرب دالتين أحدهما دالة محدودة بين -١ ، ١ والأخرى دالة نهايتها ٥ حيث $e^{-\infty} = 0$

---

(31) $\lim_{x\to -\infty} \left(x^4 + x^5 \right)$

عامل مشترك x^4

$= \lim_{x\to -\infty} x^4 \cdot (1 + x)$

تعويض مباشر

$= (-\infty)^4 \cdot (1 - \infty)$

$= (\infty) \cdot (-\infty) = \boxed{-\infty}$

---

(33) $\lim_{x\to\infty} \frac{1 - e^x}{1 + 2e^x} = \frac{-\infty}{\infty}$

بالقسمة على e^x بسطاً ومقاماً

$= \lim_{x\to\infty} \frac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 2}$

$= \frac{\frac{1}{\infty} - 1}{\frac{1}{\infty} + 2} = \frac{0 - 1}{0 + 2} = \boxed{\frac{-1}{2}}$

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(34) $\lim_{x \to \infty} \tan^{-1}(x^2 - x^4)$ عامل مشترك x^2

$= \lim_{x \to \infty} \tan^{-1}[x^2(1 - x^2)]$

$= \tan^{-1}[\infty \cdot (1 - \infty)]$

$= \tan^{-1}[\infty \cdot -\infty] = \tan^{-1}[-\infty] = -\frac{\pi}{2}$

---

(36) $\lim_{x \to (\pi/2)^+} e^{\tan x} = e^{-\infty} = 0$

الربع الثاني

* tan سالبة

* $\tan 90 = \infty$

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(57) Find: $\lim_{x \to \infty} f(x)$ if $\frac{10e^x - 21}{2e^x} < f(x) < \frac{5\sqrt{x}}{\sqrt{x} - 1}$

by: Sandwich theorem
باستخدام نظرية الساندويتش

* $\lim_{x \to \infty} \frac{10e^x - 21}{2e^x} = \frac{10}{2} = \boxed{5}$

$\Rightarrow \lim_{x \to \infty} f(x) = \boxed{5}$

* $\lim_{x \to \infty} \frac{5\sqrt{x}}{\sqrt{x} - 1} = \frac{5}{1} = \boxed{5}$

Horizontal asymptotes
خطوط التقارب الأفقية

$y = \lim_{x \to \pm\infty} f(x)$

- L → $\therefore y = L$ is h. asym.
- $\pm L$ → $\therefore y = \pm L$ are h. asym.
- $+\infty$ أو $-\infty$ → $\therefore$ No h. asym.

Example: Find horizontal asymptotes

① $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 4}$

(درجة البسط = درجة المقام)

$$y = \lim_{x \to \pm\infty} \frac{x^2 - 5x + 6}{x^2 - 4} = \frac{1}{1} = 1$$

$\therefore y = 1$ is horizontal asymptote

$\Longrightarrow$ ($y = 1$ is h. asym.)

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② $f(x) = \dfrac{2x-1}{\sqrt{x^2+1}}$

* لاحظ أن :
$\sqrt{x^2} = |x| = \begin{cases} x & \text{عندما } x \to \infty \\ -x & \text{عندما } x \to -\infty \end{cases}$

* $y = \lim\limits_{x \to \infty} \dfrac{2x-1}{\sqrt{x^2}+1} = \dfrac{+2}{+1} = \boxed{2}$

* $y = \lim\limits_{x \to -\infty} \dfrac{2x-1}{\sqrt{x^2}+1} = \dfrac{+2}{-1} = \boxed{-2}$

$\Rightarrow$ $y = 2$ ، $y = -2$ are h. asym.

---

③ $F(x) = \dfrac{|x+2|}{x+4}$ ← لابد من اعاده تعريفا المطلقه

$x + 2 = 0 \Rightarrow x = -2$

عكس | -2 | مثل
$-(x+2)$ | | $+(x+2)$

* $y = \lim\limits_{x \to \infty} \dfrac{x+2}{x+4} = \dfrac{1}{1} = \boxed{1}$

* $y = \lim\limits_{x \to -\infty} \dfrac{-(x+2)}{x+4} = \dfrac{-1}{1} = \boxed{-1}$

$\Rightarrow$ $y = 1$ ، $y = -1$ are h. asym.

---

④ $f(x) = \dfrac{x^4}{|x|}$

* $y = \lim\limits_{x \to \infty} \dfrac{x^4}{x} = \lim\limits_{x \to \infty} x^3 = \boxed{\infty}$

* $y = \lim\limits_{x \to -\infty} \dfrac{x^4}{-x} = \lim\limits_{x \to -\infty} -x^3 = -(-\infty) = \boxed{\infty}$

$\Rightarrow$ F(x) has not h. asym. ليس لها خطوط تقارب افقيه.

Vertical asymptotes
خطوط التقارب الرأسية

* خطوات ايجاد خطوط التقارب الرأسية V. asym.
  - نوجد أصفار مقام الداله المعطاه $f(x)$ ولتكن a, b
  - نعوض بـ a ، b في الداله نفسها :

مثلاً	مثلاً
$f(a) = \frac{\text{عدد}}{0}$	$f(b) = \frac{0}{0}$
$\therefore x = a$	$\therefore x = b$
is V. asym.	not V. asym.
$x = a$ يمثل خط تقارب رأسي	$x = b$ لا يمثل خط تقارب رأسي

Example :

find the vertical asymptotes:

① $f(x) = \frac{2x-1}{x-2}$

* اصفار المقام
$x - 2 = 0$
$x = 2$

$f(2) = \frac{2(2)-1}{2-2} = \frac{3}{0} = \frac{\text{عدد}}{0}$

$\Rightarrow x = 2$ is V. asym.

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② $f(x) = \frac{x^2 - 5x + 6}{x^2 - 4}$

* اصفار المقام
$x^2 - 4 = 0$
$x^2 = 4$
$x = \pm 2$

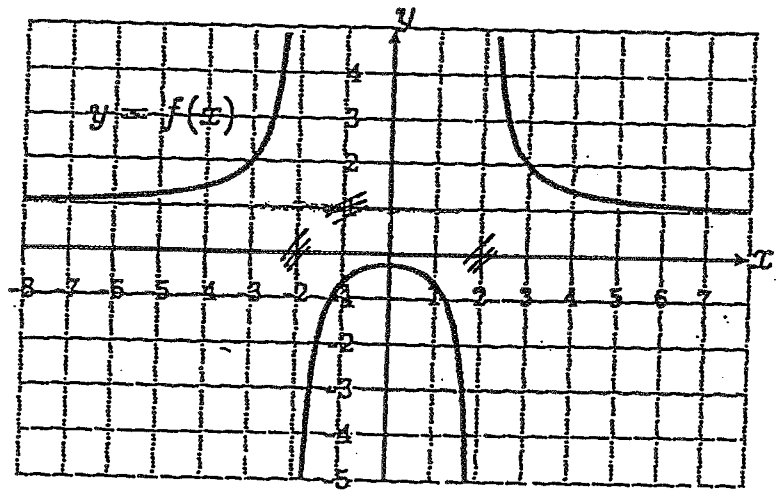
* $f(2) = \frac{4 - 10 + 6}{4 - 4} = \frac{0}{0}$

$\Rightarrow X = 2$ is not V. asym.

* $f(-2) = \frac{4 + 10 + 6}{4 - 4} = \frac{20}{0} = \frac{\text{عدد}}{0}$

$\Rightarrow X = -2$ is V. asym.

The horizontal and vertical asymptotes of f are



(a) $y = -2, \quad y = 2, \quad x = 1$

✓ (b) $x = -2, \quad x = 2, \quad y = 1$

(c) $x = -2, \quad x = 0, \quad y = 1$

(d) $x = 0, \quad x = 2, \quad y = 1$

For the function g whose graph is given, state the following.

(a) $\lim_{x \to \infty} g(x) = 2$ (b) $\lim_{x \to -\infty} g(x) = -2$

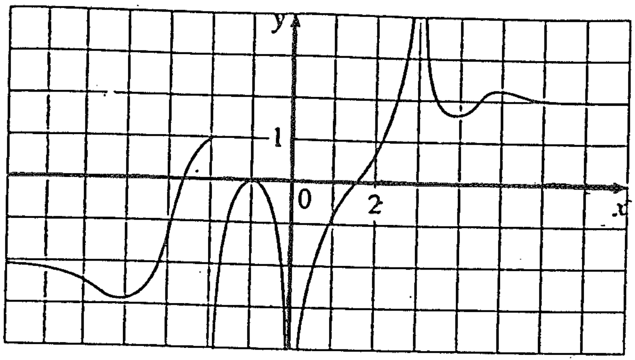
(c) $\lim_{x \to 3} g(x) = \infty$ (d) $\lim_{x \to 0} g(x) = -\infty$

(e) $\lim_{x \to -2^+} g(x) = -\infty$ (f) The equations of the asymptotes ⟹ * V. asymptotes

$x = -2$, $x = 0$, $x = 3$

* H. asymptotes

$y = -2$, $y = 2$



For the function f whose graph is given, state the following.

(a) $\lim_{x \to 2} f(x) = \infty$ (b) $\lim_{x \to -1^-} f(x) = \infty$

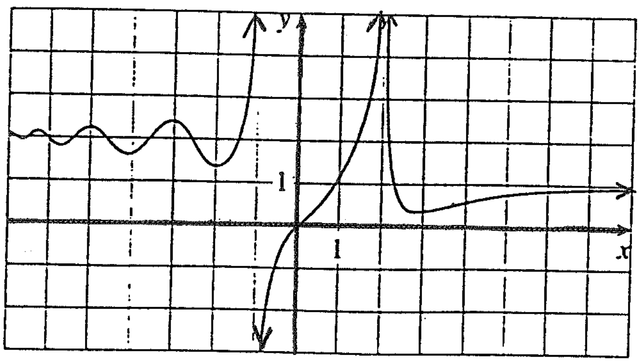
(c) $\lim_{x \to -1^+} f(x) = -\infty$ (d) $\lim_{x \to \infty} f(x) = 1$

(e) $\lim_{x \to -\infty} f(x) = 2$ (f) The equations of the asymptotes ⟹ * V. asymptotes

$x = -1$, $x = 2$

* H. asymptotes

$y = 1$, $y = 2$



2.7

- Derivatives.
- Rates of change.

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Notes

- التركيز على المفاهيم الاساسية.
- شرح أبواب المنهج حسب الخطة.
- أمثلة توضيحية وتدريبات.
- نماذج اختبارات.

السعدي

رياضيات ١١٠
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2.7

Derivatives and Rates of change

المشتقات ومعدلات التغير

The slope of tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$

is
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

صيغة أخرى

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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Example :

① Find the equation of the tangent line to $y = x^2$ at the point $P(1, 1)$

($1 \to a$, $1 \to f(a)$)

Solution

a , $f(a) = f(1) = 1^2 = 1$

* Slope $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0}$ (I.f.)

(by L.H.R)

$= \lim_{x \to 1} \frac{2x}{1} = \boxed{2}$

* eq. of the tangent line

$y = m(x - x_1) + y_1$

$y = 2(x - 1) + 1$

$y = 2x - 2 + 1$

$\boxed{y = 2x - 1}$

حل آخر : أسهل وأسرع

$\because y = x^2$ at $(\overset{x_1}{1}, \overset{y_1}{1})$

$y' = 2x$

* $m = 2(1) = \boxed{2}$

* $y = m(x - x_1) + y_1$

$y = 2(x - 1) + 1$

$y = 2x - 2 + 1$

$\boxed{y = 2x - 1}$

② Find the equation of the tangent line to $y = \frac{3}{x}$ at the point $(\overset{x_1}{3}, \overset{y_1}{1})$

Solution

• قاعده
مشتقه $\frac{a}{x}$
هو $\frac{-a}{x^2}$

$\because y = \frac{3}{x}$

$\Rightarrow y' = \frac{-3}{x^2}$ at $(3, 1)$

$\Rightarrow m = \frac{-3}{(3)^2} = \frac{-1}{3}$

* eq. of tangent line

$y = m(x - x_1) + y_1$

$y = -\frac{1}{3}(x - 3) + 1$

$y = -\frac{1}{3}x + 1 + 1$

$y = -\frac{1}{3}x + 2$

$y + \frac{1}{3}x - 2 = 0$ بالضرب في 3

$3y + x - 6 = 0$

* على حسب الاختيارات تكون صوره الجواب النهائي

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Velocit

* Displace
ـــــفه

* Average
المتوسطه.

* Instantan

Note th

Example: (16) page 151

The displacement (in meters) is given by

$$S = t^2 - 8t + 18$$

① Find the average velocity over the intervals

(a) $[3, 4]$

$\because S = t^2 - 8t + 18$

السرعة هي مشتقة المسافة

$V = \frac{dS}{dt} = 2t - 8$

* Average velocity

$= \frac{V(4) - V(3)}{4 - 3}$

$= \frac{(2(4)-8) - (2(3)-8)}{4 - 3}$

$= \frac{0 - (-2)}{1} = \frac{2}{1} = \boxed{2 \text{ m/sec}}$

(b) $[4, 4.5]$

$\because S = t^2 - 8t + 18$

السرعة هي مشتقة المسافة

$V = \frac{dS}{dt} = 2t - 8$

* Average Velocity

$= \frac{V(4.5) - V(4)}{4.5 - 4}$

$= \frac{(2(4.5)-8) - (2(4)-8)}{4.5 - 4}$

$= \frac{9 - 8}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2 \text{ m/sec}}$

② Find the instantaneous velocity when $t=4$

$\because S = t^2 - 8t + 18$ اشتقه

$\therefore V = 2t - 8$

instantaneous velocity at $t = 4$

$V(4) = 2(4) - 8 = \boxed{0}$

توقف لحظياً حيث أن السرعة Zero عندما $t = 4$

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Derivatives المشتقات

The derivative of the function $\underline{\underline{f}}$ at the number $\underline{\underline{a}}$ is denoted by :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example

Find the derivative اوجد المشتقة of the function $F(x) = x^2 - 8x + 9$ by definition. باستخدام التعريف. and $F'(3)$.

↘ باستخدام تعريف المشتقة

$$F'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2 - 8(x+h) + 9] - [x^2 - 8x + 9]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{8x} - 8h + \cancel{9} - \cancel{x^2} + \cancel{8x} - \cancel{9}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 8h}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x + h - 8)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x + h - 8) = 2x - 8$$

$\Rightarrow$ $\boxed{F'(x) = 2x - 8}$ $\Rightarrow$ $\boxed{F'(3) = -2}$

∵ الاختبار اختيارات فهناك حل اسرع

$\because F(x) = x^2 - 8x + 9$

$\therefore F'(x) = 2x - 8$

$\Rightarrow F'(3) = 2(3) - 8$

$= 6 - 8$

$= -2$

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Each limit represents the derivative of some function **f** at some number **a**

* State such **f** and **a** in each case.

(31) $\lim_{h \to 0} \frac{(1+h)^{10} - 1}{h}$

* للحصول على f(x)
ما تحت الأس يفرض بـ x

تناظر الصيغة

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

* العدد الموجود مع h
هو a
$\therefore a = 1$

$\Rightarrow f(x) = x^{10} \quad ، \quad a = 1$

---

(32) $\lim_{h \to 0} \frac{\sqrt[4]{16+h} - 2}{h}$

ما تحت الجذر يفرض بـ x

العدد الموجود مع h
هو a
$\therefore a = 16$

$\Rightarrow F(X) = \sqrt[4]{X} \quad ، \quad a = 16$

(33) $\lim_{x \to 5} \frac{2^x - 32}{x - 5}$

نناظر الصيغة

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$\Rightarrow f(x) = 2^x \quad , \quad a = 5 \leftarrow$ a : العدد الموجود بعد السهم

(34) $\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}}$

$\Rightarrow f(x) = \tan x \quad , \quad a = \frac{\pi}{4} \leftarrow$ a: العدد الموجود بعد السهم

(35) $\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$

$\Rightarrow f(x) = \cos x \quad , \quad a = \pi \leftarrow h$ a: العدد الموجود مع h

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Rates of change معدلات التغير

$$* \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \begin{matrix} \rightarrow \text{التغير في } y \\ \rightarrow \text{التغير في } x \end{matrix}$$

* Instantaneous rate of change معدل التغير اللحظي

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

* استخدام قواعد الاشتقاق (الاشتقاق المباشر) اسهل وأسرع من استخدام قانون التعريف.
* اذا طلب $f'(a)$ نوجد $f'(x)$ ثم نعوض عن x ب a.

Example: page 151 Find $f'(a)$?

(25) $f(x) = 3 - 2x + 4x^2$ الاشتقاق المباشر

$f'(x) = -2 + 8x$ (استبدلنا x ب a)

$f'(a) = -2 + 8a$

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(27) $f(t) = \dfrac{2t + 1}{t + 3}$

$f'(t) = \dfrac{\text{البسط . مشتقة المقام − المقام . مشتقة البسط}}{\text{مربع المقام}}$ ← قانون مشتقة الدالة الكسرية

$= \dfrac{2\cdot(t+3) - 1\cdot(2t+1)}{(t+3)^2}$

$= \dfrac{2t + 6 - 2t - 1}{(t+3)^2} = \dfrac{5}{(t+3)^2}$

$\Longrightarrow f'(a) = \dfrac{5}{(a+3)^2}$ ← استبدال t بـ a

---

(30) $f(x) = \sqrt{3x+1}$

$f'(x) = \dfrac{\text{مشتقة ما تحت الجذر}}{\text{نفس الجذر × 2}} = \dfrac{3}{2\sqrt{3x+1}}$

$\Longrightarrow f'(a) = \dfrac{3}{2\sqrt{3a+1}}$

(22) Page 151

If: $g(x) = 1 - x^3$ find $g'(0)$?

and use it to find the equation of tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$

$(0 \to x_1,\ 1 \to y_1)$

Solution

$\because g(x) = 1 - x^2$

$\therefore g'(x) = -3x^2$

من النقطه المعطاه

$m = g'(0) = -3(0) = 0$

$\therefore$ m الميل $= 0$

حفظ

eq. of tangent line
معادله المماس

$y = m(x - x_1) + y_1$

m → slope الميل (المشتقه)

x_1, y_1 → النقطه المعطاه

$\Rightarrow$ eq. of tangent line

is $y = m(x - x_1) + y_1$ → $(0, 1)$ من النقطه المعطاه

$y = 0(x - 0) + 1$

$y = 0 + 1 \Rightarrow \boxed{y = 1}$

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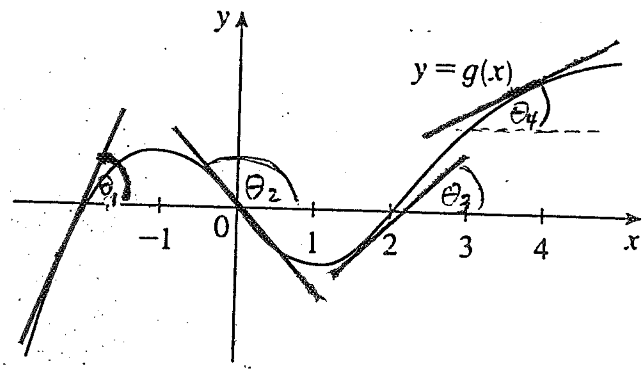
For the function g whose graph is given arrange the following numbers in increasing order:

$$0 \quad g'(-2) \quad g'(0) \quad g'(2) \quad g'(4)$$

and explain reasoning.



● الزاويه : التي يصنعها المماس للمنحنى مع الاتجاه الموجب لمحور X من أعلى

حاده : معناه الميل موجب $f'(x)>0$ الداله تزايديه

كلما زاد قياس الزاويه الحاده كلما ازداد الميل اى زادت المشتقه

θ_1 ، θ_3 ، θ_4 كلها زوايا حاده ∴ المشتقات موجبه. الرسم يوضح أن

$\theta_4 < \theta_3 < \theta_1 \Rightarrow g'(4) < g'(2) < g'(-2) \rightarrow$ ①

عند θ_4 عند θ_3 عند θ_1

منفرجه : معناه الميل سالب $f'(x) < 0$ الداله تناقص

θ_2 زاويه منفرجه ∴ المشتقه عندها سالبه

$\Rightarrow g'(0) < 0 \rightarrow$ ②

من ① ، ② يتضح أن الترتيب تصاعديًا هو

$g'(0) < 0 < g'(4) < g'(2) < g'(-2)$

(18) page 151

ⓐ Find an eq. of the tangent line to $y = g(x)$ at $\boxed{x = 5}$ if $\boxed{g(5) = -3}$ and $\boxed{g'(5) = 4}$

Solution

** (x_1, y_1) → $x_1 = 5$, $y_1 = g(5) = -3$

** $m = g'(5) = 4$

eq. of tangent line is:

$$y = m(x - x_1) + y_1$$

$$y = 4(x - 5) + (-3)$$

$$y = 4x - 20 - 3 \Rightarrow \boxed{y = 4x - 23}$$

ⓑ If the tangent to $y = f(x)$ at $(4, 3)$ نقطة التماس passes through the point $(0, 2)$ Find $F(4)$ and $f'(4)$?

$(4, 3)$ → x_1, y_1

$(0, 2)$ → x, y تقع على المنحنى ∴ تحقق معادلته

Solution

* $\boxed{F(4) = 3}$ ⇐ ∵ نقطة $(4, 3)$ نقطة تماس تقع على المنحنى $(4, F(4))$

* eq. of tangent: $y = m(x - x_1) + y_1$

$$y = F'(4)(x - 4) + 3$$

يمر بالنقطة $(0, 2)$ ∴ تحقق المعادلة

$$2 = F'(4)(0 - 4) + 3$$

$$\Rightarrow 2 = -4F'(4) + 3$$

$$\Rightarrow 4F'(4) = 3 - 2$$

$$\Rightarrow 4F'(4) = 1 \Rightarrow \boxed{F'(4) = \frac{1}{4}}$$

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(7) Find the eq. of the tangent line to the curve at the given point

$y = \sqrt{x}$, $(1, 1)$ → x_1 = 1, y_1 = 1

Solution

$$y' = \frac{1}{2\sqrt{x}}$$

عوض عن X ب 1 في المشتقة تحصل على m

$$\Rightarrow m = \frac{1}{2\sqrt{1}} = \boxed{\frac{1}{2}} \longrightarrow \text{slope.}$$

∴ eq. of the tangent line :

$$y = m(x - x_1) + y_1$$

$$y = \frac{1}{2}(x - 1) + 1$$

$$y = \frac{1}{2}x - \frac{1}{2} + 1 \Rightarrow \boxed{y = \frac{1}{2}x + \frac{1}{2}}$$

كل التمنيات بالنجاح والتوفيق

2.8

13

Notes

- التركيز على المفاهيم الاساسية.
- شرح ابواب المنهج حسب الخطة.
- أمثلة توضيحية وتدريبات.
- نماذج اختبارات.

السعدي

رياضيات ١١٠
Math.110

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2.8

The derivative as a function

The derivative of a function f at a fixed number a is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If: we replace a by a variable x

we obtain f' as a new function

called the derivative of f and defined by

equation: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: If: $f(x) = 3x^2 - 1$ find $f'(x)$?

by def. (بالتعريف)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-1} - \cancel{3x^2} \cancel{+1}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} = \lim_{h \to 0} (6x + 3h) = \boxed{6x}$$

by rule (بقاعده الاشتقاق المباشر)

$$y = 3x^2 - 1$$

$$y' = \boxed{6x}$$

- If: $y = f(x)$

رموز ... المشتقة

The notations for the derivative are:

$$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D f(x)$$

قابله للاشتقاق ... موجوده

- $\underline{f}$ is differentiable at $\underline{a}$ if $f'(a)$ exist.

مفتوحه ... فتره

- $\underline{f}$ is differentiable on open interval:

(a, b) or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$

قابله للاشتقاق ... عند ... كل ... عدد

if $\underline{it}$ is differentiable at every number

داخل الفتره

in the interval.

---

* اذا كانت الداله قابله للاشتقاق عند a فإنها تكون متصله عند a

نظريه ... قابله للاشتقاق

- Theorem: If $\underline{f}$ is differentiable at $\underline{a}$

متصله

then $\underline{f}$ is continuous at $\underline{a}$

---

* توجد دوال متصله ولكنها غير قابله للاشتقاق مثال: $f(x) = |x|$

- There are functions that are continuous but not differentiable.

For example: $f(x) = |x|$ is continuous at $x = 0$

غير قابله للاشتقاق

but not differentiable at $x = 0$

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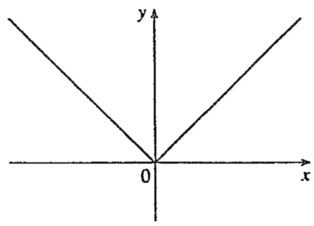
متصله

- $F(x) = |x|$ is continuous at $x = 0$

because : $\lim_{x \to 0^+}(x) = \lim_{x \to 0^-}(-x) = F(0) = 0 \Rightarrow$ الداله متصله عند $x = 0$

النهايه اليمنى — النهايه اليسرى — قيمه الداله

$$F(x) = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

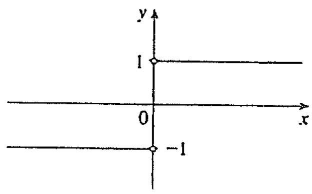


غير قابله للاشتقاق

- $F(x) = |x|$ is not differentiable at $x = 0$

because :

$$F'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



$$F'(0^+) \neq F'(0^-)$$

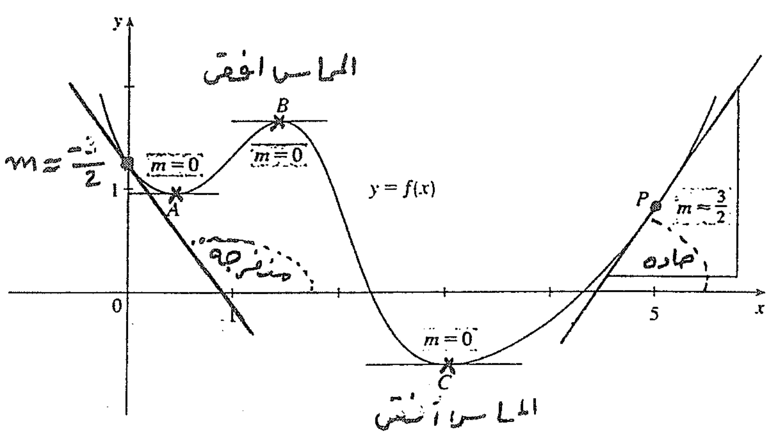
المشتقه اليمنى $\neq$ المشتقه اليسرى $\Longrightarrow$ الداله غير قابله للاشتقاق عند $x = 0$

Notes

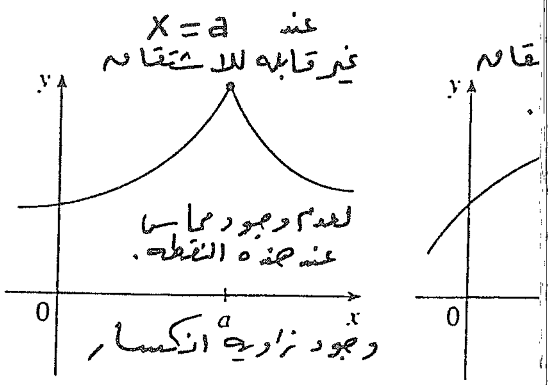
* اذا كان المماس أفقي
$F'(x) = m = 0$ (ثابته)

* اذا كان المماس يصنع زاويه حاده
موجب $0 < (F'(x) = m)$ (تزايديه)

* اذا كانت المماس يصنع زاويه منفرجه
سالب $0 > (F'(x) = m)$ (تناقصيه)



• This functions are



(a) A corner or Kink (b) A

35–38 The graph of f is given. State, with reasc
at which f is not differentiable.

(35) F(x) is not differential
at X = −4 → (Corner
X = 0 → (disconti
X ≈ 2.3 → (Vertica

(36) F(x) is not different
at X = 0 → (disconti
X = 3 → (corner

(37) F(x) is not differen
at X = 4 → (corner

(38) F(x) is not differen
at X = −1 → (discon
X = 2 → (corn

Higher order derivative
المشتقات من الرتب العليا

الدالة

$y = f(x)$

المشتقة الأولى

$y' = f'(x) = \frac{df}{dx}$

المشتقة الثانية

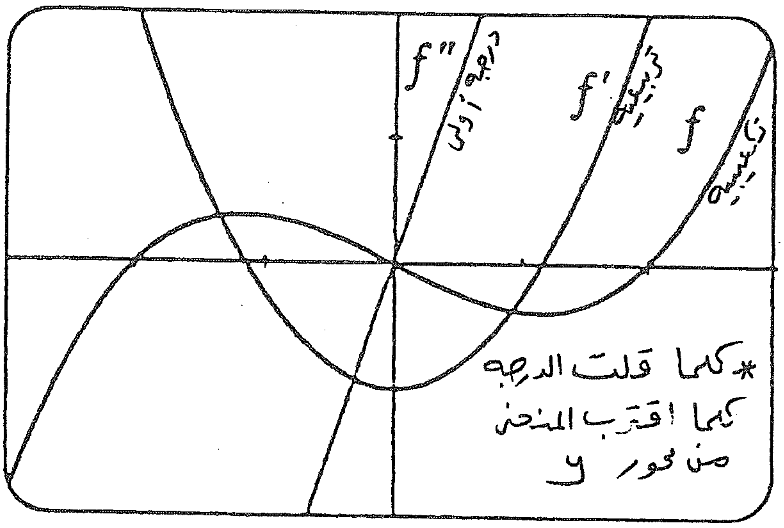
$y'' = f''(x) = \frac{d^2 f}{dx^2}$

المشتقة الثالثة

$y''' = f'''(x) = \frac{d^3 f}{dx^3}$

المشتقة الرابعة

$y^{(4)} = F^{(4)}(x) = \frac{d^4 f}{dx^4}$



$y = x^5 - 3x^3 + 4x^2 - 2x + 1$ Find $y^{(5)}$?

$y' = 5x^4 - 9x^2 + 8x - 2$

$y'' = 20x^3 - 18x + 8$

$y''' = 60x^2 - 18$

$y^{(4)} = 120x$

$y^{(5)} = 120$

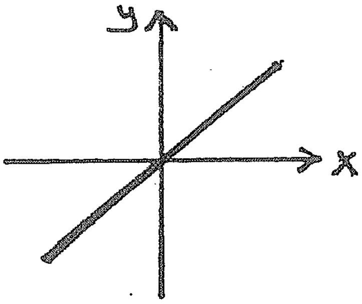
ملحوظة!
اذا زادت رتبة المشتقة عن درجة كثيرة الحدود فإن المشتقة = zero

اذا طلب $y^{(6)}$ الناتج مباشرة ← $y^{(6)} = 0$

لأن رتبة المشتقة 6 زادت عن درجة الدالة y وهي 5.

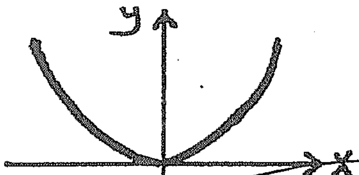
رسم الدوال المشهوره ومشتقاتها.

① $y = x$



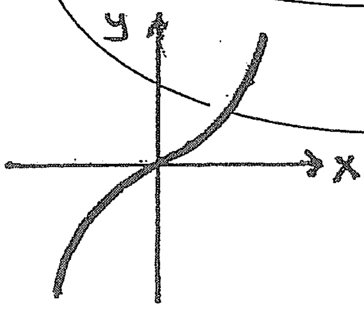
المشتقة

② $y = x^2$



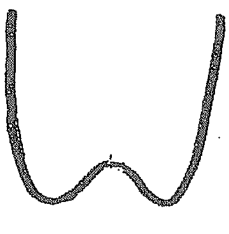
المشتقة

③ $y = x^3$

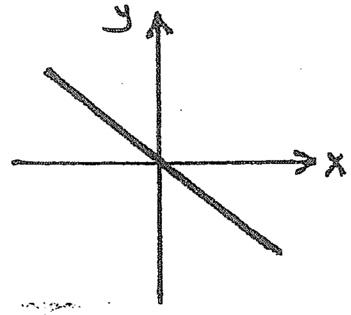


المشتقة

④ $y = x^4$

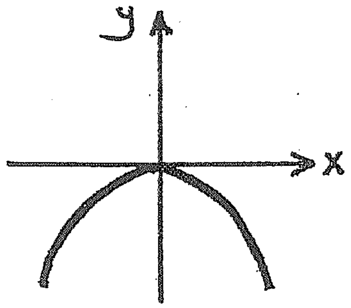


⑤ $y = -x$



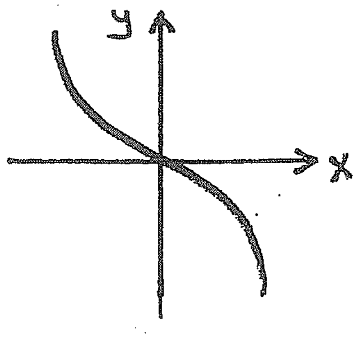
المشتقة

⑥ $y = -x^2$



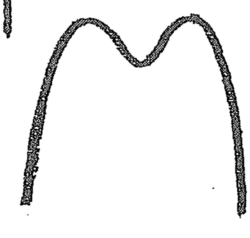
المشتقة

⑦ $y = -x^3$



المشتقة

⑧ $y = -x^4$

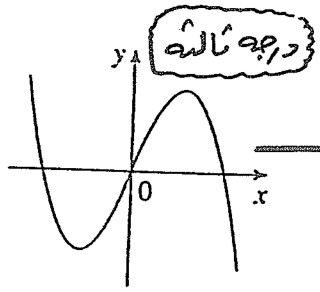


ه من الدرجة n − 1

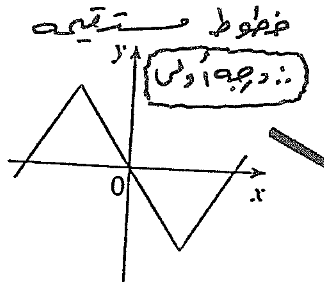
3. Match the graph of each function in (a)–(c) with the graph of its derivative in 1–3 . Give reasons for your choices.

الدوال

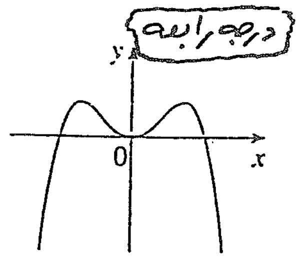
(a)



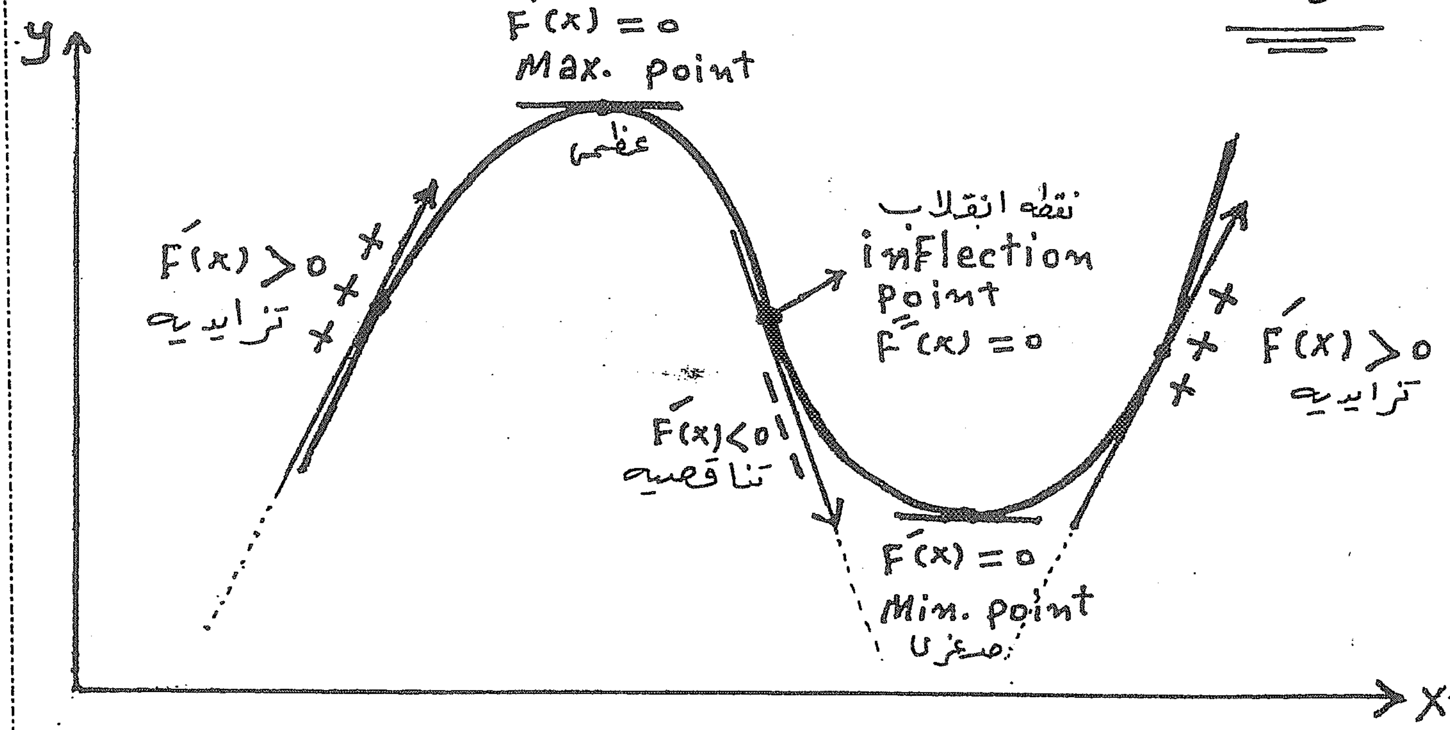
(b)



(c)

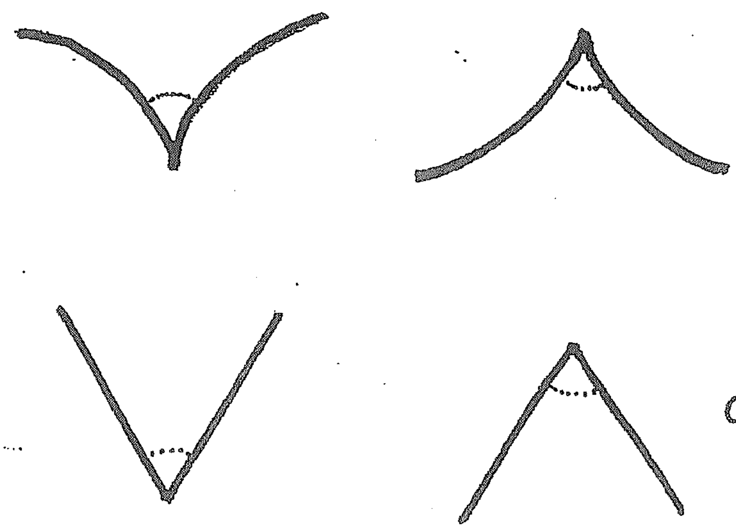


ملحوظة ①

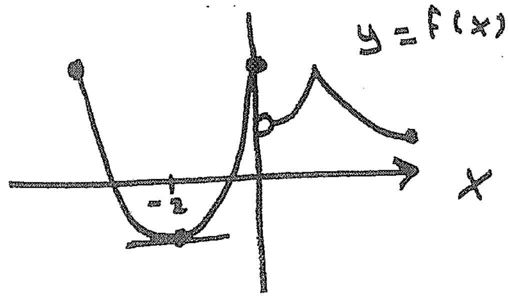


ملحوظة ②

الرسم يتكون من فرعين
بينهما زاوية (انكسار)
corner
تكون دوال متصلة
ولكنها غير قابلة للاشتقاق



Continuous but not diff.

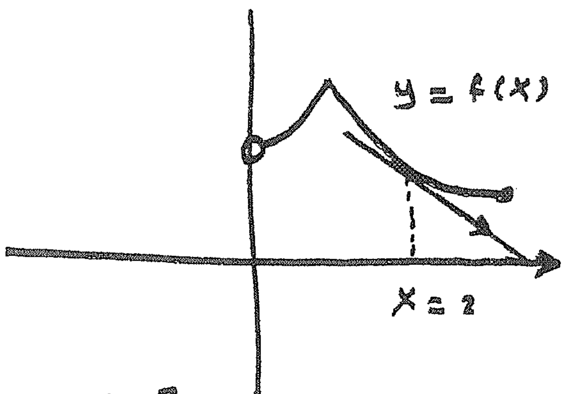


$f'(-2)$

عند العدد $x = -2$

المماس لمنحنى الدالة افقي
Horizontal

$\therefore f'(-2) = 0$



$f'(2) > 0$

المماس لأسفل
(يصنع زاوية منفرجة مع محور X)
في الاتجاه الموجب

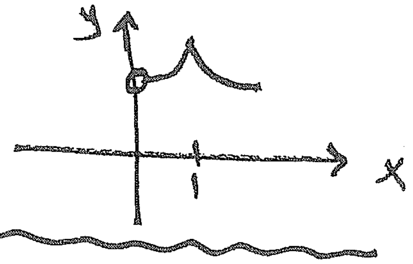
تناقصية $\therefore f'(2) < 0$

$\therefore f'(2) > 0$ **False**

f(x) is differentiable

at $x = 1$

False

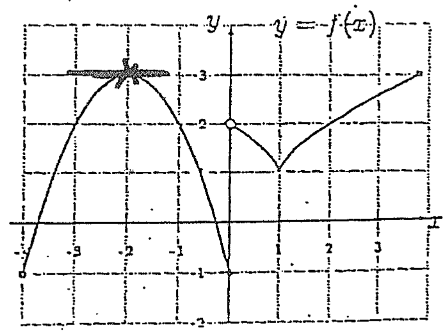


F(x) not diff. ✓
because: there is corner.

The accompanying figure shows the graph of $y = f(x)$. Then $f'(-2) =$.

(a) -3 (b) 0

(c) 1 (d) 3



$f'(-2)$

عند العدد $x = -2$

المماس لمنحنى الداله افقي

Horizontal

$\therefore f'(-2) = 0$

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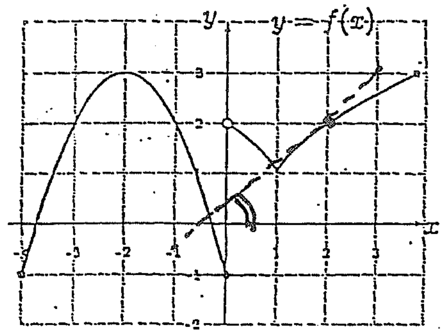
The accompanying figure shows the graph of $y = f(x)$. Then $f'(2) > 0$.

(a) True

(b) False



المماس للمنحنى عند $x = 2$

يصنع زاويه حاده مع

الاتجاه الموجب لمحور x

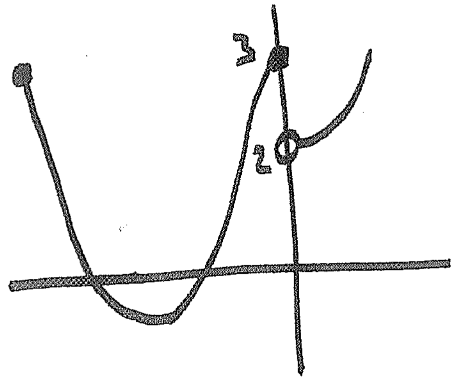
∴ الداله تزايديه

$\therefore f'(2) > 0$

$\therefore f'(2) > 0$ True

متصلة عند $x=0$

* f(x) is continuous at $x=0$ ✗ (Jump)



$$\lim_{x \to 0^+} f(x) = 2$$

$$\lim_{x \to 0^-} f(x) = 3$$

$\therefore$ f(x) is discontinuous at $x = 0$

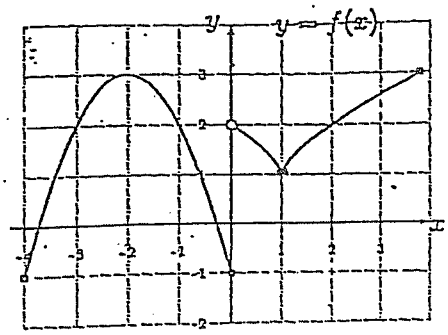
لأن النهاية اليمنى $\neq$ النهاية اليسرى.

---

f is differentiable at $x = 1$.

(a) True

(b) False



f(x) is differentiable at $x = 1$

(False)

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because: f(x) is not diff. at $x = 1$ $\Rightarrow$ There is corner.

كل التمنيات بالنجاح والتوفيق

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