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0 BEFORE CALCULUS

The development of calculus in the seventeenth and eighteenth centuries was motivated by the need to understand physical phenomena such as the tides, the phases of the moon, the nature of light, and gravity.

One of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a "function," which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus, including polynomials, trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions.

0.1 FUNCTIONS

In this section we will define and develop the concept of a "function," which is the basic mathematical object that scientists and mathematicians use to describe relationships between variable quantities. Functions play a central role in calculus and its applications.

DEFINITION OF A FUNCTION

Many scientific laws and engineering principles describe how one quantity depends on another. This idea was formalized in 1673 by Gottfried Wilhelm Leibniz (see p. xx) who coined the term *function* to indicate the dependence of one quantity on another, as described in the following definition.

0.1.1 DEFINITION If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that ***y is a function of x***.

Four common methods for representing functions are:

- Numerically by tables
- Algebraically by formulas
- Geometrically by graphs
- Verbally

Table 0.1.1

INDIANAPOLIS 500 QUALIFYING SPEEDS

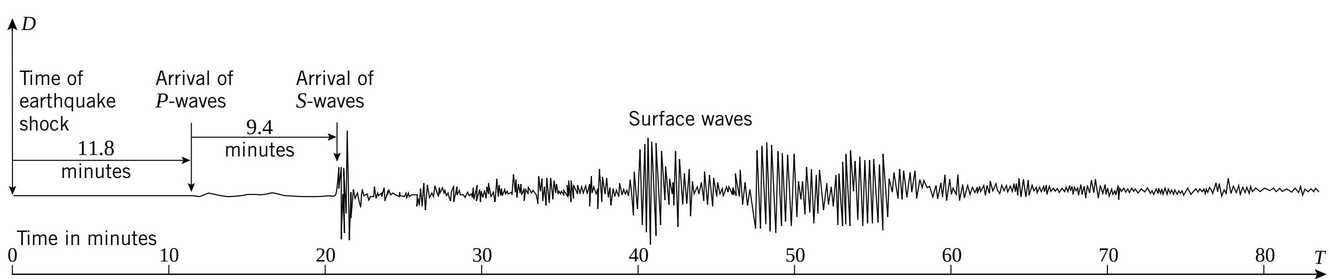
YEAR t	SPEED S (mi/h)
1994	228.011
1995	231.604
1996	233.100
1997	218.263
1998	223.503
1999	225.179
2000	223.471
2001	226.037
2002	231.342
2003	231.725
2004	222.024
2005	227.598
2006	228.985
2007	225.817
2008	226.366
2009	224.864
2010	227.970
2011	227.472

The method of representation often depends on how the function arises. For example:

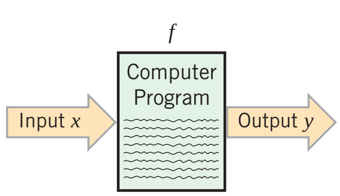
- Table 0.1.1 shows the top qualifying speed S for the Indianapolis 500 auto race as a function of the year t. There is exactly one value of S for each value of t.
- Figure 0.1.1 is a graphical record of an earthquake recorded on a seismograph. The graph describes the deflection D of the seismograph needle as a function of the time T elapsed since the wave left the earthquake's epicenter. There is exactly one value of D for each value of T.
- Some of the most familiar functions arise from formulas; for example, the formula $C = 2\pi r$ expresses the circumference C of a circle as a function of its radius r. There is exactly one value of C for each value of r.
- Sometimes functions are described in words. For example, Isaac Newton's Law of Universal Gravitation is often stated as follows: The gravitational force of attraction between two bodies in the Universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This is the verbal description of the formula

$$F = G\frac{m_1 m_2}{r^2}$$

in which F is the force of attraction, m_1 and m_2 are the masses, r is the distance between them, and G is a constant. If the masses are constant, then the verbal description defines F as a function of r. There is exactly one value of F for each value of r.



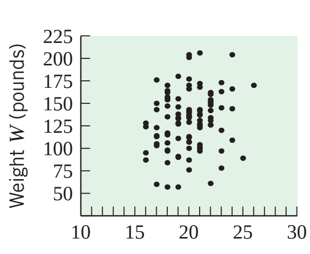
▲ **Figure 0.1.1**



▲ **Figure 0.1.2**

In the mid-eighteenth century the Swiss mathematician Leonhard Euler (pronounced "oiler") conceived the idea of denoting functions by letters of the alphabet, thereby making it possible to refer to functions without stating specific formulas, graphs, or tables. To understand Euler's idea, think of a function as a computer program that takes an *input* x, operates on it in some way, and produces exactly one *output* y. The computer program is an object in its own right, so we can give it a name, say f. Thus, the function f (the computer program) associates a unique output y with each input x (Figure 0.1.2). This suggests the following definition.

0.1.2 DEFINITION A ***function*** f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by $f(x)$ (read "f of x").



▲ **Figure 0.1.3**

In this definition the term *unique* means "exactly one." Thus, a function cannot assign two different outputs to the same input. For example, Figure 0.1.3 shows a plot of weight versus age for a random sample of 100 college students. This plot does *not* describe W as a function of A because there are some values of A with more than one corresponding value of W. This is to be expected, since two people with the same age can have different weights.

INDEPENDENT AND DEPENDENT VARIABLES

For a given input x, the output of a function f is called the ***value*** of f at x or the ***image*** of x under f. Sometimes we will want to denote the output by a single letter, say y, and write

$$y = f(x)$$

This equation expresses y as a function of x; the variable x is called the ***independent variable*** (or ***argument***) of f, and the variable y is called the ***dependent variable*** of f. This terminology is intended to suggest that x is free to vary, but that once x has a specific value a corresponding value of y is determined. For now we will only consider functions in which the independent and dependent variables are real numbers, in which case we say that f is a ***real-valued function of a real variable***. Later, we will consider other kinds of functions.

Table 0.1.2

x	0	1	2	3
y	3	4	−1	6

Example 1 Table 0.1.2 describes a functional relationship $y = f(x)$ for which

$f(0) = 3$ — f associates $y = 3$ with $x = 0$.

$f(1) = 4$ — f associates $y = 4$ with $x = 1$.

$f(2) = -1$ — f associates $y = -1$ with $x = 2$.

$f(3) = 6$ — f associates $y = 6$ with $x = 3$. ◀

Example 2 The equation

$$y = 3x^2 - 4x + 2$$

has the form $y = f(x)$ in which the function f is given by the formula

$$f(x) = 3x^2 - 4x + 2$$

Leonhard Euler (1707–1783) Euler was probably the most prolific mathematician who ever lived. It has been said that "Euler wrote mathematics as effortlessly as most men breathe." He was born in Basel, Switzerland, and was the son of a Protestant minister who had himself studied mathematics. Euler's genius developed early. He attended the University of Basel, where by age 16 he obtained both a Bachelor of Arts degree and a Master's degree in philosophy. While at Basel, Euler had the good fortune to be tutored one day a week in mathematics by a distinguished mathematician, Johann Bernoulli. At the urging of his father, Euler then began to study theology. The lure of mathematics was too great, however, and by age 18 Euler had begun to do mathematical research. Nevertheless, the influence of his father and his theological studies remained, and throughout his life Euler was a deeply religious, unaffected person. At various times Euler taught at St. Petersburg Academy of Sciences (in Russia), the University of Basel, and the Berlin Academy of Sciences. Euler's energy and capacity for work were virtually boundless. His collected works form more than 100 quarto-sized volumes and it is believed that much of his work has been lost. What is particularly astonishing is that Euler was blind for the last 17 years of his life, and this was one of his most productive periods! Euler's flawless memory was phenomenal. Early in his life he memorized the entire *Aeneid* by Virgil, and at age 70 he could not only recite the entire work but could also state the first and last sentence on each page of the book from which he memorized the work. His ability to solve problems in his head was beyond belief. He worked out in his head major problems of lunar motion that baffled Isaac Newton and once did a complicated calculation in his head to settle an argument between two students whose computations differed in the fiftieth decimal place.

Following the development of calculus by Leibniz and Newton, results in mathematics developed rapidly in a disorganized way. Euler's genius gave coherence to the mathematical landscape. He was the first mathematician to bring the full power of calculus to bear on problems from physics. He made major contributions to virtually every branch of mathematics as well as to the theory of optics, planetary motion, electricity, magnetism, and general mechanics.

[*Image: http://commons.wikimedia.org/wiki/File:Leonhard_Euler_by_Handmann_.png*]

For each input x, the corresponding output y is obtained by substituting x in this formula. For example,

$$f(0) = 3(0)^2 - 4(0) + 2 = 2$$ f associates $y = 2$ with $x = 0$.

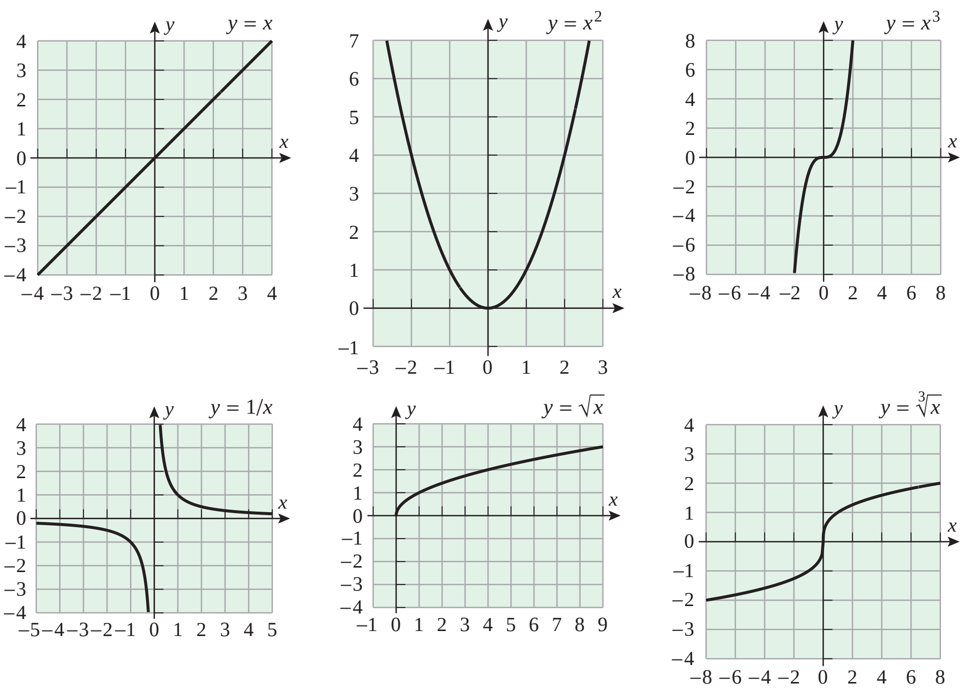
$$f(-1.7) = 3(-1.7)^2 - 4(-1.7) + 2 = 17.47$$ f associates $y = 17.47$ with $x = -1.7$.

$$f(\sqrt{2}) = 3(\sqrt{2})^2 - 4\sqrt{2} + 2 = 8 - 4\sqrt{2}$$ f associates $y = 8 - 4\sqrt{2}$ with $x = \sqrt{2}$.

GRAPHS OF FUNCTIONS

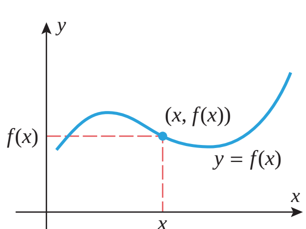
If f is a real-valued function of a real variable, then the ***graph*** of f in the xy-plane is defined to be the graph of the equation $y = f(x)$. For example, the graph of the function $f(x) = x$ is the graph of the equation $y = x$, shown in Figure 0.1.4. That figure also shows the graphs of some other basic functions that may already be familiar to you. In Appendix A we discuss techniques for graphing functions using graphing technology.

Figure 0.1.4 shows only portions of the graphs. Where appropriate, and unless indicated otherwise, it is understood that graphs shown in this text extend indefinitely beyond the boundaries of the displayed figure.



▲ **Figure 0.1.4**

Since $\sqrt{x}$ is imaginary for negative values of x, there are no points on the graph of $y = \sqrt{x}$ in the region where $x < 0$.



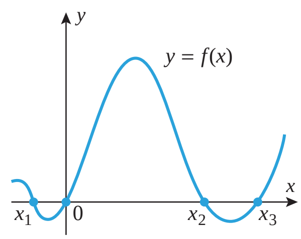
▲ **Figure 0.1.5** The y-coordinate of a point on the graph of $y = f(x)$ is the value of f at the corresponding x-coordinate.

Graphs can provide valuable visual information about a function. For example, since the graph of a function f in the xy-plane is the graph of the equation $y = f(x)$, the points on the graph of f are of the form $(x, f(x))$; that is, *the y-coordinate of a point on the graph of f is the value of f at the corresponding x-coordinate* (Figure 0.1.5). The values of x for which $f(x) = 0$ are the x-coordinates of the points where the graph of f intersects the x-axis (Figure 0.1.6). These values are called the ***zeros*** of f, the ***roots*** of $f(x) = 0$, or the ***x-intercepts*** of the graph of $y = f(x)$.

THE VERTICAL LINE TEST

Not every curve in the xy-plane is the graph of a function. For example, consider the curve in Figure 0.1.7, which is cut at two distinct points, (a, b) and (a, c), by a vertical line. This curve cannot be the graph of $y = f(x)$ for any function f; otherwise, we would have

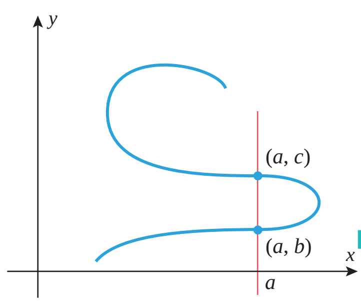
$$f(a) = b \quad \text{and} \quad f(a) = c$$



▲ **Figure 0.1.6** f has zeros at x_1, 0, x_2, and x_3.

which is impossible, since f cannot assign two different values to a. Thus, there is no function f whose graph is the given curve. This illustrates the following general result, which we will call the ***vertical line test***.

> **0.1.3 THE VERTICAL LINE TEST** *A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.*



▲ **Figure 0.1.7** This curve cannot be the graph of a function.

▶ **Example 3** The graph of the equation

$$x^2 + y^2 = 25$$

is a circle of radius 5 centered at the origin and hence there are vertical lines that cut the graph more than once (Figure 0.1.8). Thus this equation does not define y as a function of x. ◀

THE ABSOLUTE VALUE FUNCTION

Recall that the ***absolute value*** or ***magnitude*** of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative. Thus,

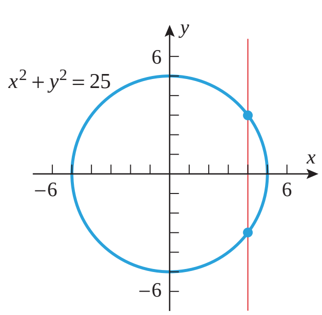
$$|5| = 5, \quad \left|-\tfrac{4}{7}\right| = \tfrac{4}{7}, \quad |0| = 0$$

Symbols such as $+x$ and $-x$ are deceptive, since it is tempting to conclude that $+x$ is positive and $-x$ is negative. However, this need not be so, since x itself can be positive or negative. For example, if x is negative, say $x = -3$, then $-x = 3$ is positive and $+x = -3$ is negative.

A more detailed discussion of the properties of absolute value is given in Web Appendix F. However, for convenience we provide the following summary of its algebraic properties.

> **0.1.4 PROPERTIES OF ABSOLUTE VALUE** *If a and b are real numbers, then*

(a)	$\lvert -a\rvert = \lvert a\rvert$	A number and its negative have the same absolute value.
(b)	$\lvert ab\rvert = \lvert a\rvert\,\lvert b\rvert$	The absolute value of a product is the product of the absolute values.
(c)	$\lvert a/b\rvert = \lvert a\rvert/\lvert b\rvert,\ b \neq 0$	The absolute value of a ratio is the ratio of the absolute values.
(d)	$\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$	The ***triangle inequality***



▲ **Figure 0.1.8**

The graph of the function $f(x) = |x|$ can be obtained by graphing the two parts of the equation

$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

separately. Combining the two parts produces the V-shaped graph in Figure 0.1.9.

Absolute values have important relationships to square roots. To see why this is so, recall from algebra that every positive real number x has two square roots, one positive and one negative. By definition, the symbol $\sqrt{x}$ denotes the *positive* square root of x.

WARNING

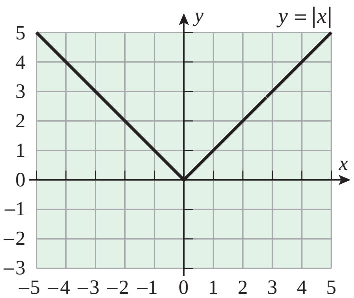
To denote the negative square root you must write $-\sqrt{x}$. For example, the positive square root of 9 is $\sqrt{9} = 3$, whereas the negative square root of 9 is $-\sqrt{9} = -3$. (Do not make the mistake of writing $\sqrt{9} = \pm 3$.)

Care must be exercised in simplifying expressions of the form $\sqrt{x^2}$, since it is *not* always true that $\sqrt{x^2} = x$. This equation is correct if x is nonnegative, but it is false if x is negative. For example, if $x = -4$, then

$$\sqrt{x^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq x$$

TECHNOLOGY MASTERY

Verify (1) by using a graphing utility to show that the equations $y = \sqrt{x^2}$ and $y = |x|$ have the same graph.



▲ **Figure 0.1.9**

A statement that is correct for all real values of x is

$$\sqrt{x^2} = |x| \tag{1}$$

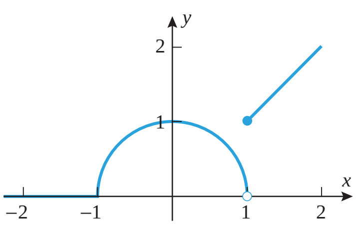
PIECEWISE-DEFINED FUNCTIONS

The absolute value function $f(x) = |x|$ is an example of a function that is defined ***piecewise*** in the sense that the formula for f changes, depending on the value of x.

▶ **Example 4** Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0, & x \leq -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \geq 1 \end{cases}$$

Solution. The formula for f changes at the points $x = -1$ and $x = 1$. (We call these the ***breakpoints*** for the formula.) A good procedure for graphing functions defined piecewise is to graph the function separately over the open intervals determined by the breakpoints, and then graph f at the breakpoints themselves. For the function f in this example the graph is the horizontal ray $y = 0$ on the interval $(-\infty, -1]$, it is the semicircle $y = \sqrt{1-x^2}$ on the interval $(-1, 1)$, and it is the ray $y = x$ on the interval $[1, +\infty)$. The formula for f specifies that the equation $y = 0$ applies at the breakpoint -1 [so $y = f(-1) = 0$], and it specifies that the equation $y = x$ applies at the breakpoint 1 [so $y = f(1) = 1$]. The graph of f is shown in Figure 0.1.10. ◀



▲ **Figure 0.1.10**

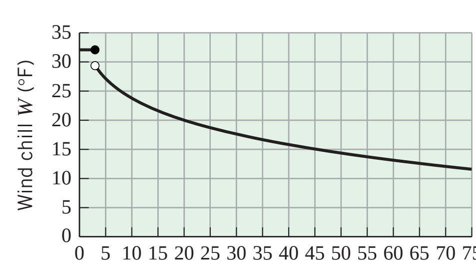
REMARK In Figure 0.1.10 the solid dot and open circle at the breakpoint $x = 1$ serve to emphasize that the point on the graph lies on the ray and not the semicircle. There is no ambiguity at the breakpoint $x = -1$ because the two parts of the graph join together continuously there.

The wind chill index measures the sensation of coldness that we feel from the combined effect of temperature and wind speed.

▶ **Example 5** Increasing the speed at which air moves over a person's skin increases the rate of moisture evaporation and makes the person feel cooler. (This is why we fan ourselves in hot weather.) The ***wind chill index*** is the temperature at a wind speed of 4 mi/h that would produce the same sensation on exposed skin as the current temperature and wind speed combination. An empirical formula (i.e., a formula based on experimental data) for the wind chill index W at 32°F for a wind speed of v mi/h is

$$W = \begin{cases} 32, & 0 \leq v \leq 3 \\ 55.628 - 22.07v^{0.16}, & 3 < v \end{cases}$$

A computer-generated graph of $W(v)$ is shown in Figure 0.1.11. ◀



▶ **Figure 0.1.11** Wind chill versus wind speed at 32°F

DOMAIN AND RANGE

If x and y are related by the equation $y = f(x)$, then the set of all allowable inputs (x-values) is called the ***domain*** of f, and the set of outputs (y-values) that result when x varies over the domain is called the ***range*** of f. For example, if f is the function defined by the table in Example 1, then the domain is the set $\{0, 1, 2, 3\}$ and the range is the set $\{-1, 3, 4, 6\}$.

Sometimes physical or geometric considerations impose restrictions on the allowable inputs of a function. For example, if y denotes the area of a square of side x, then these variables are related by the equation $y = x^2$. Although this equation produces a unique value of y for every real number x, the fact that lengths must be nonnegative imposes the requirement that $x \geq 0$.

One might argue that a physical square cannot have a side of length zero. However, it is often convenient mathematically to allow zero lengths, and we will do so throughout this text where appropriate.

When a function is defined by a mathematical formula, the formula itself may impose restrictions on the allowable inputs. For example, if $y = 1/x$, then $x = 0$ is not an allowable input since division by zero is undefined, and if $y = \sqrt{x}$, then negative values of x are not allowable inputs because they produce imaginary values for y and we have agreed to consider only real-valued functions of a real variable. In general, we make the following definition.

0.1.5 DEFINITION If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the ***natural domain*** of the function.

The domain and range of a function f can be pictured by projecting the graph of $y = f(x)$ onto the coordinate axes as shown in Figure 0.1.12.

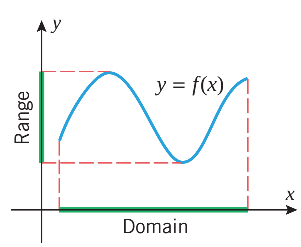


Figure 0.1.12 The projection of $y = f(x)$ on the x-axis is the set of allowable x-values for f, and the projection on the y-axis is the set of corresponding y-values.

Example 6 Find the natural domain of

(a) $f(x) = x^3$ (b) $f(x) = 1/[(x-1)(x-3)]$
(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

Solution (a). The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

Solution (b). The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

Solution (c). Since $f(x) = \tan x = \sin x / \cos x$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\pi/2$. Thus, the natural domain consists of all real numbers except

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

For a review of trigonometry see Appendix B.

Solution (d). The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

$$x^2 - 5x + 6 = (x-3)(x-2) \geq 0$$

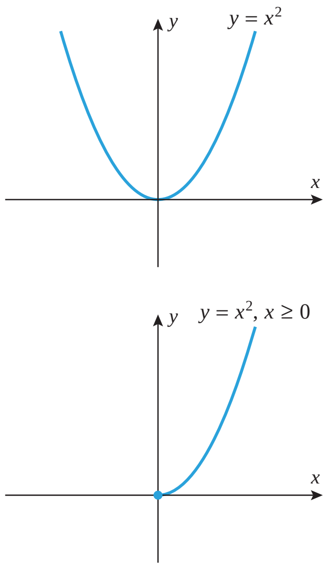
This inequality is satisfied if $x \leq 2$ or $x \geq 3$ (verify), so the natural domain of f is

$$(-\infty, 2] \cup [3, +\infty)$$

In some cases we will state the domain explicitly when defining a function. For example, if $f(x) = x^2$ is the area of a square of side x, then we can write

$$f(x) = x^2, \quad x \geq 0$$

to indicate that we take the domain of f to be the set of nonnegative real numbers (Figure 0.1.13).



▲ **Figure 0.1.13**

THE EFFECT OF ALGEBRAIC OPERATIONS ON THE DOMAIN

Algebraic expressions are frequently simplified by canceling common factors in the numerator and denominator. However, care must be exercised when simplifying formulas for functions in this way, since this process can alter the domain.

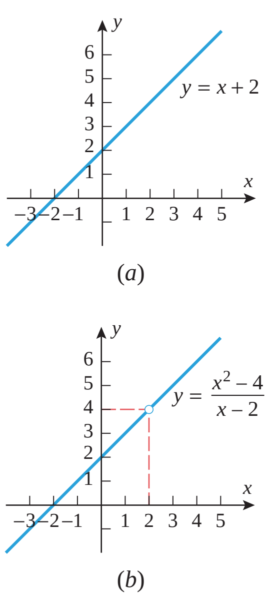
▶ **Example 7** The natural domain of the function

$$f(x) = \frac{x^2 - 4}{x - 2} \tag{2}$$

consists of all real x except $x = 2$. However, if we factor the numerator and then cancel the common factor in the numerator and denominator, we obtain

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2 \tag{3}$$

Since the right side of (3) has a value of $f(2) = 4$ and $f(2)$ was undefined in (2), the algebraic simplification has changed the function. Geometrically, the graph of (3) is the line in Figure 0.1.14*a*, whereas the graph of (2) is the same line but with a hole at $x = 2$, since the function is undefined there (Figure 0.1.14*b*). In short, the geometric effect of the algebraic cancellation is to eliminate the hole in the original graph. ◀



▲ **Figure 0.1.14**

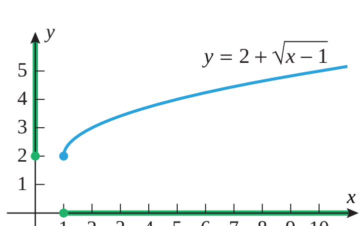
Sometimes alterations to the domain of a function that result from algebraic simplification are irrelevant to the problem at hand and can be ignored. However, if the domain must be preserved, then one must impose the restrictions on the simplified function explicitly. For example, if we wanted to preserve the domain of the function in Example 7, then we would have to express the simplified form of the function as

$$f(x) = x + 2, \quad x \neq 2$$

▶ **Example 8** Find the domain and range of

(a) $f(x) = 2 + \sqrt{x - 1}$ (b) $f(x) = (x + 1)/(x - 1)$

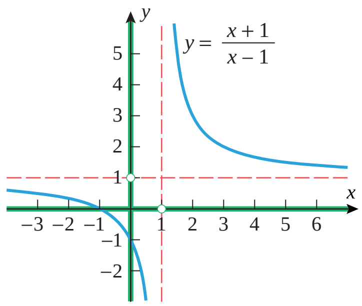
Solution (a). Since no domain is stated explicitly, the domain of f is its natural domain, $[1, +\infty)$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $f(x) = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$, which is the range of f. The domain and range are highlighted in green on the x- and y-axes in Figure 0.1.15.



▲ **Figure 0.1.15**

Solution (b). The given function f is defined for all real x, except $x = 1$, so the natural domain of f is

$$\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$



▲ **Figure 0.1.16**

To determine the range it will be convenient to introduce a dependent variable

$$y = \frac{x+1}{x-1} \tag{4}$$

Although the set of possible y-values is not immediately evident from this equation, the graph of (4), which is shown in Figure 0.1.16, suggests that the range of f consists of all y, except $y = 1$. To see that this is so, we solve (4) for x in terms of y:

$$\begin{aligned} (x-1)y &= x+1 \\ xy - y &= x+1 \\ xy - x &= y+1 \\ x(y-1) &= y+1 \\ x &= \frac{y+1}{y-1} \end{aligned}$$

It is now evident from the right side of this equation that $y = 1$ is not in the range; otherwise we would have a division by zero. No other values of y are excluded by this equation, so the range of the function f is $\{y : y \neq 1\} = (-\infty, 1) \cup (1, +\infty)$, which agrees with the result obtained graphically. ◀

DOMAIN AND RANGE IN APPLIED PROBLEMS

In applications, physical considerations often impose restrictions on the domain and range of a function.

▶ **Example 9** An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (Figure 0.1.17*a*).

(a) Let V be the volume of the box that results when the squares have sides of length x. Find a formula for V as a function of x.

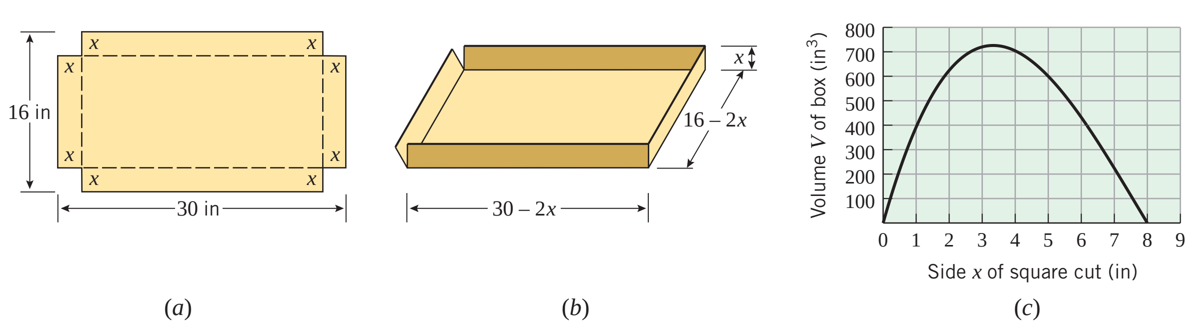
(b) Find the domain of V.

(c) Use the graph of V given in Figure 0.1.17*c* to estimate the range of V.

(d) Describe in words what the graph tells you about the volume.

Solution (a). As shown in Figure 0.1.17*b*, the resulting box has dimensions $16 - 2x$ by $30 - 2x$ by x, so the volume $V(x)$ is given by

$$V(x) = (16-2x)(30-2x)x = 480x - 92x^2 + 4x^3$$



▲ **Figure 0.1.17**

Solution (b). The domain is the set of x-values and the range is the set of V-values. Because x is a length, it must be nonnegative, and because we cannot cut out squares whose sides are more than 8 in long (why?), the x-values in the domain must satisfy

$$0 \le x \le 8$$

Solution (c). From the graph of V versus x in Figure 0.1.17c we estimate that the V-values in the range satisfy

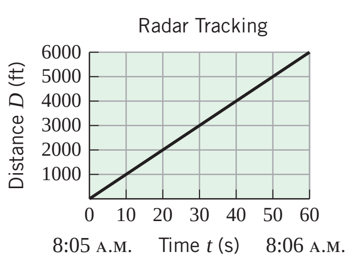
$$0 \le V \le 725$$

Note that this is an approximation. Later we will show how to find the range exactly.

Solution (d). The graph tells us that the box of maximum volume occurs for a value of x that is between 3 and 4 and that the maximum volume is approximately 725 in^3. The graph also shows that the volume decreases toward zero as x gets closer to 0 or 8, which should make sense to you intuitively. ◀

In applications involving time, formulas for functions are often expressed in terms of a variable t whose starting value is taken to be $t = 0$.

▶ **Example 10** At 8:05 A.M. a car is clocked at 100 ft/s by a radar detector that is positioned at the edge of a straight highway. Assuming that the car maintains a constant speed between 8:05 A.M. and 8:06 A.M., find a function $D(t)$ that expresses the distance traveled by the car during that time interval as a function of the time t.



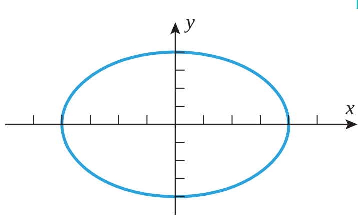
▲ **Figure 0.1.18**

Solution. It would be clumsy to use the actual clock time for the variable t, so let us agree to use the *elapsed* time in seconds, starting with $t = 0$ at 8:05 A.M. and ending with $t = 60$ at 8:06 A.M. At each instant, the distance traveled (in ft) is equal to the speed of the car (in ft/s) multiplied by the elapsed time (in s). Thus,

$$D(t) = 100t, \quad 0 \le t \le 60$$

The graph of D versus t is shown in Figure 0.1.18. ◀

ISSUES OF SCALE AND UNITS



The circle is squashed because 1 unit on the y-axis has a smaller length than 1 unit on the x-axis.

▲ **Figure 0.1.19**

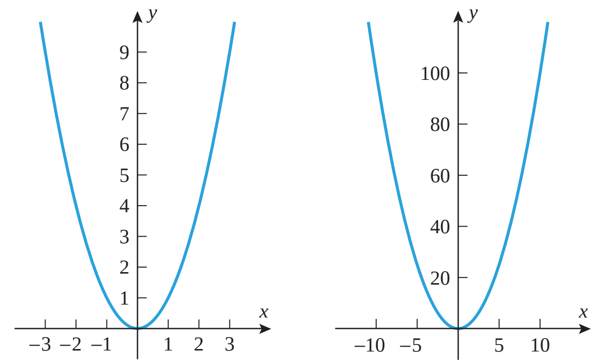
In geometric problems where you want to preserve the "true" shape of a graph, you must use units of equal length on both axes. For example, if you graph a circle in a coordinate system in which 1 unit in the y-direction is smaller than 1 unit in the x-direction, then the circle will be squashed vertically into an elliptical shape (Figure 0.1.19).

However, sometimes it is inconvenient or impossible to display a graph using units of equal length. For example, consider the equation

$$y = x^2$$

If we want to show the portion of the graph over the interval $-3 \le x \le 3$, then there is no problem using units of equal length, since y only varies from 0 to 9 over that interval. However, if we want to show the portion of the graph over the interval $-10 \le x \le 10$, then there is a problem keeping the units equal in length, since the value of y varies between 0 and 100. In this case the only reasonable way to show all of the graph that occurs over the interval $-10 \le x \le 10$ is to compress the unit of length along the y-axis, as illustrated in Figure 0.1.20.

In applications where the variables on the two axes have unrelated units (say, centimeters on the y-axis and seconds on the x-axis), then nothing is gained by requiring the units to have equal lengths; choose the lengths to make the graph as clear as possible.



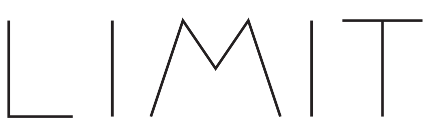
▶ **Figure 0.1.20**

✔ QUICK CHECK EXERCISES 0.1

(See page 15 for answers.)

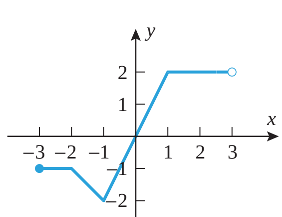
1. Let $f(x) = \sqrt{x+1} + 4$.
(a) The natural domain of f is ________.
(b) $f(3) =$ ________
(c) $f(t^2 - 1) =$ ________
(d) $f(x) = 7$ if $x =$ ________
(e) The range of f is ________.

2. Line segments in an xy-plane form "letters" as depicted.



(a) If the y-axis is parallel to the letter I, which of the letters represent the graph of $y = f(x)$ for some function f?
(b) If the y-axis is perpendicular to the letter I, which of the letters represent the graph of $y = f(x)$ for some function f?

3. The accompanying figure shows the complete graph of $y = f(x)$.
(a) The domain of f is ________.
(b) The range of f is ________.
(c) $f(-3) =$ ________
(d) $f\left(\frac{1}{2}\right) =$ ________
(e) The solutions to $f(x) = -\frac{3}{2}$ are $x =$ ________ and $x =$ ________.



◀ **Figure Ex-3**

4. The accompanying table gives a 5-day forecast of high and low temperatures in degrees Fahrenheit (°F).
(a) Suppose that x and y denote the respective high and low temperature predictions for each of the 5 days. Is y a function of x? If so, give the domain and range of this function.
(b) Suppose that x and y denote the respective low and high temperature predictions for each of the 5 days. Is y a function of x? If so, give the domain and range of this function.

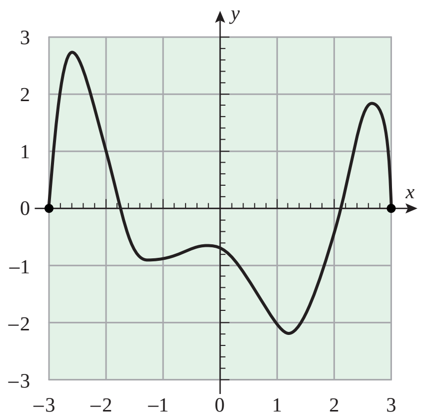
	MON	TUE	WED	THURS	FRI
HIGH	75	71	65	70	73
LOW	52	56	48	50	52

▲ **Table Ex-4**

5. Let l, w, and A denote the length, width, and area of a rectangle, respectively, and suppose that the width of the rectangle is half the length.
(a) If l is expressed as a function of w, then $l =$ ________.
(b) If A is expressed as a function of l, then $A =$ ________.
(c) If w is expressed as a function of A, then $w =$ ________.

EXERCISE SET 0.1 Graphing Utility

1. Use the accompanying graph to answer the following questions, making reasonable approximations where needed.
(a) For what values of x is $y = 1$?
(b) For what values of x is $y = 3$?
(c) For what values of y is $x = 3$?
(d) For what values of x is $y \leq 0$?
(e) What are the maximum and minimum values of y and for what values of x do they occur?



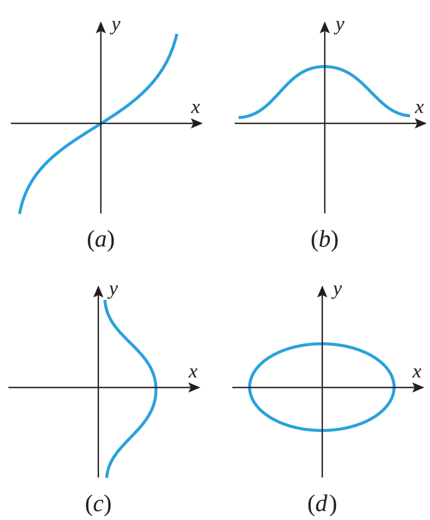
◀ **Figure Ex-1**

2. Use the accompanying table to answer the questions posed in Exercise 1.

x	–2	–1	0	2	3	4	5	6
y	5	1	–2	7	–1	1	0	9

▲ **Table Ex-2**

3. In each part of the accompanying figure, determine whether the graph defines y as a function of x.



▲ **Figure Ex-3**

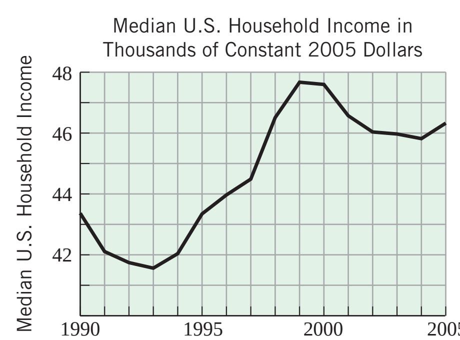
4. In each part, compare the natural domains of f and g.

(a) $f(x) = \dfrac{x^2 + x}{x + 1}$; $g(x) = x$

(b) $f(x) = \dfrac{x\sqrt{x} + \sqrt{x}}{x + 1}$; $g(x) = \sqrt{x}$

FOCUS ON CONCEPTS

5. The accompanying graph shows the median income in U.S. households (adjusted for inflation) between 1990 and 2005. Use the graph to answer the following questions, making reasonable approximations where needed.
(a) When was the median income at its maximum value, and what was the median income when that occurred?
(b) When was the median income at its minimum value, and what was the median income when that occurred?
(c) The median income was declining during the 2-year period between 2000 and 2002. Was it declining more rapidly during the first year or the second year of that period? Explain your reasoning.



Source: U.S. Census Bureau, August 2006.

▲ **Figure Ex-5**

6. Use the median income graph in Exercise 5 to answer the following questions, making reasonable approximations where needed.
(a) What was the average yearly growth of median income between 1993 and 1999?
(b) The median income was increasing during the 6-year period between 1993 and 1999. Was it increasing more rapidly during the first 3 years or the last 3 years of that period? Explain your reasoning.
(c) Consider the statement: "After years of decline, median income this year was finally higher than that of last year." In what years would this statement have been correct?

7. Find $f(0)$, $f(2)$, $f(-2)$, $f(3)$, $f(\sqrt{2})$, and $f(3t)$.

(a) $f(x) = 3x^2 - 2$ (b) $f(x) = \begin{cases} \dfrac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$

8. Find $g(3)$, $g(-1)$, $g(\pi)$, $g(-1.1)$, and $g(t^2 - 1)$.

(a) $g(x) = \dfrac{x+1}{x-1}$ (b) $g(x) = \begin{cases} \sqrt{x+1}, & x \ge 1 \\ 3, & x < 1 \end{cases}$

9–10 Find the natural domain and determine the range of each function. If you have a graphing utility, use it to confirm that your result is consistent with the graph produced by your graphing utility. [*Note:* Set your graphing utility in radian mode when graphing trigonometric functions.]

9. (a) $f(x) = \dfrac{1}{x-3}$ (b) $F(x) = \dfrac{x}{|x|}$

(c) $g(x) = \sqrt{x^2 - 3}$ (d) $G(x) = \sqrt{x^2 - 2x + 5}$

(e) $h(x) = \dfrac{1}{1 - \sin x}$ (f) $H(x) = \sqrt{\dfrac{x^2 - 4}{x - 2}}$

10. (a) $f(x) = \sqrt{3 - x}$ (b) $F(x) = \sqrt{4 - x^2}$

(c) $g(x) = 3 + \sqrt{x}$ (d) $G(x) = x^3 + 2$

(e) $h(x) = 3 \sin x$ (f) $H(x) = (\sin \sqrt{x})^{-2}$

FOCUS ON CONCEPTS

11. (a) If you had a device that could record the Earth's population continuously, would you expect the graph of population versus time to be a continuous (unbroken) curve? Explain what might cause breaks in the curve.

(b) Suppose that a hospital patient receives an injection of an antibiotic every 8 hours and that between injections the concentration C of the antibiotic in the bloodstream decreases as the antibiotic is absorbed by the tissues. What might the graph of C versus the elapsed time t look like?

12. (a) If you had a device that could record the temperature of a room continuously over a 24-hour period, would you expect the graph of temperature versus time to be a continuous (unbroken) curve? Explain your reasoning.

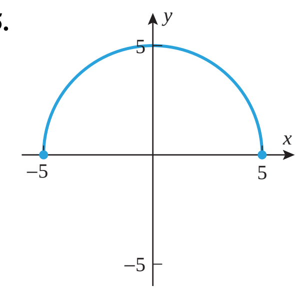
(b) If you had a computer that could track the number of boxes of cereal on the shelf of a market continuously over a 1-week period, would you expect the graph of the number of boxes on the shelf versus time to be a continuous (unbroken) curve? Explain your reasoning.

13. A boat is bobbing up and down on some gentle waves. Suddenly it gets hit by a large wave and sinks. Sketch a rough graph of the height of the boat above the ocean floor as a function of time.

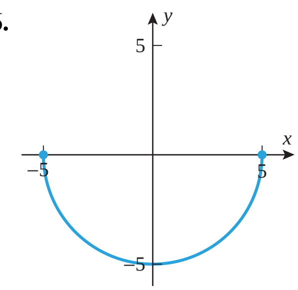
14. A cup of hot coffee sits on a table. You pour in some cool milk and let it sit for an hour. Sketch a rough graph of the temperature of the coffee as a function of time.

15–18 As seen in Example 3, the equation $x^2 + y^2 = 25$ does not define y as a function of x. Each graph in these exercises is a portion of the circle $x^2 + y^2 = 25$. In each case, determine whether the graph defines y as a function of x, and if so, give a formula for y in terms of x.

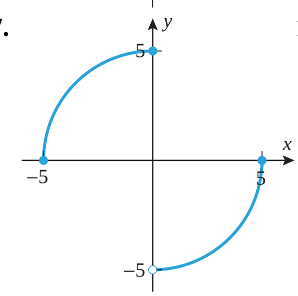
15.



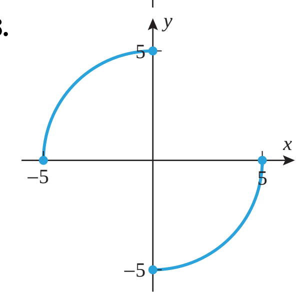
16.



17.



18.



19–22 True–False Determine whether the statement is true or false. Explain your answer.

19. A curve that crosses the x-axis at two different points cannot be the graph of a function.

20. The natural domain of a real-valued function defined by a formula consists of all those real numbers for which the formula yields a real value.

21. The range of the absolute value function is all positive real numbers.

22. If $g(x) = 1/\sqrt{f(x)}$, then the domain of g consists of all those real numbers x for which $f(x) \ne 0$.

23. Use the equation $y = x^2 - 6x + 8$ to answer the following questions.

(a) For what values of x is $y = 0$?

(b) For what values of x is $y = -10$?

(c) For what values of x is $y \ge 0$?

(d) Does y have a minimum value? A maximum value? If so, find them.

24. Use the equation $y = 1 + \sqrt{x}$ to answer the following questions.

(a) For what values of x is $y = 4$?

(b) For what values of x is $y = 0$?

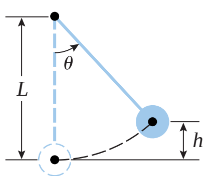
(c) For what values of x is $y \ge 6$?

(*cont.*)

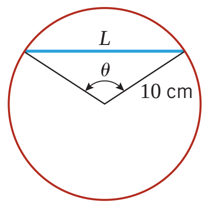
(d) Does y have a minimum value? A maximum value? If so, find them.

25. As shown in the accompanying figure, a pendulum of constant length L makes an angle θ with its vertical position. Express the height h as a function of the angle θ.

26. Express the length L of a chord of a circle with radius 10 cm as a function of the central angle θ (see the accompanying figure).



▲ **Figure Ex-25**



▲ **Figure Ex-26**

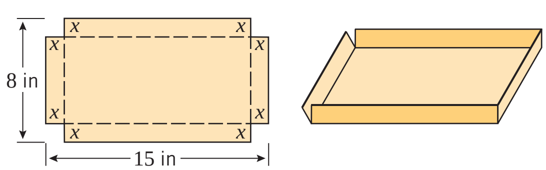
27–28 Express the function in piecewise form without using absolute values. [*Suggestion:* It may help to generate the graph of the function.] ■

27. (a) $f(x) = |x| + 3x + 1$ (b) $g(x) = |x| + |x - 1|$

28. (a) $f(x) = 3 + |2x - 5|$ (b) $g(x) = 3|x - 2| - |x + 1|$

29. As shown in the accompanying figure, an open box is to be constructed from a rectangular sheet of metal, 8 in by 15 in, by cutting out squares with sides of length x from each corner and bending up the sides.

(a) Express the volume V as a function of x.
(b) Find the domain of V.
(c) Plot the graph of the function V obtained in part (a) and estimate the range of this function.
(d) In words, describe how the volume V varies with x, and discuss how one might construct boxes of maximum volume.



▲ **Figure Ex-29**

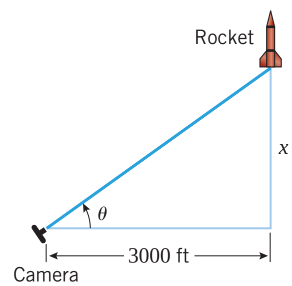
30. Repeat Exercise 29 assuming the box is constructed in the same fashion from a 6-inch-square sheet of metal.

31. A construction company has adjoined a 1000 ft^2 rectangular enclosure to its office building. Three sides of the enclosure are fenced in. The side of the building adjacent to the enclosure is 100 ft long and a portion of this side is used as the fourth side of the enclosure. Let x and y be the dimensions of the enclosure, where x is measured parallel to the building, and let L be the length of fencing required for those dimensions.

(a) Find a formula for L in terms of x and y.
(b) Find a formula that expresses L as a function of x alone.
(c) What is the domain of the function in part (b)?
(d) Plot the function in part (b) and estimate the dimensions of the enclosure that minimize the amount of fencing required.

32. As shown in the accompanying figure, a camera is mounted at a point 3000 ft from the base of a rocket launching pad. The rocket rises vertically when launched, and the camera's elevation angle is continually adjusted to follow the bottom of the rocket.

(a) Express the height x as a function of the elevation angle θ.
(b) Find the domain of the function in part (a).
(c) Plot the graph of the function in part (a) and use it to estimate the height of the rocket when the elevation angle is $\pi/4 \approx 0.7854$ radian. Compare this estimate to the exact height.



◀ **Figure Ex-32**

33. A soup company wants to manufacture a can in the shape of a right circular cylinder that will hold 500 cm^3 of liquid. The material for the top and bottom costs 0.02 cent/cm^2, and the material for the sides costs 0.01 cent/cm^2.

(a) Estimate the radius r and the height h of the can that costs the least to manufacture. [*Suggestion:* Express the cost C in terms of r.]
(b) Suppose that the tops and bottoms of radius r are punched out from square sheets with sides of length $2r$ and the scraps are waste. If you allow for the cost of the waste, would you expect the can of least cost to be taller or shorter than the one in part (a)? Explain.
(c) Estimate the radius, height, and cost of the can in part (b), and determine whether your conjecture was correct.

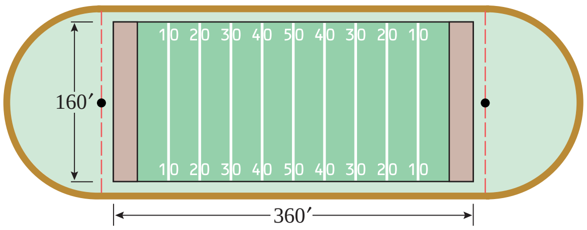
34. The designer of a sports facility wants to put a quarter-mile (1320 ft) running track around a football field, oriented as in the accompanying figure on the next page. The football field is 360 ft long (including the end zones) and 160 ft wide. The track consists of two straightaways and two semicircles, with the straightaways extending at least the length of the football field.

(a) Show that it is possible to construct a quarter-mile track around the football field. [*Suggestion:* Find the shortest track that can be constructed around the field.]
(b) Let L be the length of a straightaway (in feet), and let x be the distance (in feet) between a sideline of the football field and a straightaway. Make a graph of L versus x.

(cont.)

(c) Use the graph to estimate the value of x that produces the shortest straightaways, and then find this value of x exactly.

(d) Use the graph to estimate the length of the longest possible straightaways, and then find that length exactly.



▲ **Figure Ex-34**

35–36 (i) Explain why the function f has one or more holes in its graph, and state the x-values at which those holes occur. (ii) Find a function g whose graph is identical to that of f, but without the holes. ■

35. $f(x) = \dfrac{(x+2)(x^2-1)}{(x+2)(x-1)}$ **36.** $f(x) = \dfrac{x^2+|x|}{|x|}$

37. In 2001 the National Weather Service introduced a new wind chill temperature (WCT) index. For a given outside temperature T and wind speed v, the wind chill temperature index is the equivalent temperature that exposed skin would feel with a wind speed of v mi/h. Based on a more accurate model of cooling due to wind, the new formula is

$$\text{WCT} = \begin{cases} T, & 0 \le v \le 3 \\ 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}, & 3 < v \end{cases}$$

where T is the temperature in °F, v is the wind speed in mi/h, and WCT is the equivalent temperature in °F. Find the WCT to the nearest degree if $T = 25$°F and

(a) $v = 3$ mi/h (b) $v = 15$ mi/h (c) $v = 46$ mi/h.

Source: Adapted from UMAP Module 658, *Windchill*, W. Bosch and L. Cobb, COMAP, Arlington, MA.

38–40 Use the formula for the wind chill temperature index described in Exercise 37. ■

38. Find the air temperature to the nearest degree if the WCT is reported as −60°F with a wind speed of 48 mi/h.

39. Find the air temperature to the nearest degree if the WCT is reported as −10°F with a wind speed of 48 mi/h.

40. Find the wind speed to the nearest mile per hour if the WCT is reported as 5°F with an air temperature of 20°F.

✔ QUICK CHECK ANSWERS 0.1

1. (a) $[-1, +\infty)$ (b) 6 (c) $|t| + 4$ (d) 8 (e) $[4, +\infty)$ **2.** (a) M (b) I **3.** (a) $[-3, 3)$ (b) $[-2, 2]$ (c) -1 (d) 1 (e) $-\frac{3}{4}$; $-\frac{3}{2}$ **4.** (a) yes; domain: $\{65, 70, 71, 73, 75\}$; range: $\{48, 50, 52, 56\}$ (b) no **5.** (a) $l = 2w$ (b) $A = l^2/2$ (c) $w = \sqrt{A/2}$

0.2 NEW FUNCTIONS FROM OLD

Just as numbers can be added, subtracted, multiplied, and divided to produce other numbers, so functions can be added, subtracted, multiplied, and divided to produce other functions. In this section we will discuss these operations and some others that have no analogs in ordinary arithmetic.

■ ARITHMETIC OPERATIONS ON FUNCTIONS

Two functions, f and g, can be added, subtracted, multiplied, and divided in a natural way to form new functions $f+g$, $f-g$, fg, and f/g. For example, $f+g$ is defined by the formula

$$(f+g)(x) = f(x) + g(x) \tag{1}$$

which states that for each input the value of $f+g$ is obtained by adding the values of f and g. Equation (1) provides a formula for $f+g$ but does not say anything about the domain of $f+g$. However, for the right side of this equation to be defined, x must lie in the domains of both f and g, so we define the domain of $f+g$ to be the intersection of these two domains. More generally, we make the following definition.