ENG. Mahdi Alsiddig
0563938879

3.2 Monotonicity and Concavity

❖ Monotonicity

- لإيجاد فترة تزايد منحى الدالة (increasing) :

$$f'(x) \geq 0$$

- لإيجاد فترة تناقص منحى الدالة (decreasing) :

$$f'(x) \leq 0$$

Example 1 : If $f(x) = 2x^3 - 3x^2 - 12x + 7$,find where f is increasing and where it is decreasing.

Solution

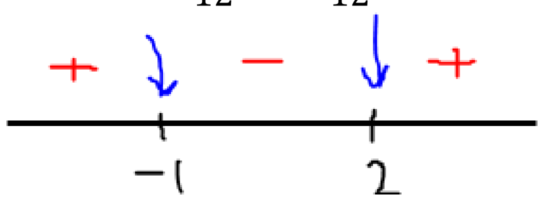
$$f'(x) = 6x^2 - 6x - 12 \geq 0$$

$$a = 6, \quad b = -6, \quad c = -12$$

$$x_{1,2} = \frac{6 \pm \sqrt{36 + 288}}{12} = \frac{6 \pm \sqrt{324}}{12} = \frac{6 \pm 18}{12}$$

$$x_1 = \frac{6+18}{12} = \frac{24}{12} = \mathbf{2}$$

$$x_2 = \frac{6-18}{12} = \frac{-12}{12} = \mathbf{-1}$$



$$\therefore f(x) \text{ is increasing on } (-\infty, -1] \cup [\,2, \infty\,)$$

$$\therefore f(x) \text{ is decreasing on } [-1, 2]$$

Example 2 : Determine where $g(x) = \frac{x}{(1+x^2)}$ is increasing and where it is decreasing.

Solution

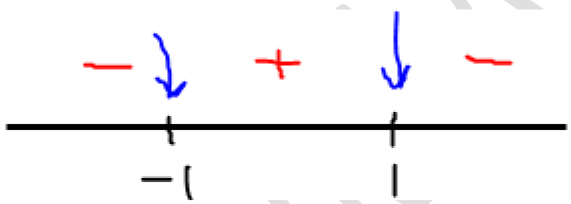
$$g'(x) = \frac{[(1)(1+x^2)] - [(x)(2x)]}{(1+x^2)^2} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} \geq 0$$

$$(1+x^2)^2 \cdot \frac{1-x^2}{(1+x^2)^2} \geq 0 \cdot (1+x^2)^2$$

$$1 - x^2 \geq 0$$

$$1 \geq x^2$$

$$\pm 1 \geq x$$



$\therefore$ ***f(x) is increasing on*** $[-1, 1]$

$\therefore$ ***f(x) is decreasing on*** $(-\infty, -1] \cup [1, \infty)$

❖ Concavity

- لإيجاد فترة تقعر منحى الدالة للأعلى (concave up) :

$$f''(x) > 0$$

- لإيجاد فترة تقعر منحى الدالة لأسفل (concave down) :

$$f''(x) < 0$$

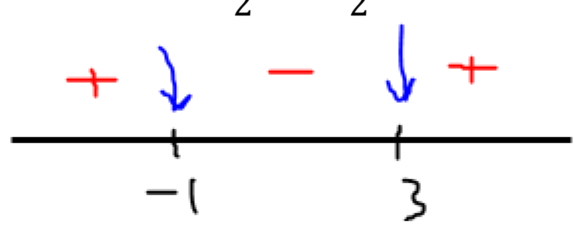
Example 3: Where is $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ increasing, decreasing, concave up and concave down ?

Solution

$$f'(x) = x^2 - 2x - 3$$
$$f''(x) = 2x - 2$$

- **اولا نوجد فترات تزايد وتناقص الدالة :**

$$f'(x) = x^2 - 2x - 3 \geq 0$$
$$a = 1, \quad b = -2, \quad c = -3$$
$$x_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2}$$
$$x_1 = \frac{2+4}{2} = \frac{6}{2} = \mathbf{3}$$
$$x_2 = \frac{2-4}{2} = \frac{-2}{2} = \mathbf{-1}$$



$$\therefore \boldsymbol{f(x)\ is\ increasing\ on}\ (-\infty, -1] \cup [3, \infty)$$

$$\therefore \boldsymbol{f(x)\ is\ decreasing\ on}\ [-1, 3]$$

- **ثانيا نوجد فترات تقعر الدالة لأعلى ولأسفل:**

$$f''(x) = 2x - 2 > 0$$
$$2x > 2$$
$$x > 1$$

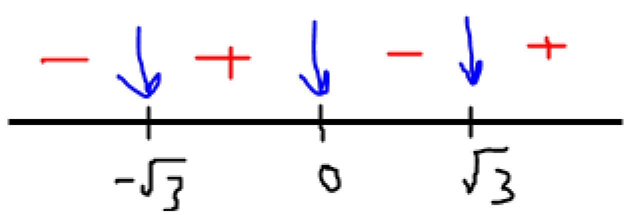
$$\therefore\ \boldsymbol{f(x)\ is\ concave\ up\ on\ (\ 1, \infty\ )}$$
$$\therefore\ \boldsymbol{f(x)\ is\ concave\ down\ on\ (-\infty, 1)}$$

Example 4 : Where is $g(x) = \frac{x}{(1+x^2)}$ is concave up and where is it concave down?

Solution

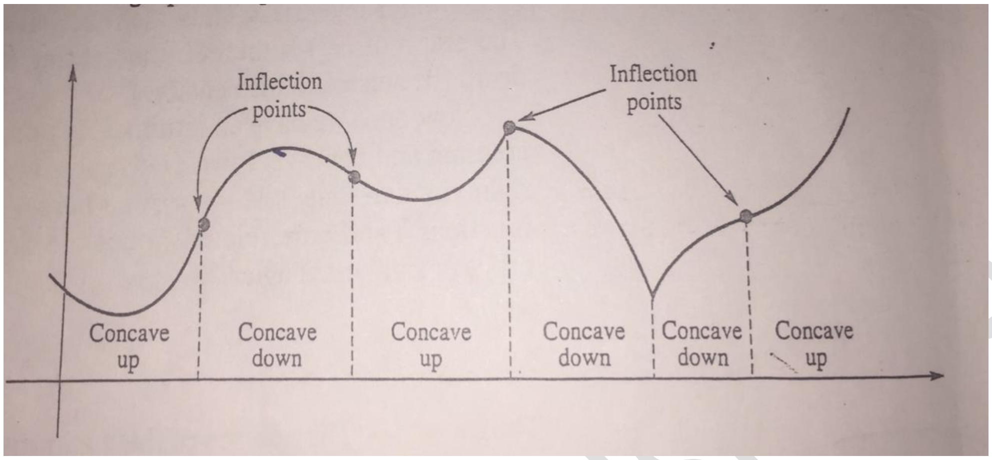
$$g'(x) = \frac{1 - x^2}{(1 + x^2)^2}$$

$$g''(x) = \frac{(-2x)(1 + x^2)^2 - (1 - x^2)2(1 + x^2)(2x)}{(1 + x^2)^4}$$
$$= \frac{(1 + x^2)[(-2x)(1 + x^2) - (1 - x^2)(4x)]}{(1 + x^2)^4}$$
$$= \frac{2x^3 - 6x}{(1 + x^2)^3} = \frac{2x(x^2 - 3)}{(1 + x^2)^3} > 0$$
$$2x(x^2 - 3) > 0$$
$$2x > 0 \quad \rightarrow x > \boldsymbol{0}$$
$$x^2 - 3 > 0 \quad \rightarrow \quad x^2 > 3 \quad \rightarrow \quad x > \pm\sqrt{\boldsymbol{3}}$$



$$\therefore\ \boldsymbol{f(x)\ is\ concave\ up\ on\ (\ -\sqrt{3}, 0\ ) \cup\ (\sqrt{3}, \infty)}$$
$$\therefore\ \boldsymbol{f(x)\ is\ concave\ down\ on\ (-\infty, -\sqrt{3})\ \cup\ (0\ , \sqrt{3})}$$

➢ Inflection points نقاط الانقلاب



- لإيجاد نقاط الانقلاب في منحى الدالة (inflection points) :

نوجد المشتقة الثانية للدالة ونساويها بالصفر , ونحل المعادلة لإيجاد قيم x

Example 7 : Find all points of inflection of $F(x) = x^{1/3} + 2$

Solution

$$F'(x) = \frac{1}{3}\, x^{-2/3}$$

$$F'(x) = -\frac{2}{9}\, x^{-5/3}$$

$\therefore$ ***The inflection point is*** : (**0** , **2**)