324 Stat
Lecture Notes

(8 One- and Two-Sample Test Of Hypothesis)

(Book*: Chapter 10 ,pg319)

Probability& Statistics for Engineers & Scientists
By Walpole, Myers, Myers, Ye

Definition:

A statistical hypothesis is a statement concerning one population or more.

The Null and The Alternative Hypotheses:

The structure of hypothesis testing will be formulated with the use of the term null hypothesis. This refers to any hypothesis we wish to test that called .
The rejection of H_0 leads to the acceptance of an alternative hypothesis H_1
A null hypothesis concerning a population parameter, will denoted by H_0 always be stated so as to specify an exact value of the parameter, θ whereas the alternative hypothesis allows for the possibility of several values. We usually test the null hypothesis: $H_0 : \theta = \theta_0$ against one of the following alternative hypothesis:

$$H_1 : \begin{Bmatrix} \theta \neq \theta_0 \\ \theta > \theta_0 \\ \theta < \theta_0 \end{Bmatrix}$$

Two Types of Errors:

	H_0 is true	H_0 is false
Accept H_0	Correct decision	Type II error, β
Reject H_0	Type I error, α	Correct decision

type I error: rejecting H_0 when H_0 is true.

Type II error: accepting H_0 when H_0 is false.

P (Type I error) =P (rejecting $H_0 \mid H_0$ is true) = α .

P (Type II error) = P (accepting $H_0 \mid H_0$ is false) =β .

Ideally we like to use a test procedure for which both the type I and type II errors are small.

* It is noticed that a reduction in β is always possible by increasing the size of the critical region, α .

* For a fixed sample size, decrease in the probability of one error will usually result in an increase in the probability of the other error.

* Fortunately the probability of committing both types of errors can be reduced by increasing the sample size.

Definition: Power of the Test:

The power of a test is the probability of rejecting H_0 given that a specific alternative hypothesis H_1 is true. The power of a test can be computed as (1-β) .

One – Tailed and Two – Tailed test:

A test of any statistical hypothesis where the alternative is one – sided such as:

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

$$or \quad H_1 : \theta < \theta_0$$

is called a one – tailed test

The critical region for the alternative hypothesis $H_1 : \theta > \theta_0$ lies entirely in the right tail of the distribution while the critical region for the alternative hypothesis $H_1 : \theta < \theta_0$ lies entirely in the left tail.

A test of any statistical hypothesis where the alternative is two – sided, such as: $H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta \neq \theta_0$ is called two – tailed test since the critical region is split into two parts having equal probabilities placed in each tail of the distribution of the test statistic.

The Use of P – Values in Decision Making:
Definition:

A p – value is the lowest level (of significance) at which the observed value of the test statistic is significant.

$P-value = 2P(Z > |Z_{obs}|)$ when H_1 is as follows: $H_1 : \theta \neq \theta_0$

$p-value = P(Z > Z_{obs})$ when H_1 is as follows: $H_1 : \theta > \theta_0$

$p-value = P(Z < Z_{obs})$ when H_1 is as follows: $H_1 : \theta < \theta_0$

H_0 is rejected if $p\text{-}value \leq \alpha$ otherwise H_0 is accepted.

EX(1):

$$H_0 : \mu = 10 \qquad vs \qquad H_1 : \mu \neq 10 \quad , \quad \alpha = 0.05 \Rightarrow Z = 1.87$$

$$P - value = 2P(Z > |1.87|) = 2P(Z > 1.87) = 2[1 - P(Z \leq 1.87)]$$

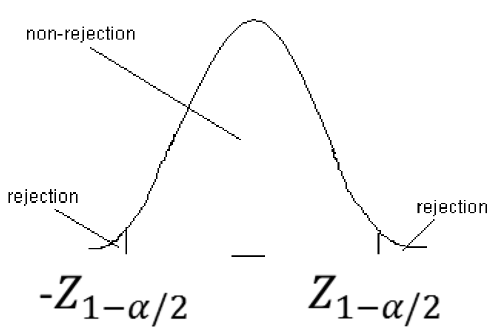
$$= 2[1 - 0.9693] = 2(0.0307) = 0.0614$$

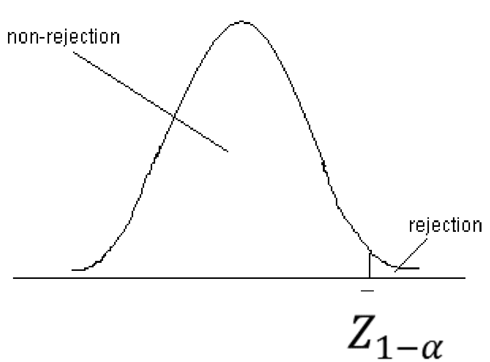
Since $P - value > \alpha$ then H_0 is accepted.

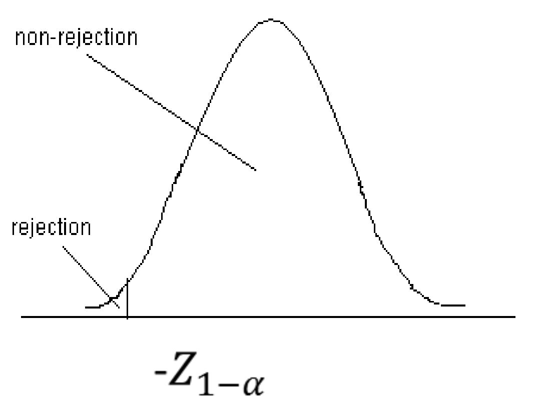
The Steps for testing a Hypothesis Concerning a Population Parameter θ (Reading):

1. Stating the null hypothesis H_0 that $\theta = \theta_0$.
2. Choosing an appropriate alternative hypothesis from one of the alternatives . $H_1 : \theta < \theta_0$ or $\theta > \theta_0$ or $\theta \neq \theta_0$
3. Choosing a significance level of size $\alpha = 0.01, 0.025, 0.05$ *or* 0.1

4. Determining the rejection or critical region (R.R.) and the acceptance region (A.R.) of H_0.







5- Selecting the appropriate test statistic and establish the critical region. If the decision is to be based on a p – value it is not necessary to state the critical region.

6. Computing the value of the test statistic from the sample data.

7. Decision rule:

A. rejecting H_0 if the value of the test statistic in the critical region or also $p - value \leq \alpha$

B. accepting H_0 if the value of the test statistic in the A.R. or if

$$p - value > \alpha$$

EX(2):

The manufacturer of a certain brand of cigarettes claims that the average nicotine content does not exceed **2.5** milligrams. State the null and alternative hypotheses to be used in testing this claim and determine where the critical region is located.

Solution:

$$H_0 : \mu = 2.5 \;\; against \;\; H_1 : \mu < 2.5$$

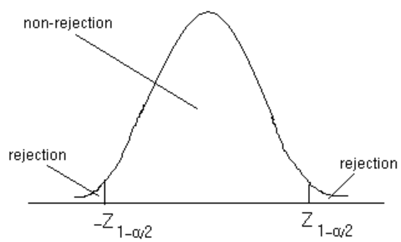
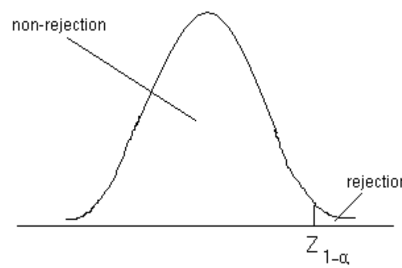
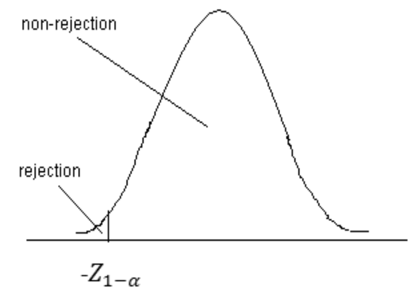
EX(3) H.w.:

A real state agent claims that 60% of all private residence being built today are **3** – bed room homes. To test this claim, a large sample of new residence is inspected, the proportion of the homes with **3** – bed rooms is recorded and used as our test statistic. State the null and alternative hypotheses to be used in this test and determine the location of the critical region.

Solution:

$$H_0 : p = 0.6 \qquad vs \qquad H_1 : p \neq 0.6$$

Tests Concerning a Single Mean

Hypothesis	$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$	$H_0 : \mu = \mu_0$ $H_1 : \mu > \mu_0$	$H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$
Test statistic (T.S.)	$Z = \dfrac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}, \sigma \; known \; or \;\; n \geq 30$		
R.R. and A.R. of H_0			
Decision	Reject H_0 (and accept H_1) at α the significance level if:		
	$Z > Z_{1-\alpha/2}$ or $Z < -Z_{1-\alpha/2}$ $Two - Sided\, Test$	$Z > Z_{1-\alpha}$ $One - Sided\, Test$	$Z < -Z_{1-\alpha}$ $One - Sided\, Test$

EX 10.3 pg 338

A random sample of **100** recorded deaths in the United States during the past year showed an average life span of **71.8** years with a standard deviation of **8.9** years. Dose this seem to indicate that the average life span today is greater than **70** years? Use a **0.05** level of significance.

Solution:

$$H_0 : \mu = 70 \quad vs \quad H_1 : \mu > 70 \quad , \quad \alpha = 0.05$$

$$n = 100, \bar{X} = 71.8, \sigma = 8.9$$

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{71.8 - 70}{8.9 / \sqrt{100}} = 2.02$$

$$Z_{1-\alpha} = Z_{0.05} = Z_{0.95} = 1.645 \rightarrow R.R.: Z > 1.645$$

$$Z = 2.02 > Z_{1-\alpha} = 1.645$$

$$Since\ Z = 2.02 \in R.R. \rightarrow we\ reject\ H_0\ at\ \alpha = 0.05$$

Reject H_0 since the value of the test statistic is in the critical region (R.R.) or

$$P-value = P(Z > 2.02) = 1 - P(Z \leq 2.02) = 1 - 0.9783 = 0.0217$$

Reject H_0 since $\alpha > p-value$

EX 10.4 pg 338

A manufacturer of sports equipment has developed a new synthetic fishing line that he claims has a standard deviation of **0.5** kilogram. Test the hypothesis that $\mu=8$ kilograms against the alternative that $\mu\neq 8$ kilograms if a random sample of **50** lines is tested and found to have a mean breaking strength of **7.8** kilograms. Use a **0.01** level of significance.

Solution:

$$H_0 : \mu = 8 \quad vs \quad H_1 : \mu \neq 8 \quad , \quad \alpha = 0.01$$

$$n = 50, \bar{X} = 7.8, \sigma = 0.5$$

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} = \frac{7.8 - 8}{0.5 / \sqrt{50}} = -2.82$$

$$Z_{1-\alpha/2} = Z_{0.995} = 2.575 \ and \ -Z_{1-\alpha/2} = -Z_{0.995} = -2.575$$

$$R.R.: Z > 2.575 \ or \ Z < -2.575$$

$$Since \ Z = -2.83 \in R.R. \rightarrow we \ reject \ H_0 \ at \ \alpha = 0.01$$

Reject H_0 since the value of **Z** is in the critical region (R.R.)

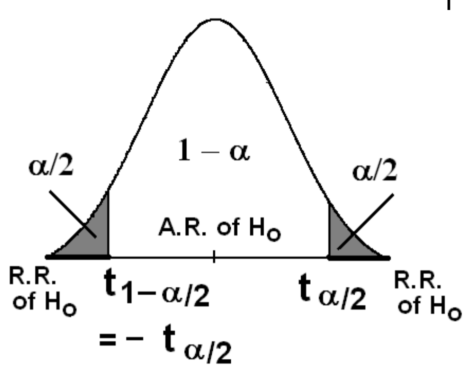
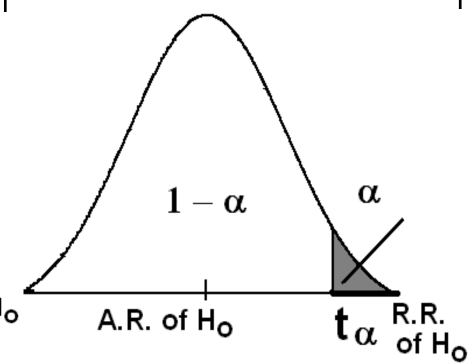
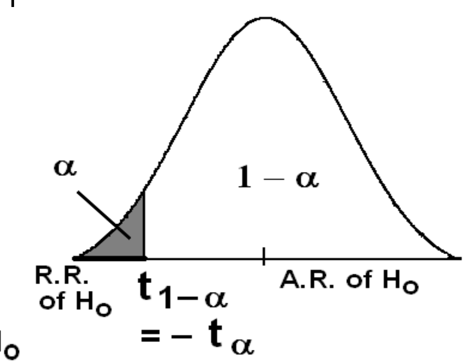
$$p - value = 2P(Z > |-2.83|) = 2P(Z > 2.83)$$

$$= 2(1 - P(Z \leq 2.83) = 2(1 - 0.9977)$$

$$= 2(0.0023) = 0.0046$$

H_0 is reject since $p - value \leq \alpha$

Tests Concerning a Single Mean (Variance Unknown)

Hypothesis	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$	$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$
Test statistic (T.S.)	$T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}, \sigma \text{ unknown or } n < 30$		
R.R. and A.R. of H_0			
Decision	Reject H_0 (and accept H_1) at the significance level α if:		
	$T > t_{\alpha/2}$ or $T < -t_{\alpha/2}$ (Two- Sided Test)	$T > t_{\alpha}$ (One- Sided Test)	$T < -t_{\alpha}$ (One- Sided Test)

EX(10.5):

If a random sample of **12** homes with a mean $\bar{X} = 42$ included in a planned study indicates that vacuum cleaners expend an average of **42** kilowatt – hours per year with standard deviation of **11.9** kilowatt hours dose this suggest at the **0.05** level of significance that vacuum cleaners expend on the average less than **46** kilowatt hours annually, assume the population of kilowatt - hours to be normal?

Solution:

$$H_0 : \mu = 46 \quad vs \quad H_1 : \mu < 46 \quad , \quad \alpha = 0.05$$

$$n = 12, \bar{X} = 42, S = 11.9$$

$$T = \frac{\bar{X} - \mu_0}{S / \sqrt{n}} = \frac{42 - 46}{11.9 / \sqrt{12}} = -1.16$$

$$v = n - 1 = 11, -t_{-\alpha} = -\ \mathrm{t}_{0.05,11=}\ -1.796$$

$$R.R.: T < -1.796$$

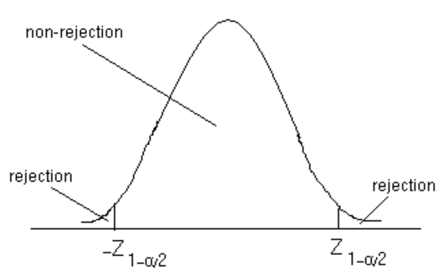
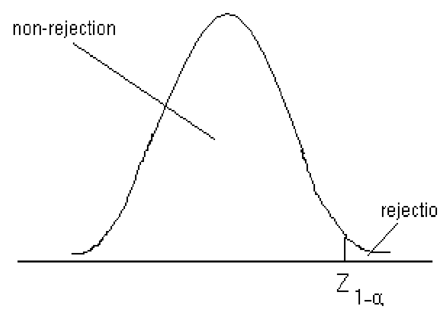
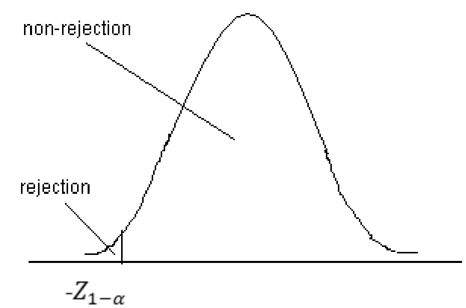
$$Since\ T = -1.16 \in A.R. \rightarrow we\ \ accept\ \ H_0\ \ at\ \ \alpha = 0.05$$

accept H_0 since the value of **t** is in the acceptance region (A.R.)

Tests Concerning Two Means

<table>
<tr>
<td>Hypothesis</td>
<td>$H_0: \mu_1 - \mu_2 = d$
$H_1: \mu_1 - \mu_2 \neq d$</td>
<td>$H_0: \mu_1 - \mu_2 = d$
$H_1: \mu_1 - \mu_2 > d$</td>
<td>$H_0: \mu_1 - \mu_2 = d$
$H_1: \mu_1 - \mu_2 < d$</td>
</tr>
<tr>
<td>Test statistic
(T.S.)</td>
<td colspan="3">$Z = \dfrac{(\bar{X}_1 - \bar{X}_2) - d}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$ $(if\ \sigma_1^2\ and\ \sigma_2^2\ are\ known)$

$or\ T = \dfrac{(\bar{X}_1 - \bar{X}_2) - d}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}},\ S_p^2 = \dfrac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$

$(if\ \sigma_1^2 = \sigma_2^2 = \sigma^2\ is\ unknown\ but\ equal)$</td>
</tr>
<tr>
<td>R.R. and A.R. of H_0</td>
<td>
</td>
<td>
</td>
<td>
</td>
</tr>
<tr>
<td>Decision</td>
<td colspan="3">Reject H_0 (and accept H_1) at the significance level α</td>
</tr>
<tr>
<td></td>
<td>$T.S. \in R.R.$
$Two - Sided\ Test$</td>
<td>$T.S. \in R.R.$
$One - Sided\ Test$</td>
<td>$T.S. \in R.R.$
$One - Sided\ Test$</td>
</tr>
</table>

EX(10.6 pg 344):

An experiment was performed to compare the abrasive wear of two different laminated materials. Twelve pieces of material **1** was tested, by exposing each piece to a machine measuring wear. Ten pieces of material **2** were similarly tested. In each case the depth of wear was observed. The samples of material **1** gave an average coded wear of **85** units with a standard deviation of **4** while the samples of material **2** gave an average coded wear of **81** and a standard deviation of **5.** Can we conclude at the **0.05** level of significance that the abrasive wear of material **1** exceeds that a material **2** by more than **2**units? Assume the population to be approximately normal with equal variances.

Solution:

Material 1	Material 2
$n_1 = 12$	$n_1 = 10$
$\bar{X}_1 = 85$	$\bar{X}_1 = 81$
$S_1 = 4$	$S_1 = 5$

$$H_0 : \mu_1 - \mu_2 = 2 \quad vs \quad H_1 : \mu_1 - \mu_2 > 2 \quad , \quad \alpha = 0.05 \quad ,$$

$$S_P = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}} = \sqrt{\frac{11(14) + 9(25)}{12 + 10 - 2}} = 4.478$$

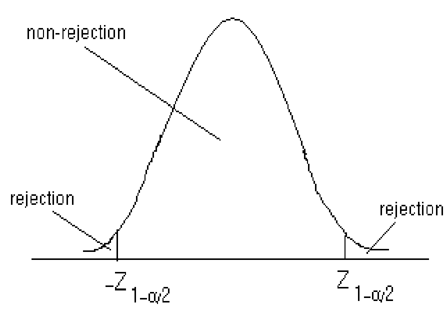
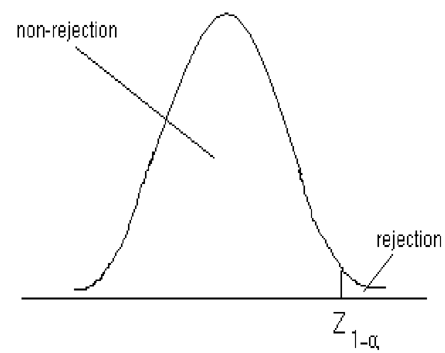
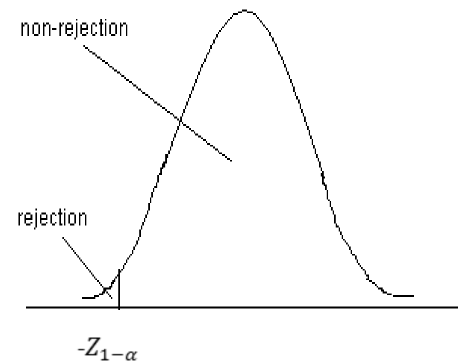
$$T = \frac{(\bar{X}_1 - \bar{X}_2) - d_0}{S_P \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(85 - 81) - 2}{4.478\sqrt{\frac{1}{12} + \frac{1}{10}}} = 1.04$$

$$v = n_1 + n_2 - 2 = 20, t_{1-0.05,20} = t_{0.95,20} = 1.725 \rightarrow R.R.: T > 1.725$$

$$Since\ T = 1.04 \in A.R. \rightarrow we\ accept\ H_0\ at\ \alpha = 0.05$$

Accept H_0 since the value of **t** is in the acceptance region.

Tests Concerning Proportions

Hypothesis	$H_0 : p = p_0$ $H_1 : p \neq p_0$	$H_0 : p = p_0$ $H_1 : p > p_0$	$H_0 : p = p_0$ $H_1 : p < p_0$
Test statistic (T.S.)	$Z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0 q_0}{n}}}$ $\quad (q_0 = 1 - p_0)$		
R.R. and A.R. of H_0			
Decision	Reject H_0 (and accept H_1) at the significance level α if:		
	$Z > Z_{1-\alpha/2}$ $or\ Z < -Z_{1-\alpha/2}$ $Two - Sided\ Test$	$Z > Z_{1-\alpha}$ $One - Sided\ Test$	$Z < -Z_{1-\alpha}$ $One - Sided\ Test$

EX 10.9 pg 362

A builder claims that heat pumps are installed in 70% of all homes being constructed today in the city of Richmond. Would you agree with this claim if a random survey of new homes in this city shows that **8** out of **15** had heat significance? (α=0.1)

Solution:

$$H_0 : p = 0.7 \quad vs \quad H_1 : p \neq 0.7 \quad , \quad \alpha = 0.1$$

$$n = 15, X = 8 \quad , \hat{p} = \frac{X}{n} = \frac{8}{15} = 0.533$$

$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} = \frac{0.533 - 0.7}{\sqrt{\frac{(0.7)(0.3)}{15}}} = -1.42$$

$$Z_{1-\alpha/2} = Z_{1-0.05} = Z_{0.95} = 1.645, -Z_{1-0.05} = -1.645$$

$$R.R.: Z > 1.645 \quad or \quad Z < -1.645$$

$$Since \quad Z = -1.42 \in \text{A.R} \rightarrow we \ accept \quad H_0 \quad at \quad \alpha = 0.1$$

$$p - value = 2P(Z > |-1.42|) = 2P(Z > 1.42)$$

$$= 2P(1 - P(Z \leq 1.42)) = 2(1 - 0.9222)$$

$$= 2(0.0778) = 0.1556$$

Accept H_0 since $\alpha < p - value$